Definition of Particle Beams

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Thanks to Chris Prior, RAL and Erk Jensen, CERN

Outline

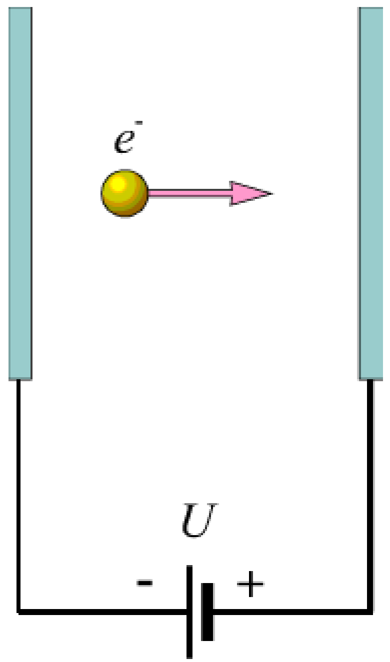
- Basic quantities and concepts in accelerator physics
- Recap: Fundamentals of relativity and electromagnetism
- A short introduction to ion optics
  - Why do require those for beam instrumentation ?

Literature

- J.D. Jackson, *Classical Electrodynamics*
- H. Wiedemann, *Particle Accelerator Physics I & II*
- K.G. Steffen, *High Energy Beam Optics*
- M. Livingston, J. Blewett, *Particle Accelerators*
- CERN *Yellow Reports*
- *...*

Particle Energies

Definition of 1 eV:



$$E = eU = 1.602 \cdot 10^{-19}\,\mathrm{J}$$

$$\Leftrightarrow \quad E = 1\mathrm{eV}$$

Common units

$$1\ \mathrm{keV} = 10^3\ \mathrm{eV},$$
$$1\ \mathrm{MeV} = 10^6\ \mathrm{eV}$$
$$1\ \mathrm{GeV} = 10^9\ \mathrm{eV}$$
$$1\ \mathrm{TeV} = 10^{12}\ \mathrm{eV}$$

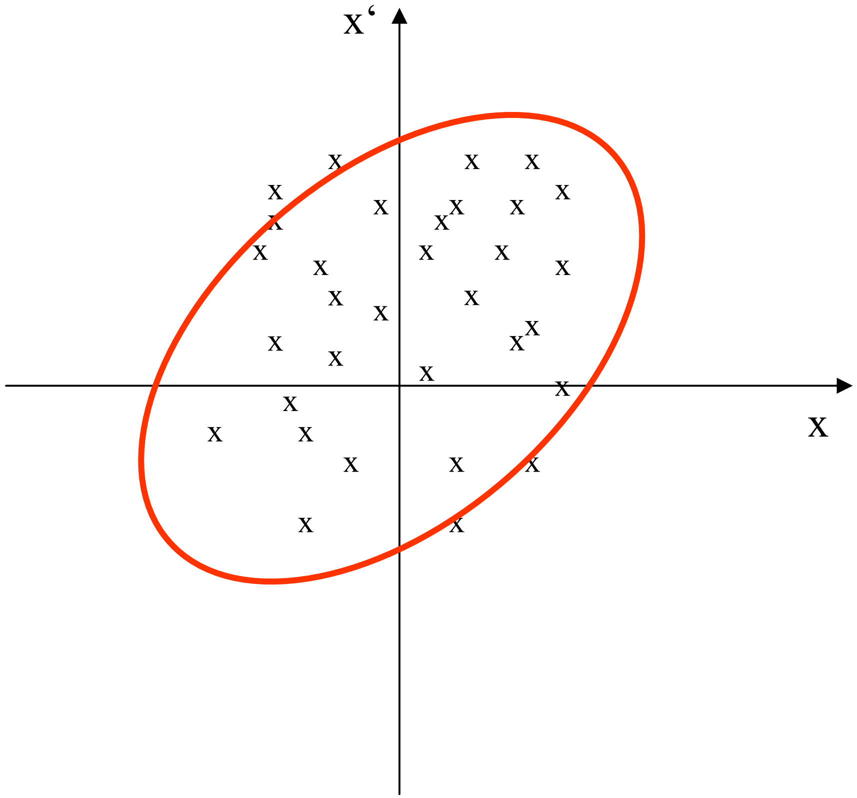
⇨ Maxwell equations,
Theory of relativity

Other Important Units

- Mass [eV/c²]
  - $1\ eV/c^2 = 1.78 \times 10^{-36}$ kg
  - Electron mass = 511 keV/c^2
  - Proton mass = 938 MeV/c^2
  - Carsten's mass $\approx 4 \times 10^{37}\ eV/c^2$

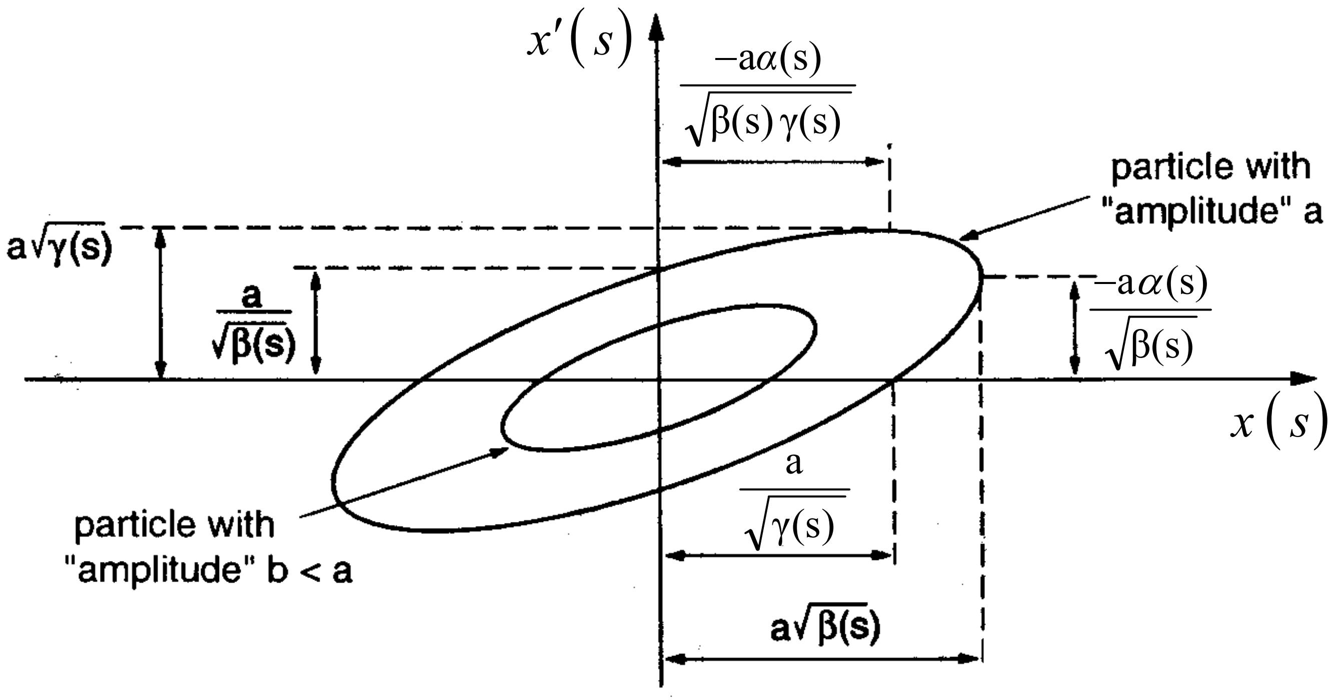
- Momentum [eV/c]
  - $1\ eV/c = 5.3 \times 10^{-28}$ kg m/s
  - Momentum of football at 70 km/h
    ≈ 10 kg m/s $\approx 2 \times 10^{28}$ eV/c

Concept of Emittance



Contain 95 % of particles.

Definition



$\alpha(s) \quad \beta(s) \quad \gamma(s)$ are called TWISS Parameters.

Shape and orientation evolve along the machine.

Area stays constant *(Liouville's theorem)*.

Remarks

- Unit of emittance is [mm · mrad]
  – Possible to confuse with ellipse area
- Area often given in publications
  – explicitly contained in [π · mm · mrad]
- No standard for percentage !
- Also: Statistical definitions available

⇨ Look carefully at specific use/definition !

How to Accelerate Particles ?!?

Force	Rel. Strength	Reach [m]	Concerned particles
Gravitation	$6 \cdot 10^{-39}$	∞	all
Electro-Magnetism	1/137	∞	charged particles
Strong Force	~ 1	10^{-15}-10^{-16}	Hadrons
Weak Force	10^{-5}	<<10^{-16}	Hadrons and Leptons

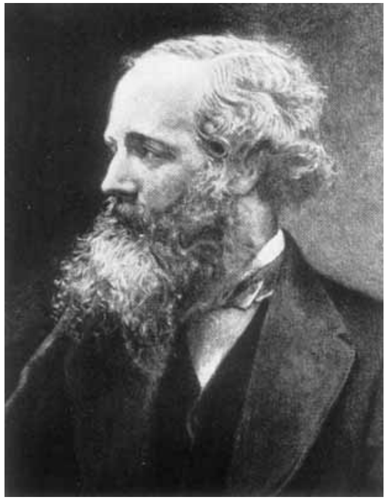
What is Electromagnetism?

- The study of Maxwell's equations, devised in 1863 to represent the relationships between electric and magnetic fields in the presence of electric charges and currents, whether steady or rapidly fluctuating, in a vacuum or in matter.

- The equations represent one of the most elegant and concise way to describe the fundamentals of electricity and magnetism. They pull together in a consistent way earlier results known from the work of Gauss, Faraday, Ampère, Biot, Savart and others.

- Remarkably, Maxwell's equations are perfectly consistent with the transformations of special relativity.

Maxwell’s Equations



- Relate Electric and Magnetic fields generated by charge and current distributions.

E = electric field

D = electric displacement

H = magnetic field

B = magnetic flux density

ρ= charge density

j = current density

μ_0 (permeability of free space) = $4\pi\ 10^{-7}$

ε_0 (permittivity of free space) = $8.854\ 10^{-12}$

c (speed of light) = $2.99792458\ 10^{8}$ m/s

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

In vacuum $\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 c^2 = 1$

The Cockcroft Institute of Accelerator Science and Technology

Maxwell's 1st Equation

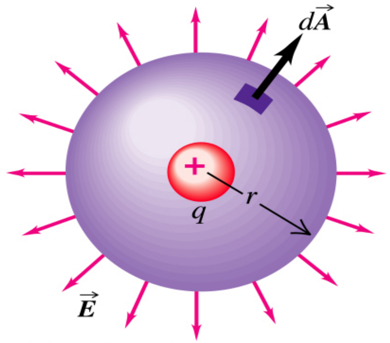
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Gauss' Flux Theorem:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Leftrightarrow \quad \iiint_V \nabla \cdot \vec{E}\, dV = \oint\!\!\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint \rho\, dV = \frac{Q}{\varepsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface.

A point charge *q* generates an electric field

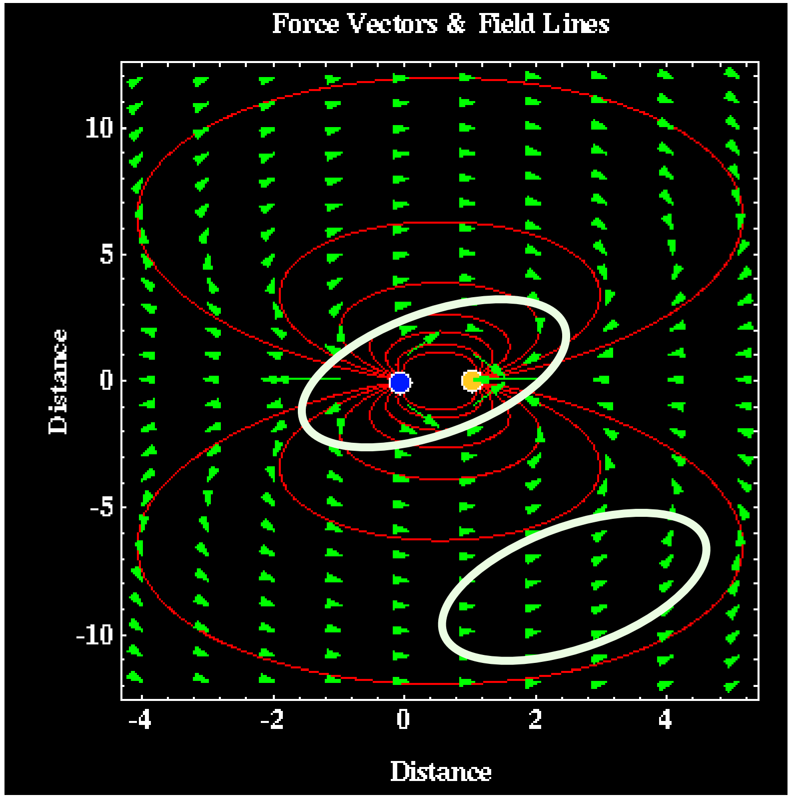


$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

$$\iint_{sphere} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0} \iint_{sphere} \frac{dS}{r^2} = \frac{q}{\varepsilon_0}$$

Maxwell's 2nd Equation

$$\nabla \cdot \vec{B} = 0$$



$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \oiint \vec{B} \cdot d\vec{S} = 0$$

A statement that

There are no magnetic monopoles

Maxwell's 3rd Equation

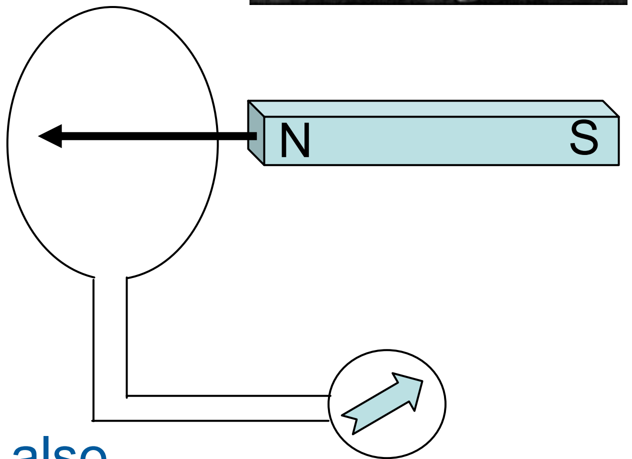
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Equivalent to Faraday's Law of Induction:

$$\iint_S \nabla \times \vec{E} \cdot d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

(for a fixed circuit *C*) $\Leftrightarrow \oint_C E \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$

The electromotive force round a circuit $\varepsilon = \oint \vec{E} \cdot d\vec{l}$ is proportional to the rate of change of flux of magnetic field, $\Phi = \iint \vec{B} \cdot d\vec{S}$ through the circuit.



Faraday's Law is the basis for electric generators. It also forms the basis for inductors and transformers.

4^{th} Equation

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Ampère

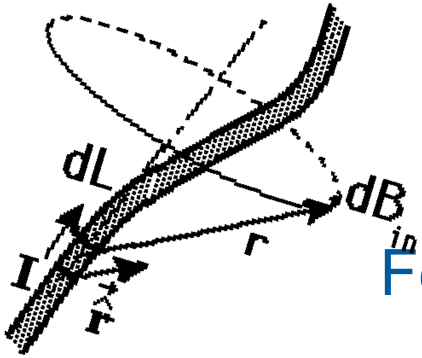
Originates from Ampère's (Circuital) Law : $\nabla \times \vec{B} = \mu_0 \vec{j}$

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_S \nabla \times \vec{B} \cdot d\vec{S} = \mu_0 \iint_S \vec{j} \cdot d\vec{S} = \mu_0 I$$

Biot

Satisfied by the field for a steady line current (Biot-Savart Law, 1820):

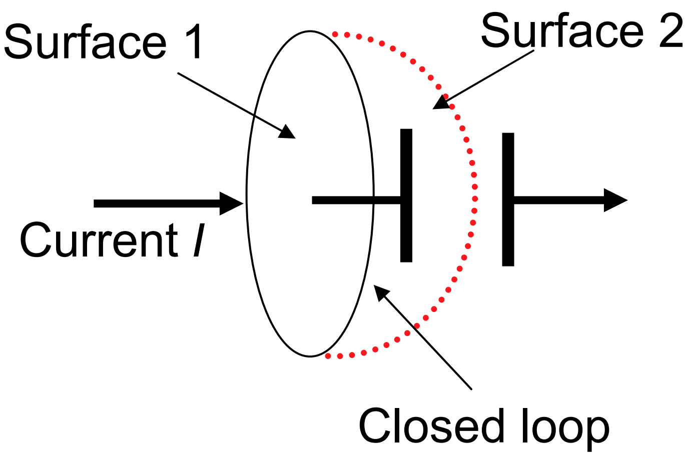
$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \wedge \vec{r}}{r^3}$$



For a straight line current $B_\theta = \frac{\mu_0 I}{2\pi r}$

Need for displacement current

- Faraday: vary B-field, generate E-field
- Maxwell: varying E-field should then produce a B-field, but not covered by Ampère's Law.



- Apply Ampère to surface 1 (flat disk): line integral of **B** = $\mu_0 I$
- Applied to surface 2, line integral is zero since no current penetrates the deformed surface.
- In capacitor, $E = \frac{Q}{\varepsilon_0 A}$, so $I = \frac{dQ}{dt} = \varepsilon_0 A \frac{dE}{dt}$
- Displacement current density is $\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \vec{j}_d \right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consistency with charge conservation

- Charge conservation: Total current flowing out of a region equals the rate of decrease of charge within the volume.

$$\oiint \vec{j} \cdot d\vec{S} = -\frac{d}{dt}\iiint \rho \, dV$$

$$\Leftrightarrow \quad \iiint \nabla \cdot \vec{j} \, dV = -\iiint \frac{\partial \rho}{\partial t} dV$$

$$\Leftrightarrow \quad \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

- From Maxwell's equations: Take divergence of (modified) Ampère's equation

$$\nabla \cdot \nabla \times \overline{B} = \mu_0 \nabla \cdot \vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla \cdot \vec{E}\right)$$

$$\Rightarrow \quad 0 = \mu_0 \nabla \cdot \vec{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right)$$

$$\Rightarrow \quad 0 = \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t}$$

Charge conservation is implicit in Maxwell's Equations

Maxwell's Equations in Vacuum

- In vacuum

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

- Source-free equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

- Source equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

- Equivalent integral forms (sometimes useful for simple geometries)

$$\oiint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint \rho \, dV$$

$$\oiint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{j} \cdot d\vec{S} + \frac{1}{c^2} \frac{d}{dt} \iint \vec{E} \cdot d\vec{S}$$

In addition: Lorentz force

A charged particle moving with velocity $\vec{v}$ through an electro-magnetic field experiences a force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \vec{v} = \frac{\vec{p}}{m\gamma}$$

The energy of the particle is $W = \sqrt{\left(mc^2\right)^2 + \left(pc\right)^2} = \gamma mc^2$

Change of *W* due to the this force (work done) ; differentiate:

$$W\mathrm{d}W = c^2\vec{p}\cdot\mathrm{d}\vec{p} = qc^2\vec{p}\cdot\left(\vec{E}+\vec{v}\times\vec{B}\right)\mathrm{d}t = qc^2\vec{p}\cdot\vec{E}\,\mathrm{d}t$$

$$\mathrm{d}W = q\vec{v}\cdot\vec{E}\,\mathrm{d}t$$

Note: no work is done by the magnetic field.

Maxwell's Equations (in vacuum)

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = \mu_0 \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \qquad \nabla \cdot \vec{E} = \mu_0 c^2 \rho$$

why not DC ?

1) DC ($\frac{\partial}{\partial t} \equiv 0$): $\nabla \times \vec{E} = 0$ which is solved by $\vec{E} = -\nabla \Phi$

Limit: If you want to gain 1 MeV, you need a potential of 1 MV!

2) Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot \mathrm{d}\vec{s} = 0$

With time-varying fields:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad \oint \vec{E} \cdot \mathrm{d}\vec{s} = -\iint \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{A}$$

Maxwell's Equation (in vacuum)

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \qquad \nabla \cdot \vec{E} = 0$$

curl of 3rd and $\frac{\partial}{\partial t}$ of 1st equation:

$$\nabla \times \nabla \times \vec{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

vector identity: $\nabla \times \nabla \times \vec{E} = \nabla\nabla \cdot \vec{E} - \Delta\vec{E}$

with 4th equation : $\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$

i.e. Laplace in 4 dimensions

TEM waves.

Cavity: $\underline{E_{||} = 0}$

$\underline{B_{\perp} = 0}$

TE or TM Modes

- TE: Electric field perpendicular to direction of propagation.

TE_{nml}

n:	azimuthal
m:	radial
l:	longitudinal component.

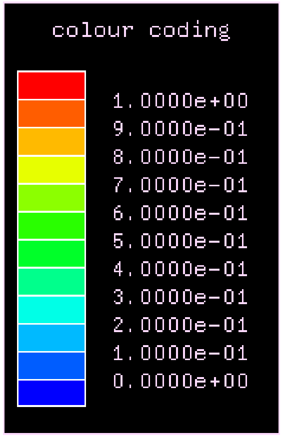
- TM: Magnetic field perpendicular to direction of propagation.

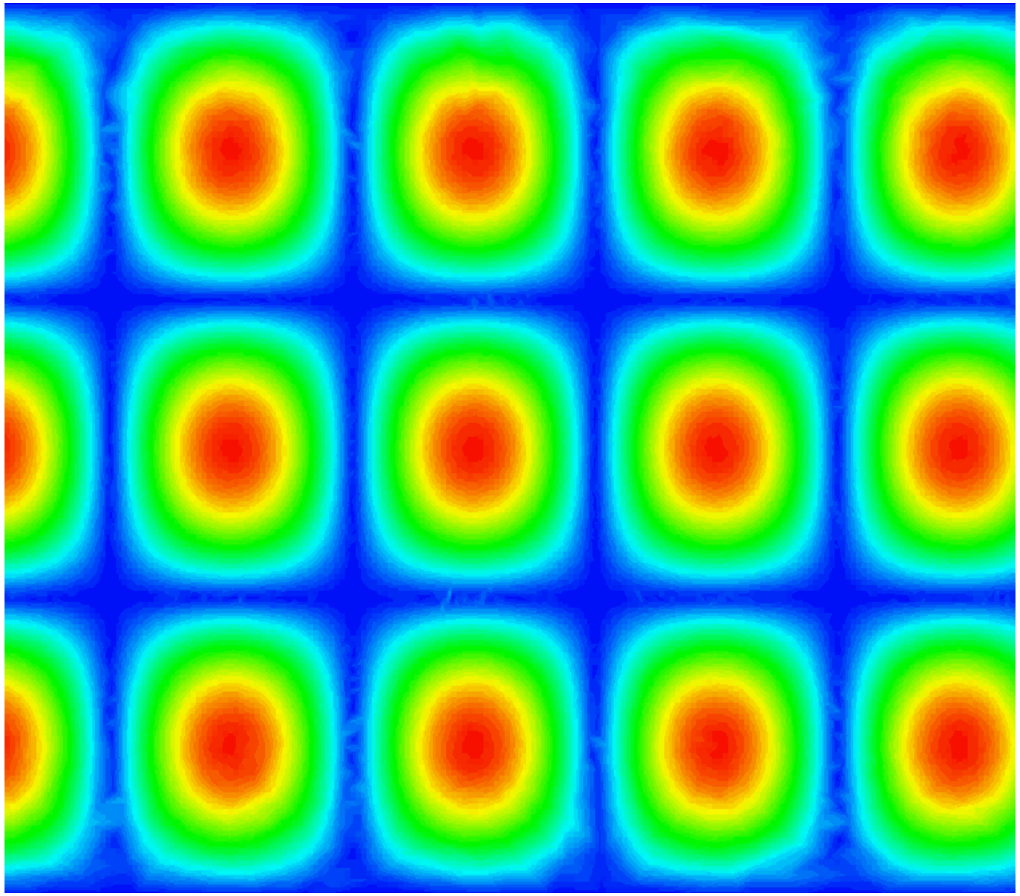
TM_{nml}

n:	azimuthal
m:	radial
l:	longitudinal component.

2 Superimposed Plane Waves

$|\vec{E}|$





Courtesy of Erk Jensen, CERN

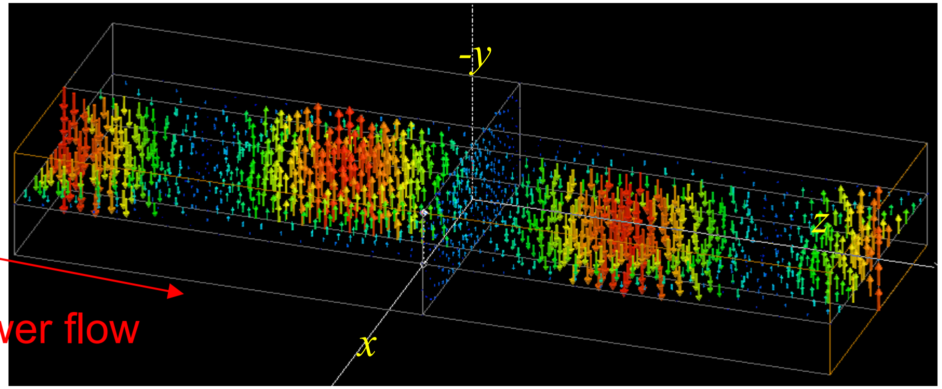
Waveguides

Fundamental (TE_{10} or H_{10}) mode
in a standard rectangular waveguide.

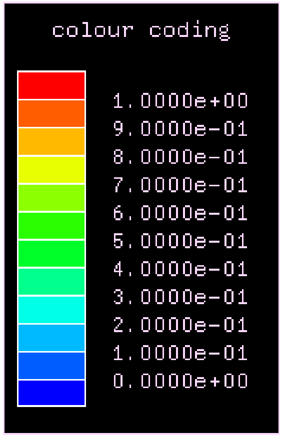
E.g. forward wave

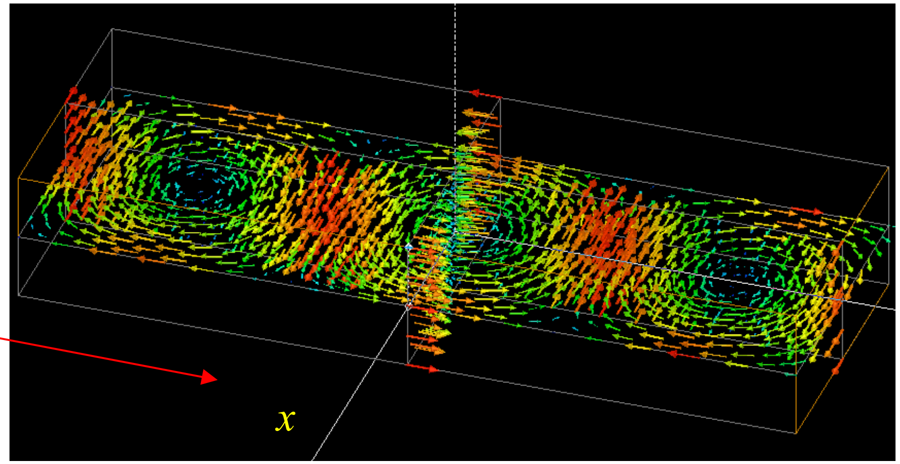
power flow: $$\frac{1}{2}\mathrm{Re}\left\{ \iint\limits_{\substack{\text{cross}\\\text{section}}} \vec{E}\times\vec{H}^{*}\cdot \mathrm{d}\vec{A} \right\}$$

Electric field



power flow





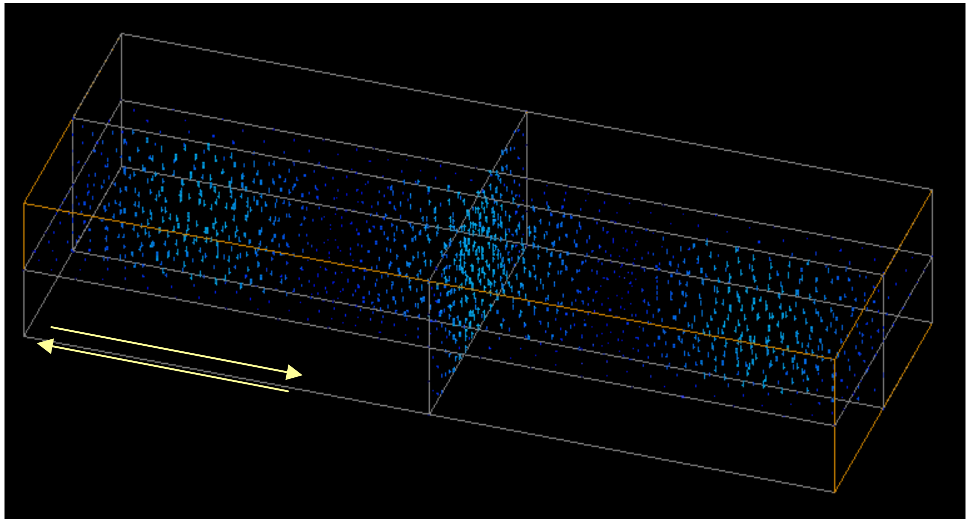
Magnetic field

Courtesy of Erk Jensen, CERN

Standing wave – resonator

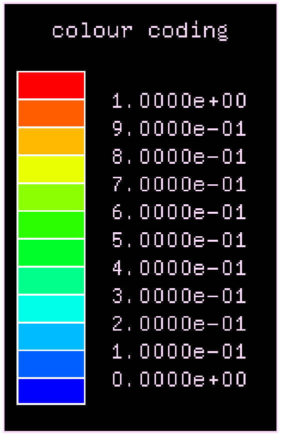
Two counter-running waves
of identical amplitude.

electric field

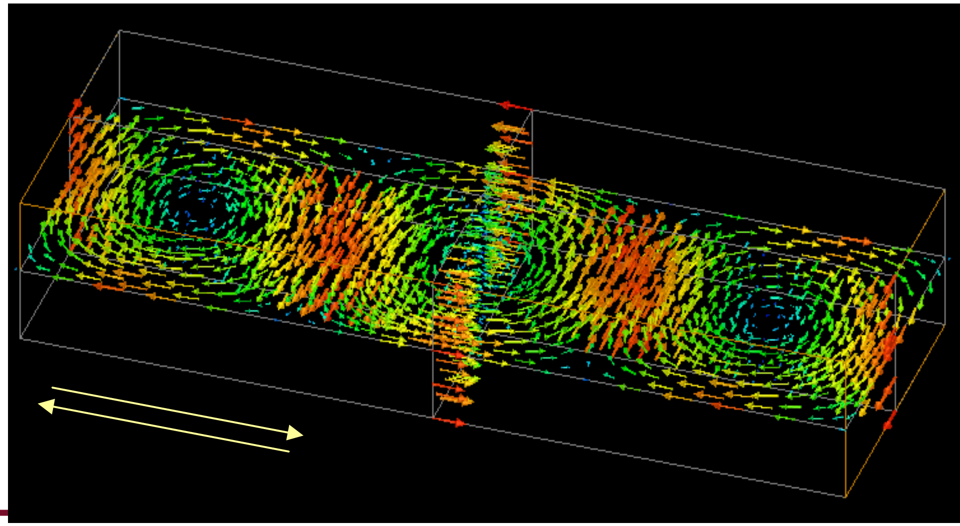


NO net power flow:

$$\frac{1}{2}\mathrm{Re}\left\{ \iint_{\substack{\text{cross}\\ \text{section}}} \vec{E}\times\vec{H}^{*}\cdot \mathrm{d}\,\vec{A} \right\} = 0$$



magnetic field
(90° out of phase)

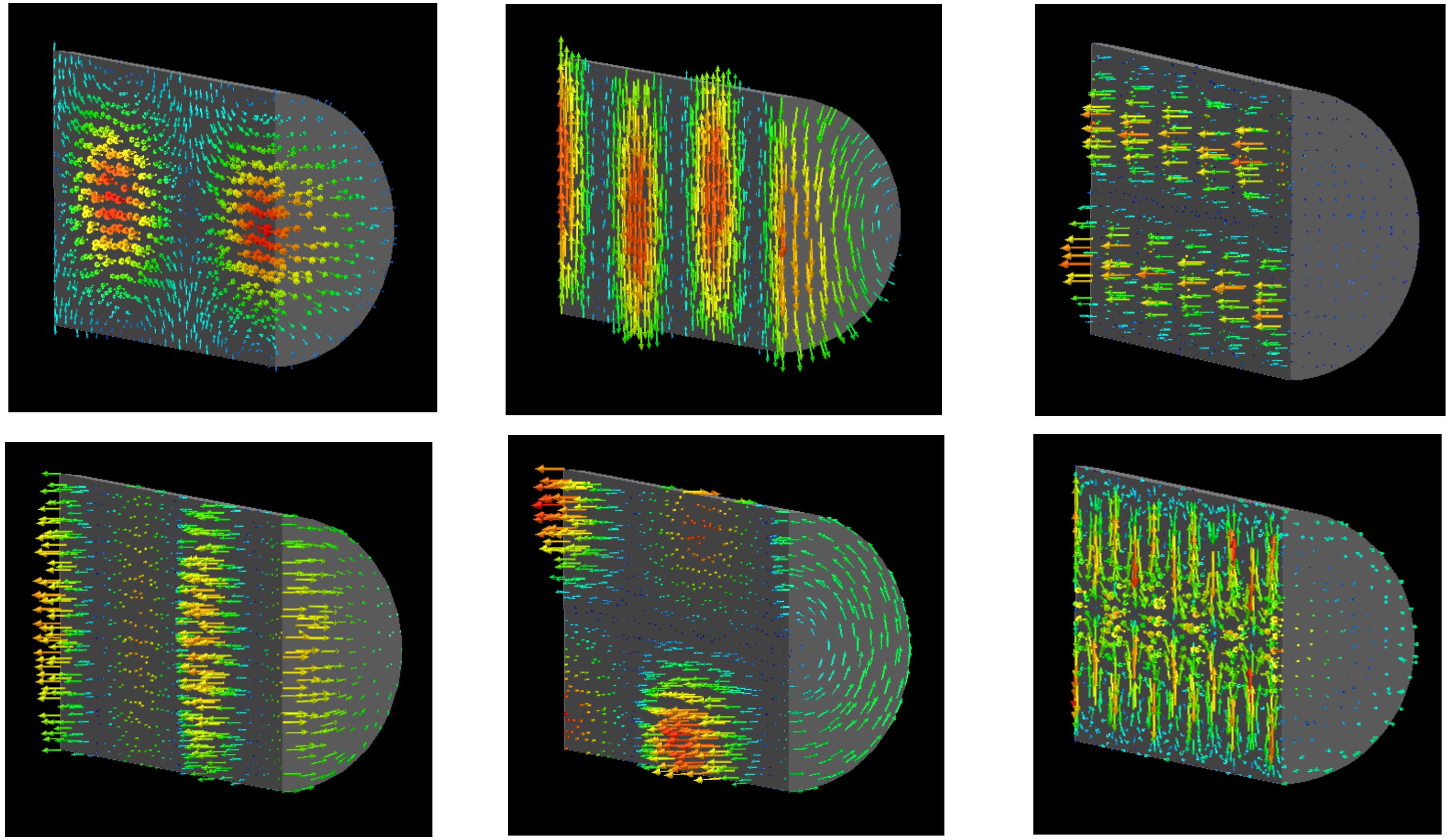


Round waveguide

parameters used in calculation:
f = 1.43, 1.09, 1.13 f_c, a: radius

$\vec{E}$

$\vec{B}$



TE$_{11}$: fundamental mode

$$\frac{f_c}{\text{GHz}} = \frac{87.85}{a/\text{mm}}$$

TM$_{01}$: axial electric field

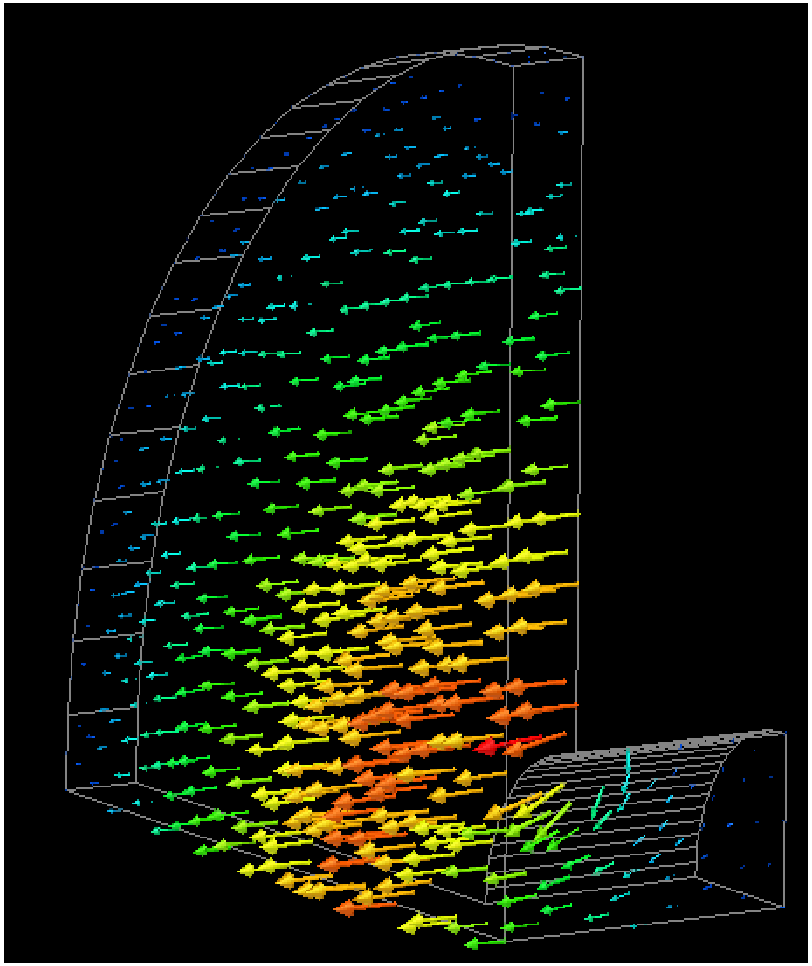
$$\frac{f_c}{\text{GHz}} = \frac{114.74}{a/\text{mm}}$$

TE$_{01}$: lowest losses!

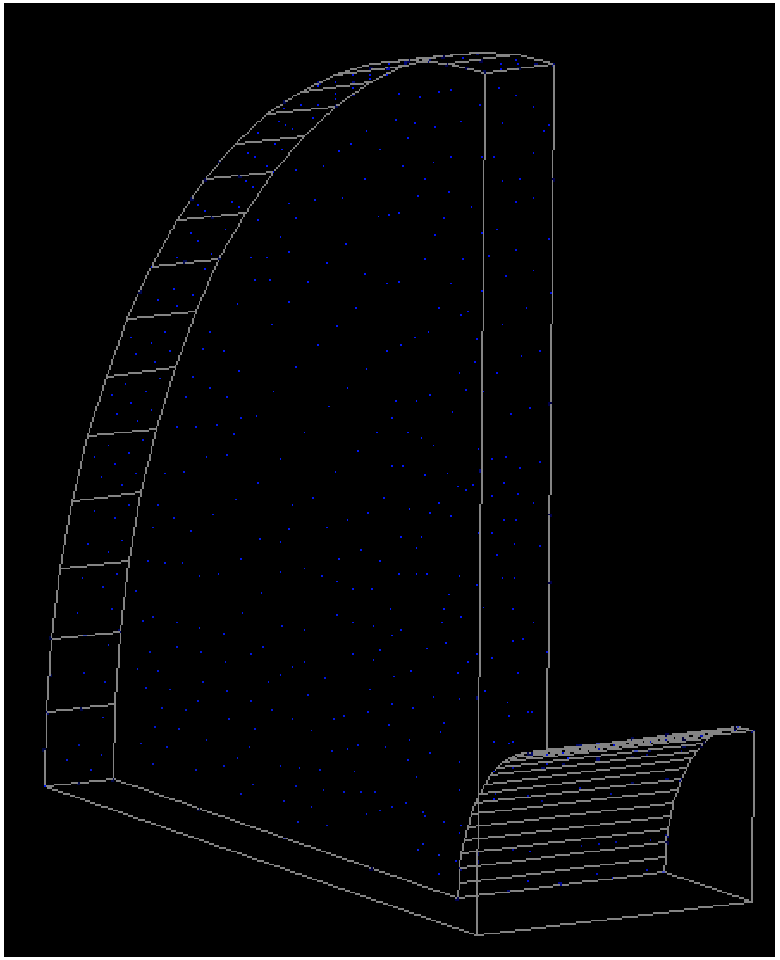
$$\frac{f_c}{\text{GHz}} = \frac{334.74}{a/\text{mm}}$$

Pillbox cavity

TM_{010}-mode (only 1/8 shown)



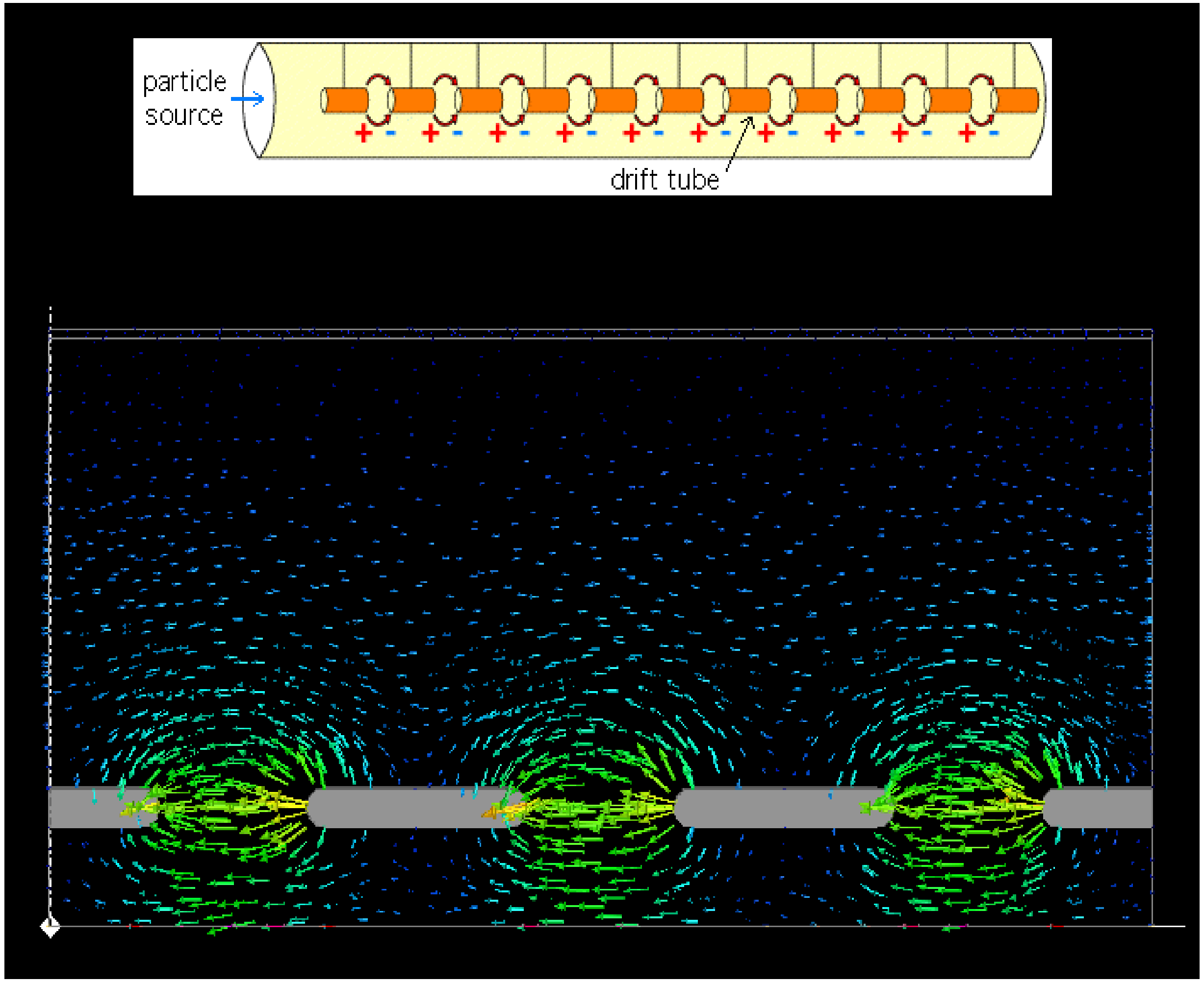
Electric field



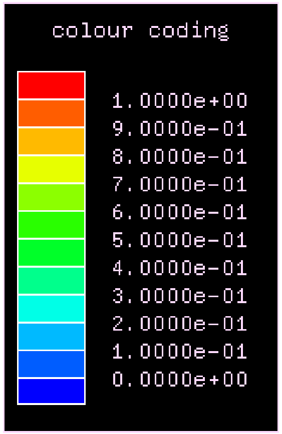
Magnetic field

Drift Tube Linac (DTL) – how it works

For slow particles the drift tube lengths can be easily adapted.



Electric field



Impact on Beam Diagnostics

- Good measurement needs to know details about distribution of E and B;
- Optimization of layout, position, choice of material;
- Important for: Beam current transformers, capacitive pick-ups, etc.
- Non-distructive methods (important !) couple to field from beam.

Relativity: Historical background

- Groundwork by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent basis.
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3\times10^8$ m/s in "ether" but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell's equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume light has speed *c* only in frames where source is at rest
  - And the ether has a small interaction with matter and is carried along with astronomical objects

Nonsense! Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

$$L(v) = L_0 \left(1 - \frac{v^2}{c^2} \right)^{1/2}$$

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

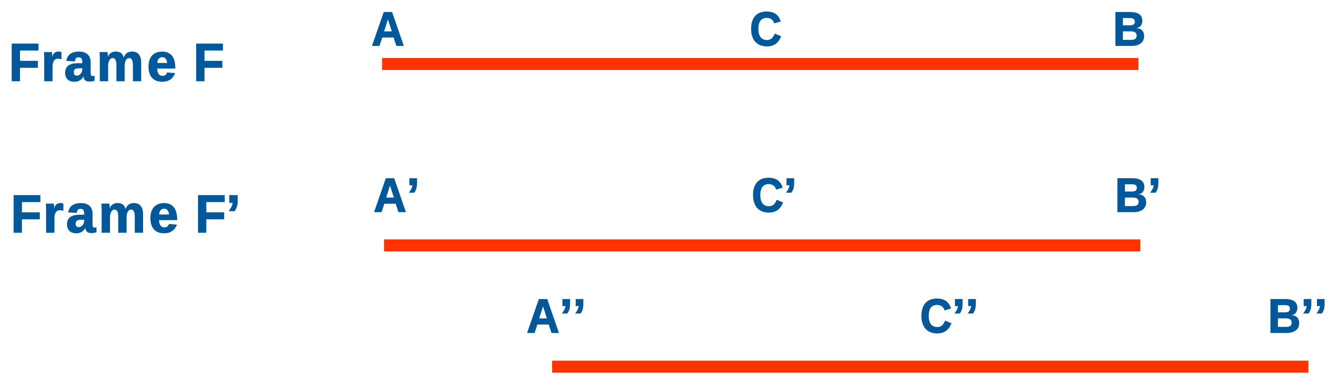
The Principle of Special Relativity

- A frame in which particles under no forces move with constant velocity is "inertial"
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another.
- Behaviour of apparatus transferred from F to F' is independent of mode of transfer
- Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.

The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.

Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB



- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is <u>not</u> absolute but frame dependent.

The Lorentz Transformation

?

- Must be linear to agree with standard Galilean transformation in low velocity limit

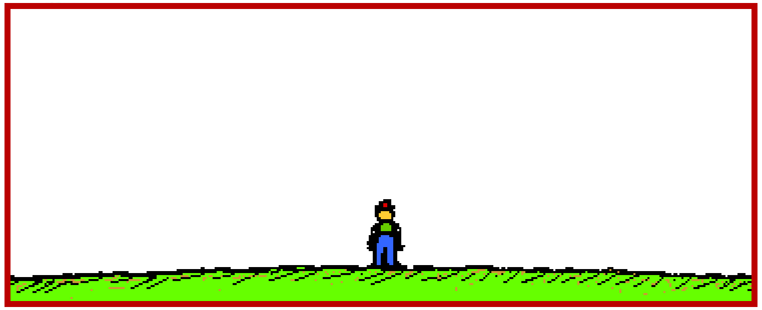
- Preserves wave fronts of pulses of light,

$$\text{i.e.} \quad P \equiv x^2 + y^2 + z^2 - c^2 t^2 = 0$$

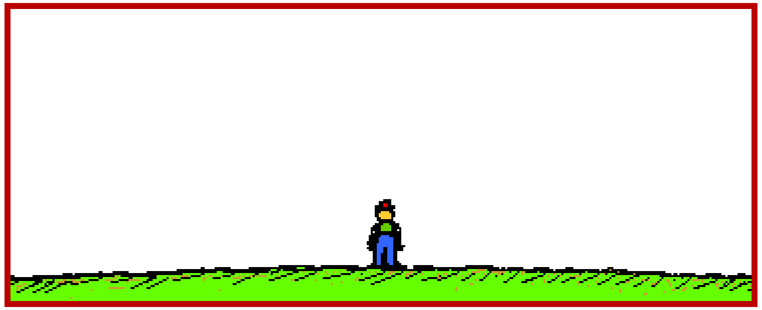
$$\text{whenever} \quad Q \equiv x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0$$

- Solution is the **Lorentz transformation** from frame F *(t,x,y,z)* to frame F'*(t',x',y',z')* moving with velocity *v* along the *x*-axis:

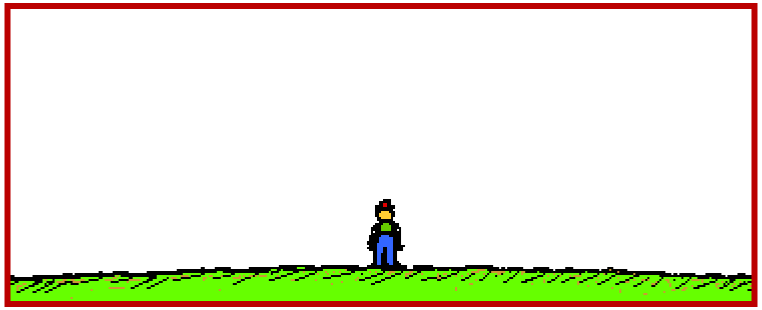
$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$



v = 0.8c



v = 0.9c



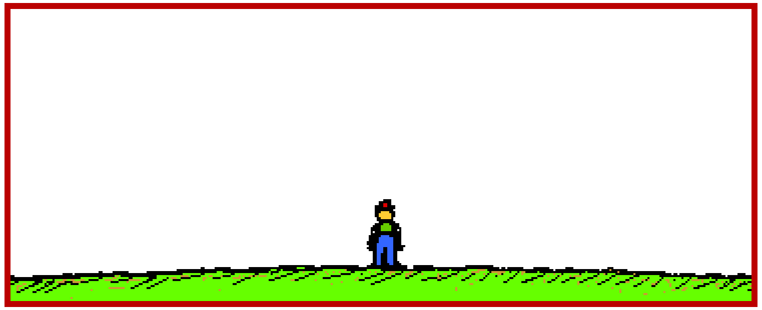
v = 0.99c

Consequences:

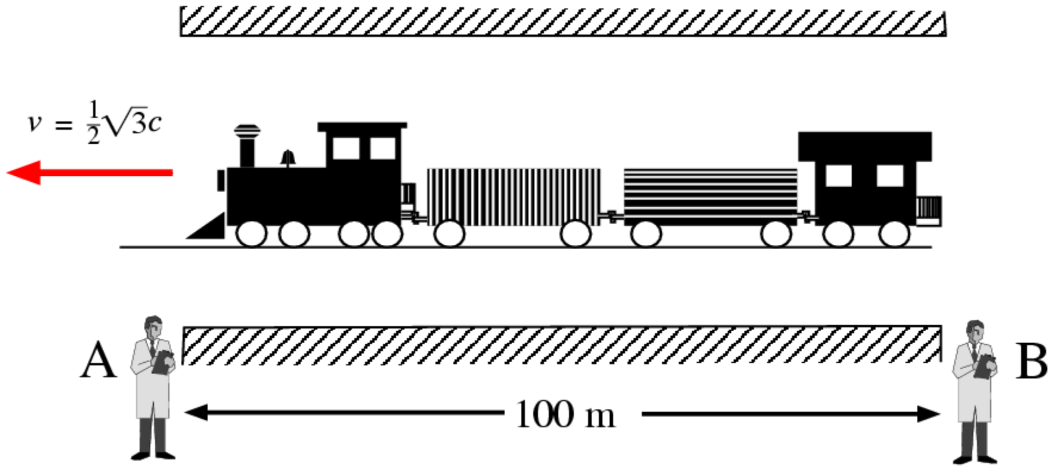
Length Contraction

Time Dilatation

v = 0.9999c



Example: High Speed Train



- All clocks synchronised.
- Observers A and B at exit and entrance of tunnel say the train is moving, has contracted and has length

$$\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = 100 \times \left(1 - \frac{3}{4}\right)^{\frac{1}{2}} = 50\,\mathrm{m}$$

- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m

Questions

- If A's clock reads zero as the driver exits tunnel, what does B's clock read when the guard goes in?

Moving train length 50m, so driver has still 50m to travel before his clock reads 0. Hence clock reading is

$$-\frac{50}{v} = -\frac{100}{\sqrt{3}c} \approx -200 \text{ ns}$$

- What does the guard's clock read as he goes in?

To the guard, tunnel is only 50m long, so driver is 50m past the exit as guard goes in. Hence clock reading is

$$+\frac{50}{v} = +\frac{100}{\sqrt{3}c} \approx +200 \text{ ns}$$

- Where is the guard when his clock reads 0?

Guard's clock reads 0 when driver's clock reads 0, which is as driver exits the tunnel.

To guard and driver, tunnel is 50m, so guard is 50m from the entrance in the train's frame, or 100m in tunnel frame.

Impact on Beam Instrumentation

- Needs to be taken into account in most electron accelerators and some ion accelerators; $v \sim c$
- Transform information from laboratory frame to beam frame;
- Optimize beam characteristics under consideration of relativistic effects.

<u>Next</u>: How to control particles in beam ??

How to guide the ions

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

Examples: B = 1T √
E=10^8 V/m

$$\frac{1}{R(x,z,s)} = \frac{e}{p} B_z(x,z,s)$$

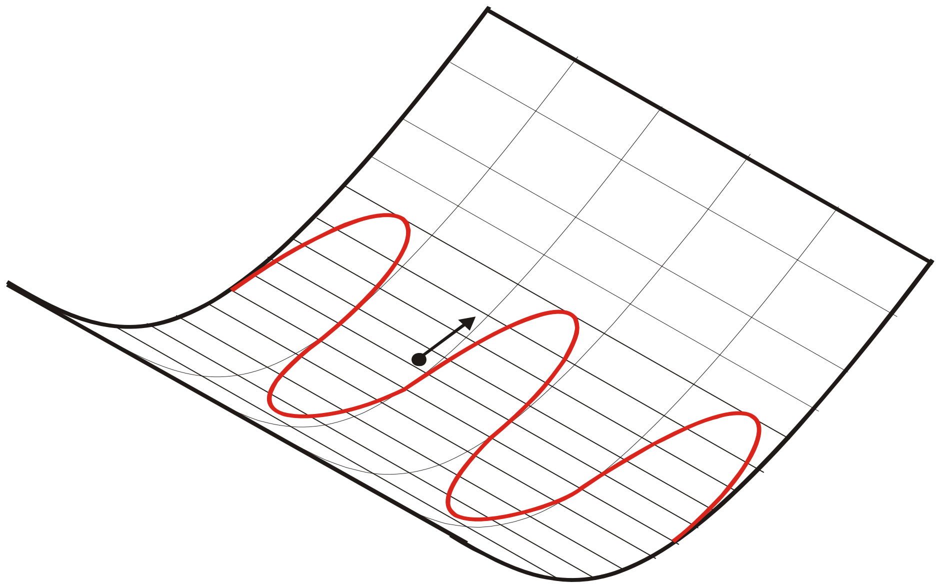
$$\frac{e}{p} B_z(x) = \frac{e}{p} B_{z0} + \frac{e}{p}\frac{dB_z}{dx} x + \frac{1}{2!}\frac{e}{p}\frac{d^2 B_z}{dx^2} x^2 + \frac{1}{3!}\frac{e}{p}\frac{d^3 B_z}{dx^3} x^3 \ldots$$

$$= \frac{1}{R} + kx + \frac{1}{2!} m x^2 + \frac{1}{3!} o x^3 \ldots$$

Transverse Particle Motion

Basis is Hill's equation

$$x''(s) \pm k(s)x(s) = 0$$

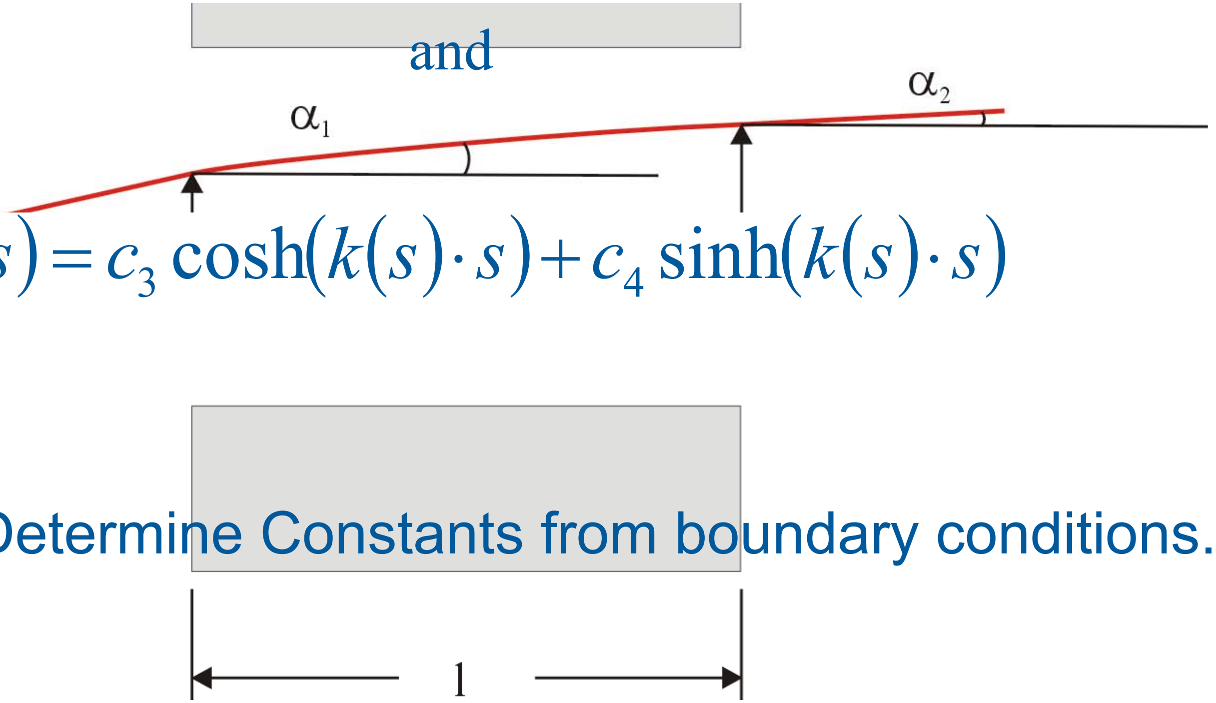


Solution of Hill's Equation

Ansatz $$x(s) = c_1 \cos(k(s) \cdot s) + c_2 \sin(k(s) \cdot s)$$

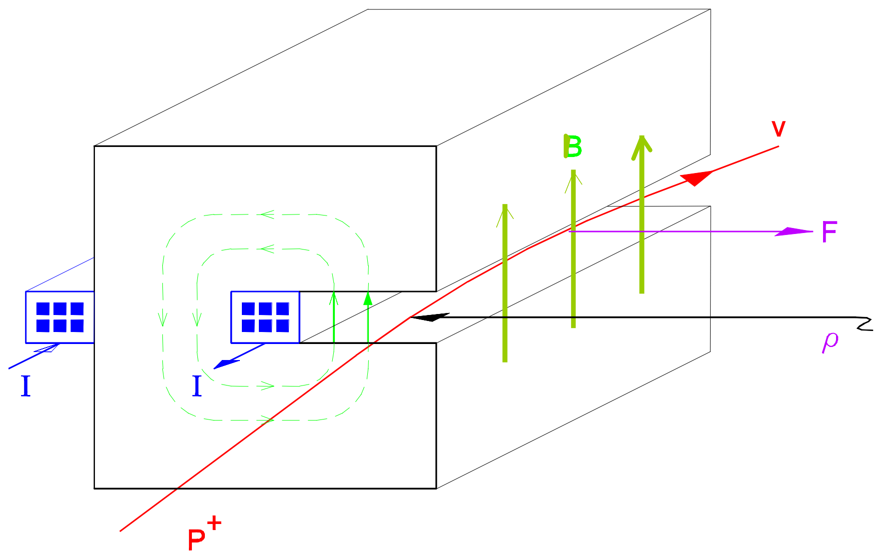
and

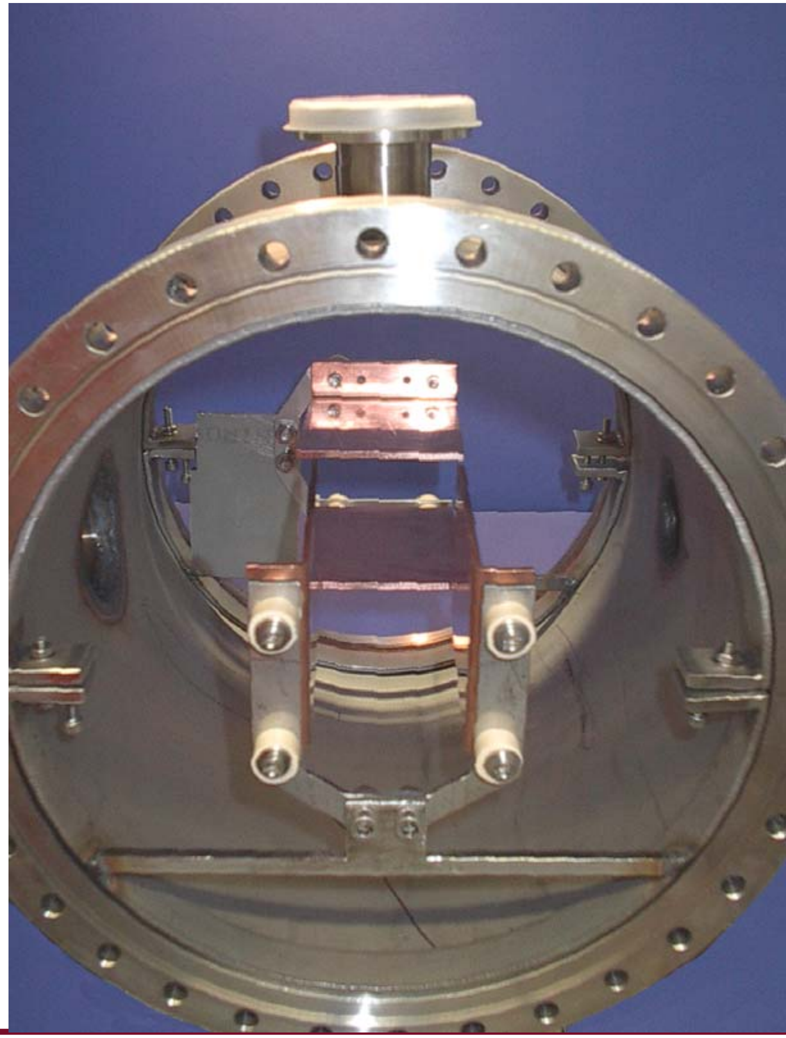
$$x(s) = c_3 \cosh(k(s) \cdot s) + c_4 \sinh(k(s) \cdot s)$$



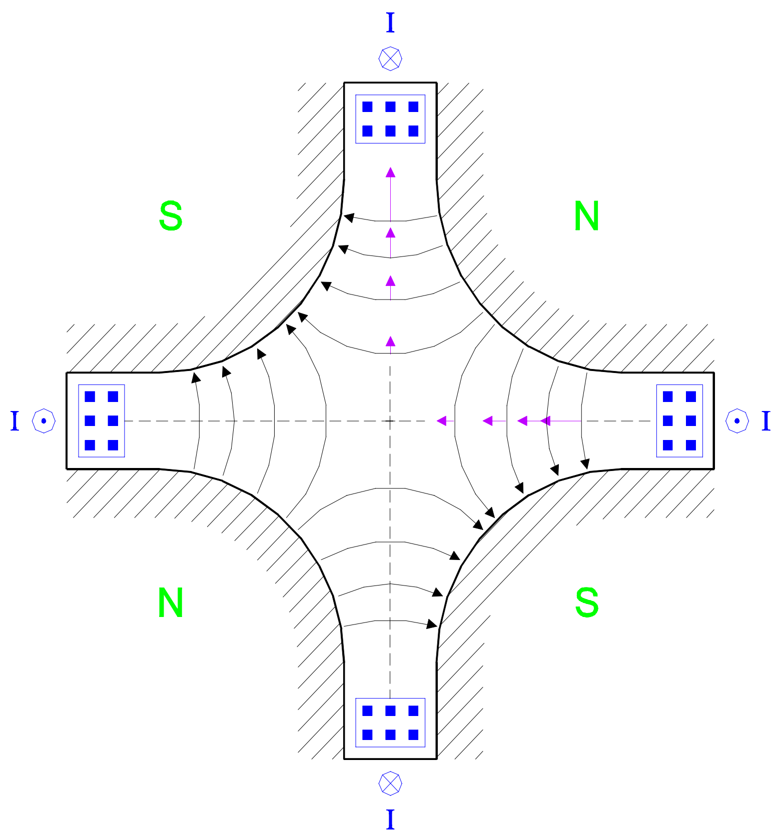
➡ Determine Constants from boundary conditions.

Keep particles on orbit: Dipole





Focusing: Quadrupoles



- Quadrupole produces a constant gradient $g=-dB_z/dx$.
  - Focusing forces increase linearly with displacement
  - Important: no coupling

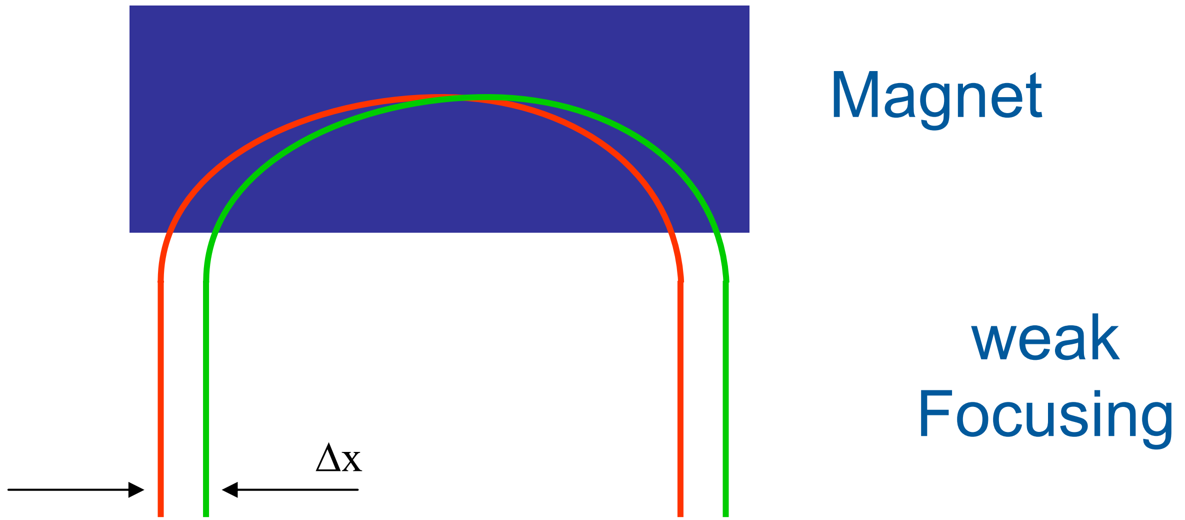
- Optical lenses are either focusing or defocusing.
- Magnetic lenses focus in one plane but are defocusing in the orthogonal plane (from Maxwell's equations)

Dipole

Compare equations: $k = 1/R^2$

$$x''(s) \pm \frac{1}{R^2}(s)\,x(s) = 0$$

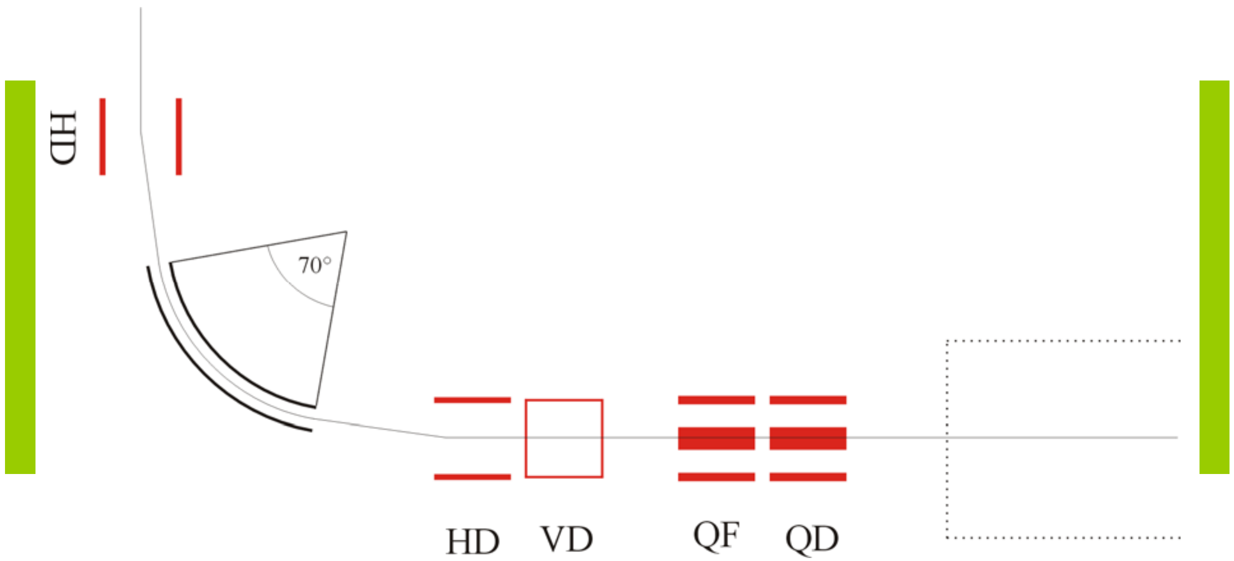
Why focusing ‘?’?’?



Complex Structures

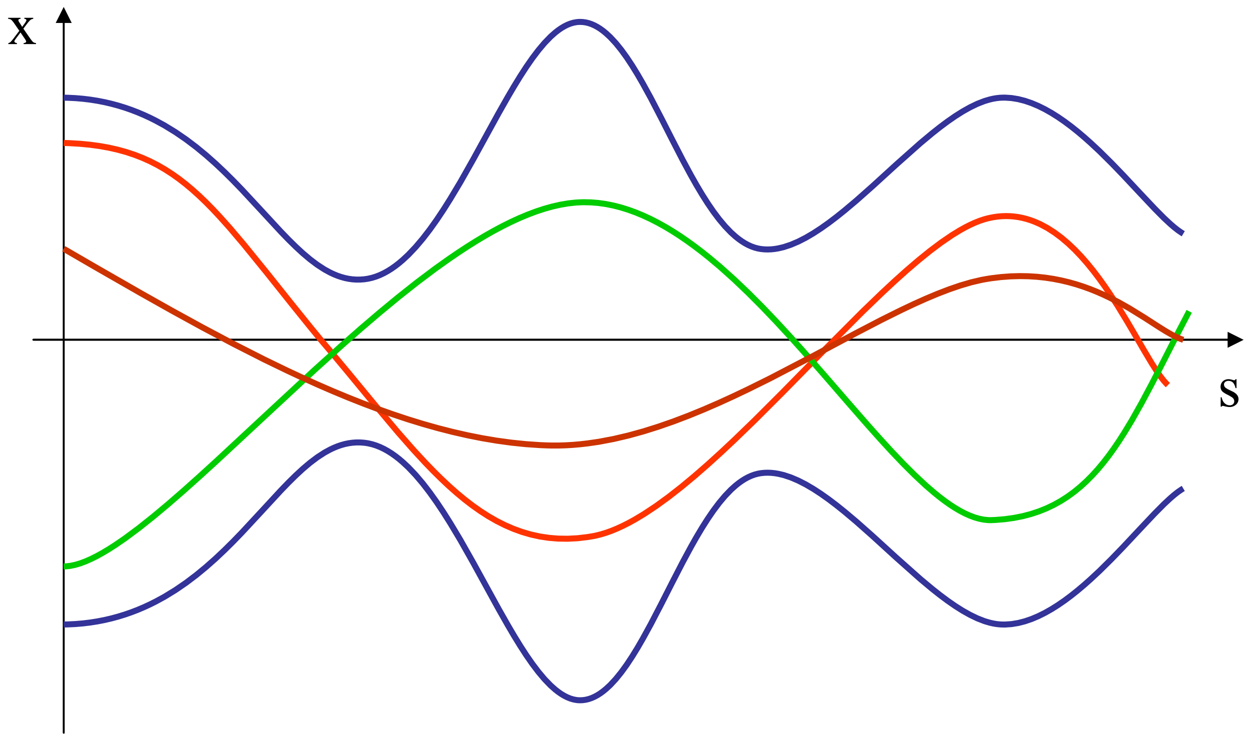
Use individual matrices

$$(M_B)(M_{QF})(M_{QD})(M_D)(M_B)(M_D)\cdots$$



Only single particles !

Normally not of major interest.



➡ Find „Envelope"…

Dispersion

Until now ideal motion with $\Delta p/p=0$

Now: linear treatment with dispersion:

$$x''(s)+\left(\frac{1}{R^2(s)}-k(s)\right)x(s)=\frac{1}{R(s)}\frac{\Delta p}{p}$$

has to be non-zero

Normally define dispersion orbit

$\Delta p / p = 1$

Momentum Compaction Factor

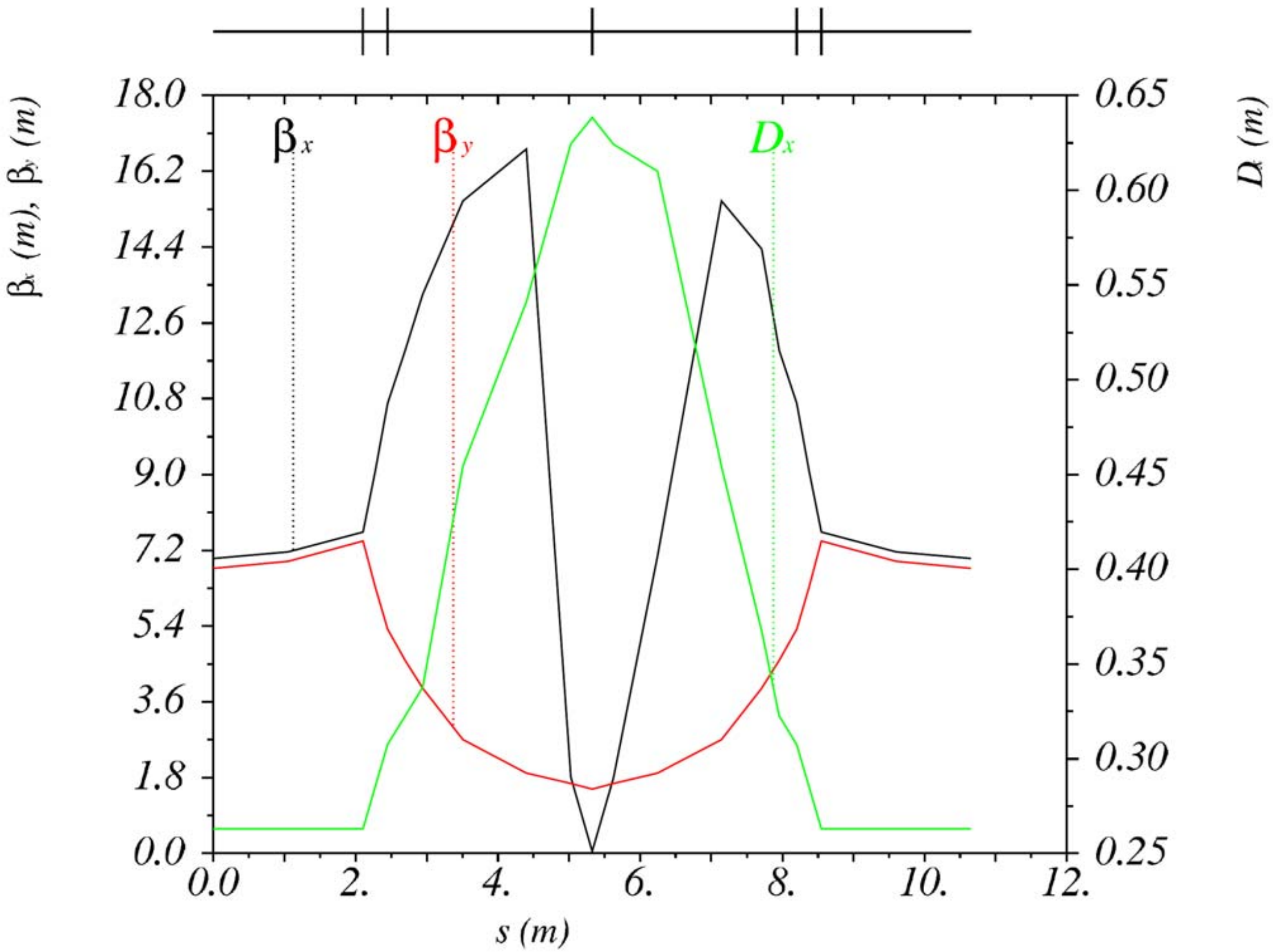
Defined as

$$\alpha = \frac{\Delta L / L}{\Delta p / p}$$

Alternative representation of design and dispersion orbit

$$\alpha = \frac{1}{L_0} \oint \frac{D(s)}{R(s)} ds$$

Machine Design: Orbit and D(s)



Conclusion

- Maxwell's equation basis for beam motion, basis also for most diagnostic methods;
- Relativistiv effects of high relevance (in particular for) electron accelerators *(v~c)*;
- Decoupled optical elements for beam transport and shaping,...exploited for instrumentation;
- Understanding the beam dynamics essential for maximizing information from beam instrumentation !