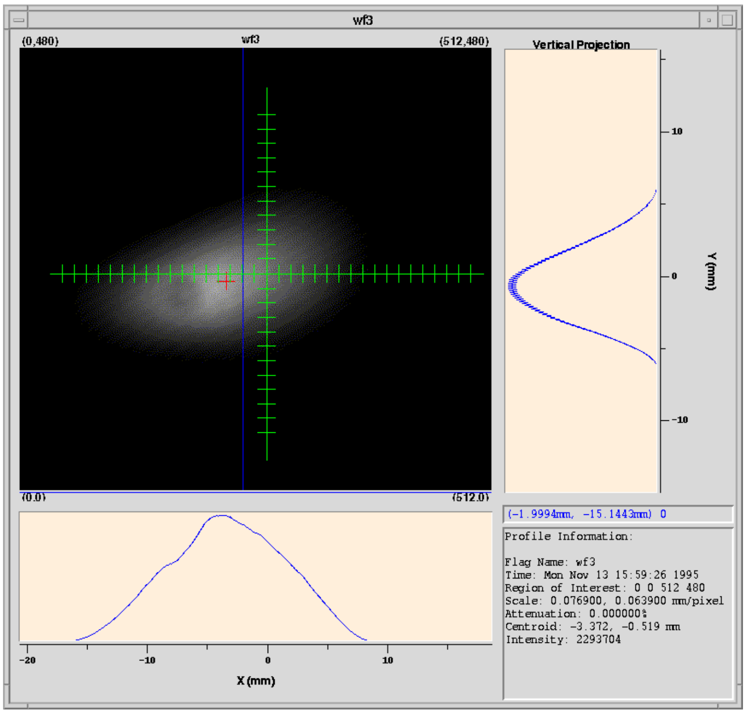
LINACS/Transport Lines Emittance Measurement



by Kay Wittenburg

LINACS/Transport Lines Emittance Measurement

In a transfer line (or Linac), the beam passes once and the shape of the emittance ellipse at the entry to the line determines its shape at the exit. Exactly the *same* transfer line injected first with one emittance ellipse and then different ellipses has to be accredited with *different* α and β, γ functions to describe the cases. Thus α and β, γ depend on the input beam and their propagation depends on the structure. Any change in the structure will only change the α and β, γ values downstream of that point. … The input ellipse must be chosen by the designer and should describe the configuration of all the particles in the beam.

1) **Explain ways of measuring the emittance of a charged particle beam in a Linear accelerator or a transport line without knowing the beam optic parameters α, β, γ.**

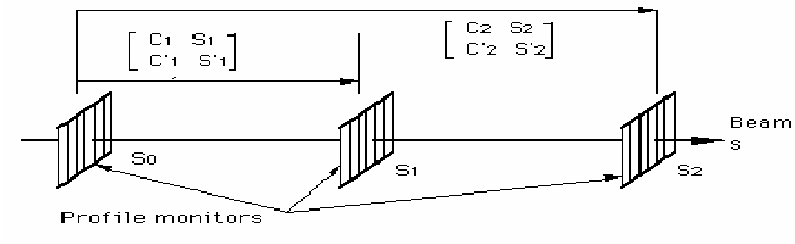
a) Exercise L1: Which one is the preferable method for a high energy proton transport line (p >5 GeV/c)?

Solution: 3 (thin) screens/SEM grids or varying quadrupole which measure the different beam widths σ. For pepper pot or slits one needs a full absorbing aperture.

b) Exercise L2: Assuming that the geometry between the measurement stations and the transport matrices M of the transport line are well defined (including magnetic elements), describe a way to get the emittance using 3 screens and the σ-matrix.

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_y^2 & \sigma_{yy'} \\ \sigma_{yy'} & \sigma_{y'}^2 \end{pmatrix} = \varepsilon_{rms} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \sigma\, matrix$$

If β is known unambiguously as in a circular machine, then a single profile measurement determines ε by $\sigma_y^2 = \varepsilon\beta$. But it is not easy to be sure in a transfer line which β to use, or rather, whether the beam that has been measured is matched to the β-values used for the line. This problem can be resolved by using three monitors (see Fig. 1), i.e. the three width measurements determine the three unknowns α, β and ε of the incoming beam.



σ elements at first Screen or Quadrupole (Ref. 1).

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_y^2 & \sigma_{yy'} \\ \sigma_{yy'} & \sigma_{y'}^2 \end{pmatrix} = \varepsilon_{rms} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \sigma\, matrix$$

Beam width$_{rms}$ of measured profile = σ_{11} = $\sqrt{\beta(s)\cdot\varepsilon}$,

L_1, L_2 = distances between screens or from Quadrupole to screen and Quadrupole field strength are given, therefore the transport matrix M is known.

Employing transfer matrix gives: $M \cdot \sigma \cdot M^t$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \cdot \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} = \sigma^{measured} = \begin{pmatrix} \sigma_y^2 & \sigma_{yy'} \\ \sigma_{y'y} & \sigma_{y'}^2 \end{pmatrix}^{measured} = \varepsilon_{rms} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11}M_{11} + \sigma_{12}M_{12} & \sigma_{11}M_{21} + \sigma_{12}M_{22} \\ \sigma_{21}M_{11} + \sigma_{22}M_{12} & \sigma_{12}M_{21} + \sigma_{22}M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11}(\sigma_{11}M_{11} + \sigma_{12}M_{12}) + M_{12}(\sigma_{21}M_{11} + \sigma_{22}M_{12}) & \cdots \\ \cdots & \cdots \end{pmatrix}$$

$\sigma_{11}^{measured} = M_{11}^2\sigma_{11} + 2M_{11}\, M_{12}\sigma_{12} + M_{12}^2\, \sigma_{22}$ $\quad(\sigma_{12} = \sigma_{21})$ (1)

Solving σ_{11} σ_{12} and σ_{22} while Matrix elements are known: Needs minimum of three different measurements, either three screens or three different Quadrupole settings with different field strength.

$\varepsilon_{rms} = \sqrt{\det \sigma} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$ (from $\beta\gamma - \alpha^2 = 1$) (2)

c) Exercise L3: In a transport line for p = 7.5 GeV/c protons are two measurement stations. The first is located exactly in the waist of the beam and shows a beam width of $\sigma_y = 3$ mm, the second at a distance of s = 10 m shows a width of $\sigma_y = 9$ mm. Assuming no optical elements in this part, calculate the emittance and the normalized emittance of the beam.

No optical elements => $M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ (3)

Waist => $\alpha = \sigma_{12} = \sigma_{21} = 0$ => $\varepsilon_{rms} = \sqrt{\sigma_{11}\sigma_{22}}$ (4)

Momentum p = 7.5 GeV/c => relativistic $\gamma\beta \approx 7.5$

Measured width at s = 0 => $(3\text{ mm})^2 = \sigma_y^2\,(s{=}0) = \sigma_{11}$ (5)

Calculate σ_{22} with width measured at s = 10 m and with (1, 4) =>
$(9\text{ mm})^2 = \sigma_y^2\,(s{=}10) = M_{11}^2 \cdot \sigma_{11} + M_{12}^2 \cdot \sigma_{22} = \sigma_{11} + s^2 \cdot \sigma_{22}$ $\quad(\sigma_{11}, \sigma_{22}$ at s=0) (6)

with (5) => $\sigma_{22} = \dfrac{\sigma_y^2(10) - \sigma_y^2(0)}{s^2}$ (7)

With (4) and (7) =>

$$\varepsilon_{rms} = \sqrt{\sigma_{11}\sigma_{22}} = \sqrt{\sigma_y^2(0) \cdot \frac{\sigma_y^2(10) - \sigma_y^2(0)}{s^2}} = \frac{\sigma_y(0)}{s}\sqrt{\sigma_y^2(10) - \sigma_y^2(0)}$$

$= \underline{2.5 \cdot 10^{-6}\text{ m rad}}$

$\varepsilon^{normalized} = \varepsilon_{rms}\, \gamma\, \beta = 19 \cdot 10^{-6}$ m rad = 19 mm mrad

Additional exercise: Calculate β(s=0 and s=10m)

Beam width σ_{rms} = $\sqrt{\beta(s)\cdot\varepsilon}$
At s=10 m: $\sigma^2 = \beta\varepsilon$ => β = 32.4 m
At s= 0 m : β = 3.6 m

What is the influence on the emittance ε assuming at s = 10m this b, a dispersion of D = 1 m and a momentum spread of $\Delta p/p = 10^{-3}$?

$$\varepsilon = \frac{\sigma^2 - \left(D\cdot\frac{\Delta p}{p}\right)^2}{\beta} = \frac{81\cdot10^{-6} - 1\cdot10^{-6}}{32.4} = 2.469\,\pi\, mm\, mrad$$

or ≈1% which is less than the typical accuracy of a profile measurement

References

S.Y. Lee, Accelerator Physics, World Scientific, pp 54-55 is attached

Emittance Measurements at the Bates Linac
K.D. Jacobs, J.B. Flanz, T. Russ
http://accelconf.web.cern.ch/accelconf/p89/PDF/PAC1989_1529.PDF
attached

Basic accelerator course, like Schmüser and Rossbach in previous CAS CERN 94-01 v 1
http://doc.cern.ch/cgi-bin/tiff2pdf?/archive/cernrep/1994/94-01/p17.tif

Beam-line instruments/ Raich, U ;
In: Joint US-CERN-Japan-Russia Particle Accelerators School on Beam Measurement, Montreux, Switzerland, 11-20 May 1998 - World Sci., Singapore, 1999. - pp.263-276

For teachers: attached: i)this document, ii) Criegee, PLIN note 88-04 (Criegee.pdf)
An other method: (P.J. Bryant, 5th CAS, Finnland)

PLIN - Note 88-04
L. Criegee
August 18, 1988

Emittance Measurements for Linac III

Principle

The measurement of the *Courant-Snyder* beam parameters $\alpha, \beta, \gamma, \epsilon$ is based on the evolution of the beam matrix σ. If the transport is described by the 2 x 2 matrix

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

with $R_{11}R_{22} - R_{12}R_{21} = 1$, the beam evolution is given by[1]

$$\sigma = \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = R \begin{pmatrix} \epsilon\beta_0 & -\epsilon\alpha_0 \\ -\epsilon\alpha_0 & \epsilon\gamma_0 \end{pmatrix} R^T .$$

$\epsilon\beta$ is the measurable squared beam envelope $\hat{x}^2$ at the end of the transport. It depends linearly on the initial beam matrix elements:

$$\hat{x}^2 = \epsilon\beta = R_{11}^2 \cdot \epsilon\beta_0 - 2R_{11}R_{12} \cdot \epsilon\alpha_0 + R_{12}^2 \cdot \epsilon\gamma_0$$

To obtain $\epsilon\alpha_0$, $\epsilon\beta_0$, $\epsilon\gamma_0$, and $\epsilon^2 = \epsilon\beta_0 \cdot \epsilon\gamma_0 - \epsilon\alpha_0 \cdot \epsilon\alpha_0$, one has set R to three (sufficiently different) values, measure the resulting $\hat{x}^2$, and solve three linear equations.

3-Position Method

One simple example is the simultaneous measurement with three profile harps as forseen for the HEBT. Here the transport matrix

$$R(S) = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$$

describes just a variable drift space. The three settings $S_1 = -L$, $S_2 = 0$, $S_3 = +L$ give the beam transport from a reference position S_2 to two symmetric ones up- and downstream. They lead to the equations

$$\left.\begin{aligned} \hat{x}_1^2 &= \epsilon\beta_2 - 2L \cdot \epsilon\alpha_2 + L^2 \cdot \epsilon\gamma_2 \\ \hat{x}_2^2 &= \epsilon\beta_2 \\ \hat{x}_3^2 &= \epsilon\beta_2 + 2L \cdot \epsilon\alpha_2 + L^2 \cdot \epsilon\gamma_2 \end{aligned}\right\} \text{ with the solution } \left\{\begin{aligned} \epsilon\beta_2 &= \hat{x}_2^2 \\ \epsilon\alpha_2 &= (\hat{x}_3^2 - \hat{x}_1^2)/(4L) \\ \epsilon\gamma_2 &= (\hat{x}_1^2 - 2\hat{x}_2^2 + \hat{x}_3^2)/(2L^2) \\ \epsilon^2 &= \epsilon\beta_2 \cdot \epsilon\gamma_2 - (\epsilon\alpha_2)^2 \end{aligned}\right.$$

The measurements should be done in the vicinity of a beam waist where the curvature due to $\epsilon\gamma$ is most noticeable. If the beam waist is exactly at S_2, we have $\hat{x}_3 = \hat{x}_1$, $\alpha_2 = 0$. The emittance formula is then reduced to

$$\epsilon = \hat{x}_2\sqrt{\hat{x}_3^2 - \hat{x}_2^2}/L \qquad (\text{for } \hat{x}_2 = minimum)$$

[1]see K.L.Brown et al., CERN 80-04 (TRANSPORT manual)

$\beta \cdot \gamma - \alpha^2 = 1$

3-Gradient Method

If the 3-Position method cannot be used, the beam parameters can be measured by varying the gradient of a quadrupole. The system of the quadrupole and a subsequent transport section with constant elements C, S, C' and S' ($CS' - C'S = 1$) is described by

$$R(k) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \cos kl & k^{-1}\sin kl \\ -k\sin kl & \cos kl \end{pmatrix} \quad \text{with} \quad \begin{cases} R_{11}(k) = & C\cos kl \quad -Sk\sin kl \\ R_{12}(k) = & Ck^{-1}\sin kl \quad +S\cos kl \end{cases}$$

The measurement of $\hat{x}^2$, as defined above, for three different quadrupole settings k yields the linear equations to determine all beam parameters.

Simplified 3-Gradient Method

The function $\hat{x}^2(k)$ is greatly simplified[2] if the quadrupole can be described as a thin lense ($kl \to 0$, $k^2 l \to q = f^{-1}$):

$$\begin{aligned} R_{11}(k) &\to R_{11}(q) = C - q \cdot S \\ R_{12}(k) &\to R_{12}(q) = S \end{aligned}$$

This leads to

$$\begin{aligned} \hat{x}^2(q) &= (C - q \cdot S)^2 \cdot \epsilon\beta_0 - 2S(C - qS) \cdot \epsilon\alpha_0 + S^2 \cdot \epsilon\gamma_0 \\ &= S^2\epsilon^2/\hat{x}_0^2 + \hat{x}_0^2 \cdot (q - q_{min})^2 \ . \end{aligned}$$

with $q_{min} = C/S - \alpha_0/\beta_0$ being the setting for minimal $\hat{x}^2$, and $\hat{x}_0^2 = \epsilon\beta_0$ the squared envelope before the quadrupole. The emittance is then given by

$$\epsilon = \frac{\hat{x}(q_{min})\,\sqrt{\hat{x}^2(q) - \hat{x}^2(q_{min})}}{S^2|q - q_{min}|} \ .$$

The other beam parameters (at the entrance of the quadrupole) are

$$\begin{aligned} \hat{x}_0 &= \sqrt{\hat{x}^2(q) - \hat{x}^2(q_{min})}/(S|q - q_{min}|) \\ \beta_0 &= \hat{x}_0^2/\epsilon \\ \alpha_0 &= \beta_0\,(C/S - q_{min}) \ . \end{aligned}$$

Some comments may be in order:

1. The formula for the emittance is simple enough to be coded in POCAL.
2. The emittance measurement after linac tank I forsees the variation of quad No 52, while No 53 is turned off [3]. The above formalism allows also measurements with No 53 powered (fixed $C' \neq 0$) and also to account for the defocussing by (linear) space charge and by RF acceleration.
3. Suitable quadrupole settings and the quality of the approximations can be investigated by beam transport calculations.

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[2] G. Jacobs, private communication

[3] S.H. Wang, PLIN - Note 88-01 (June 24, 1988)

An other method: (P.J. Bryant, 5th CAS, Finnland)

By definition, Eq. (4),

$$\varepsilon = \pi \frac{\sigma_0^2}{\beta_0} = \pi \frac{\sigma_1^2}{\beta_1} = \pi \frac{\sigma_2^2}{\beta_2} \tag{5}$$

where β_0, β_1 and β_2 are the β-values corresponding to the beam and are therefore uncertain. Although we may not know β and α, we do know the transfer matrices and how β and α propagate through the structure (see lectures by K. Steffen in these proceedings).

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1 = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \tag{6}$$

where $\gamma = (1 + \alpha^2)/\beta$. Thus, from Eq. (6)

$$\beta_1 = C_1^2 \beta_0 - 2C_1 S_1 \alpha_0 + \frac{S_1^2}{\beta_0}\left(1 + \alpha_0^2\right) \tag{7}$$

$$\beta_2 = C_2^2 \beta_0 - 2C_2 S_2 \alpha_0 + \frac{S_2^2}{\beta_0}\left(1 + \alpha_0^2\right) \tag{8}$$

and from Eq. (5),

$$\beta_0 = \pi \frac{\sigma_0^2}{\varepsilon} \tag{9}$$

$$\beta_1 = \left(\frac{\sigma_1}{\sigma_0}\right)^2 \beta_0 \tag{10}$$

$$\beta_2 = \left(\frac{\sigma_2}{\sigma_0}\right)^2 \beta_0 \tag{11}$$

From Eqs. (7) and (8), we can find α_0 and using Eqs. (10) and (11), we can express α_0 as,

$$\alpha_0 = \frac{1}{2}\beta_0 \Gamma \tag{12}$$

where

$$\Gamma = \frac{\left(\sigma_2/\sigma_0\right)^2/S_2^2 - \left(\sigma_1/\sigma_0\right)^2/S_1^2 - \left(C_2/S_2\right)^2 + \left(C_1/S_1\right)^2}{\left(C_1/S_1\right) - \left(C_2/S_2\right)} \ . \tag{13}$$

Since Γ is fully determined, direct substitution back into Eq. (7) or Eq. (8), using Eq. (10) or Eq. (11) to re-express β_1 or β_2, yields β_0 which via Eq. (9) gives the emittance,

$$\beta_0 = 1 \Big/ \left|\sqrt{\left(\sigma_{\frac{1}{2}}/\sigma_0\right)^2/S_{\frac{1}{2}}^2 - \left(C_{\frac{1}{2}}/S_{\frac{1}{2}}\right)^2 + \left(C_{\frac{1}{2}}/S_{\frac{1}{2}}\right)\Gamma - \Gamma^2/4}\right| \tag{14A}$$

$$\varepsilon = \left(\pi\sigma_0^2\right)\left|\sqrt{\left(\sigma_{\frac{1}{2}}/\sigma_0\right)^2/S_{\frac{1}{2}}^2 - \left(C_{\frac{1}{2}}/S_{\frac{1}{2}}\right)^2 + \left(C_{\frac{1}{2}}/S_{\frac{1}{2}}\right)\Gamma - \Gamma^2/4}\right| \ . \tag{14B}$$