SCALE SETTING: LESSONS FROM SCET

Scales, Cambridge

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Two questions

1. What central value of the scales should we choose?

- Goal is to have best agreement with data
  - Minimal Scale Sensitivity?
  - Best convergence?
  - Physically motivated?

2. How much should we vary around that scale?

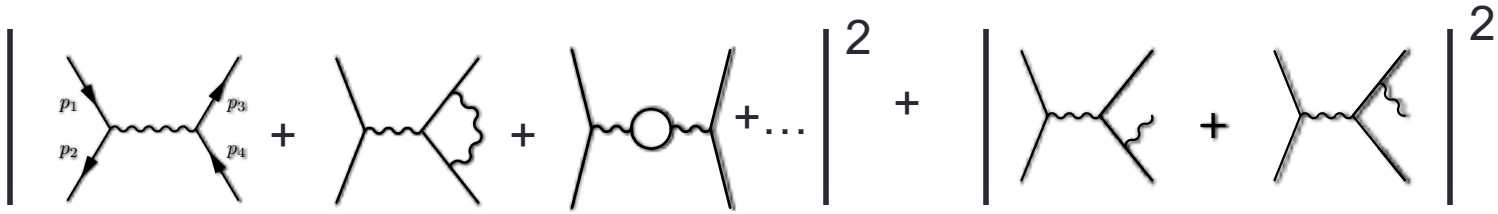
- Goal is to produce uncertainty treatable like statistical uncertainty
  - Want 95% (or 68%?) confidence that next order will be within uncertaint

- At minimum, want bands to overlap…

CENTRAL SCALE CHOICE

Scale setting for inclusive observables

$$\sigma_{\rm tot} = \sigma(e^+e^- \to \mu^+\mu^-) + \sigma(e^+e^- \to \mu^+\mu^-\gamma)$$

$=$ 

$$= \frac{4\pi\alpha_e(\mu)}{3Q^2}\left[1 + \frac{\alpha_e(\mu)}{\pi}\left(\frac{3}{4} + \beta_0 \ln\frac{Q}{\mu}\right) + \mathcal{O}(\alpha_e^2(\mu))\right]$$

- Only one scale, so choosing $\mu = Q = E_{CM}$ turns σ_{tot} into a series in $\alpha(Q)$

$$\sigma_{\rm tot} = \frac{4\pi\alpha(Q)}{3Q^2}\left[1 + 3\left(\frac{\alpha(Q)}{4\pi}\right) + (-\frac{3}{2} - 1.38 n_f)\left(\frac{\alpha(Q)}{4\pi)}\right)^2 + \cdots\right]$$

- Varying μ adds terms at higher order in α, example of higher order effects

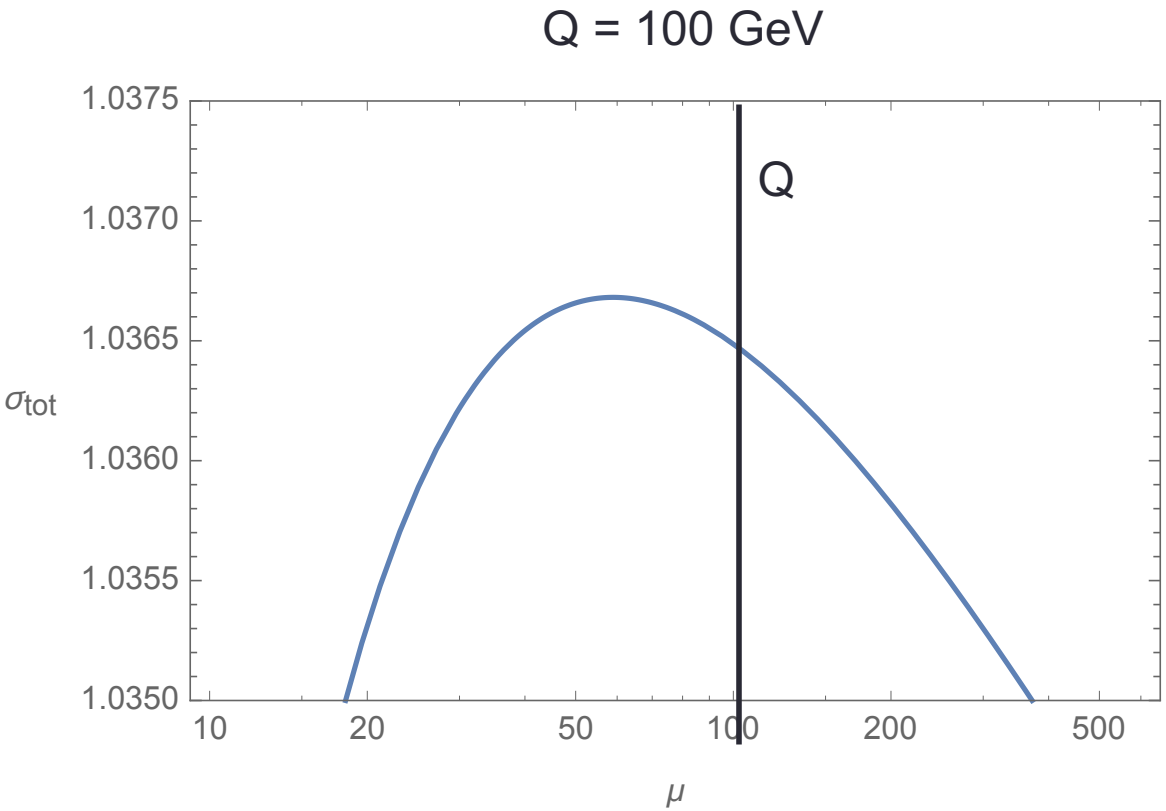
$$\frac{4\pi\alpha(2Q)}{3Q^2} = \frac{4\pi\alpha(Q)}{3Q^2}\left[1 + \frac{1}{3}n_f \ln 2\left(\frac{\alpha(Q)}{4\pi}\right) + \cdots\right]$$

$$\frac{\ln 2}{3} n_f = 3 \text{ for } n_f \approx 13$$

$$\frac{\ln x}{3} = 3 \text{ for } x = 8103.1$$

- Doesn't even have the right scaling with group factors
  - With one flavor, we need μ = 8000 Q to get NLO effect right

Total cross section in e^+e^-



2-loop has form

$$\sigma \approx 1 + \frac{\alpha_s(\mu)}{4\pi}\sigma_1 + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \left(\sigma_2 + 2\sigma_1\beta_0 \ln\frac{\mu}{Q}\right)$$

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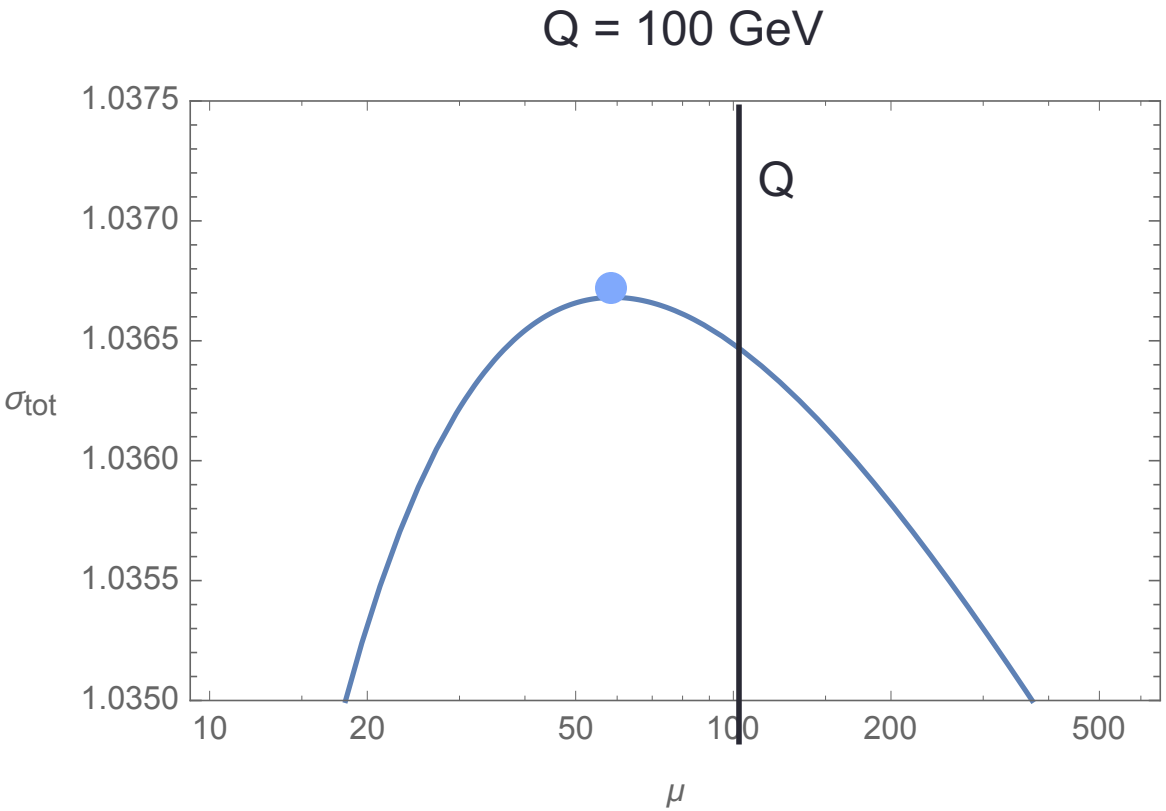

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with an extremum at

$$\mu = Qe^{-\frac{\beta_1}{4\beta_0} - \frac{\sigma_2}{2\beta_0\sigma_1}}$$
$$= 0.58Q = 58\text{GeV}$$

Is this why $\mu \sim Q/2$ is common?

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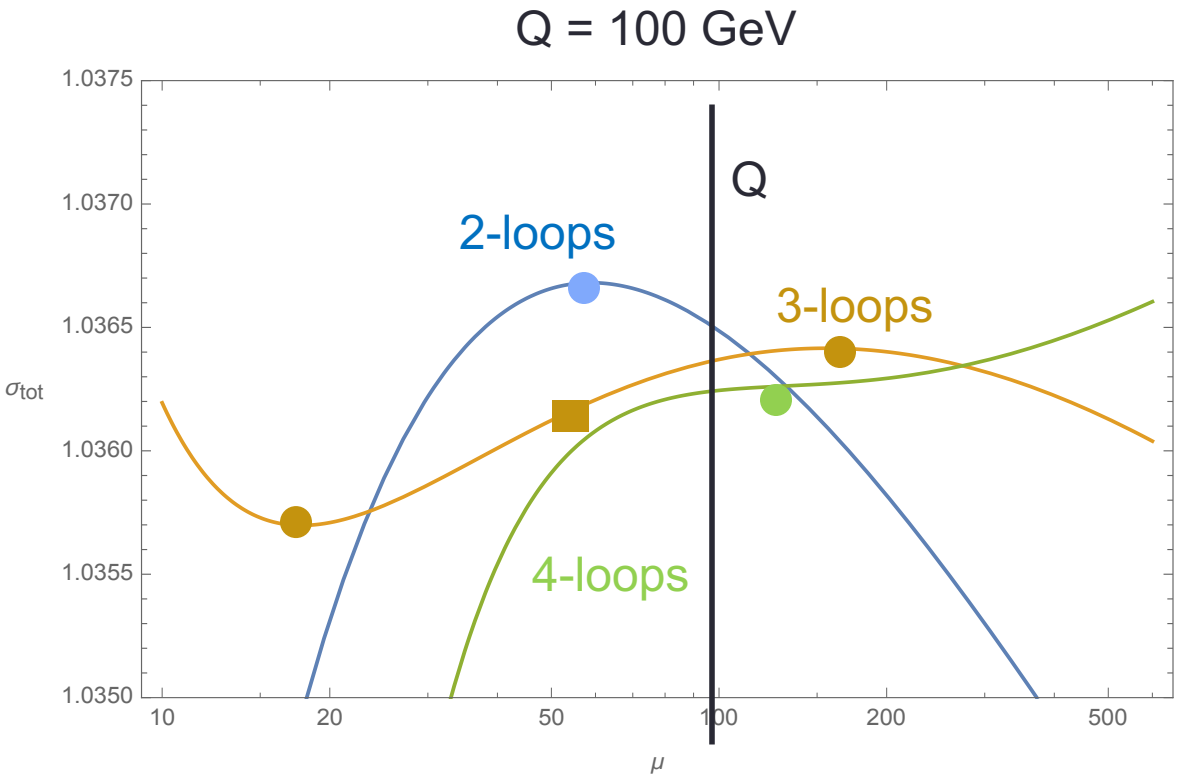

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- Region where variation is flattest tells us natural scale (circles)
- In flat region, curves are polynomials in log μ
  - Center of polynomial symmetrizes uncertanties (squares)
- Curves' intersection minimizes corrections
  - Can be dangerous is correction is large

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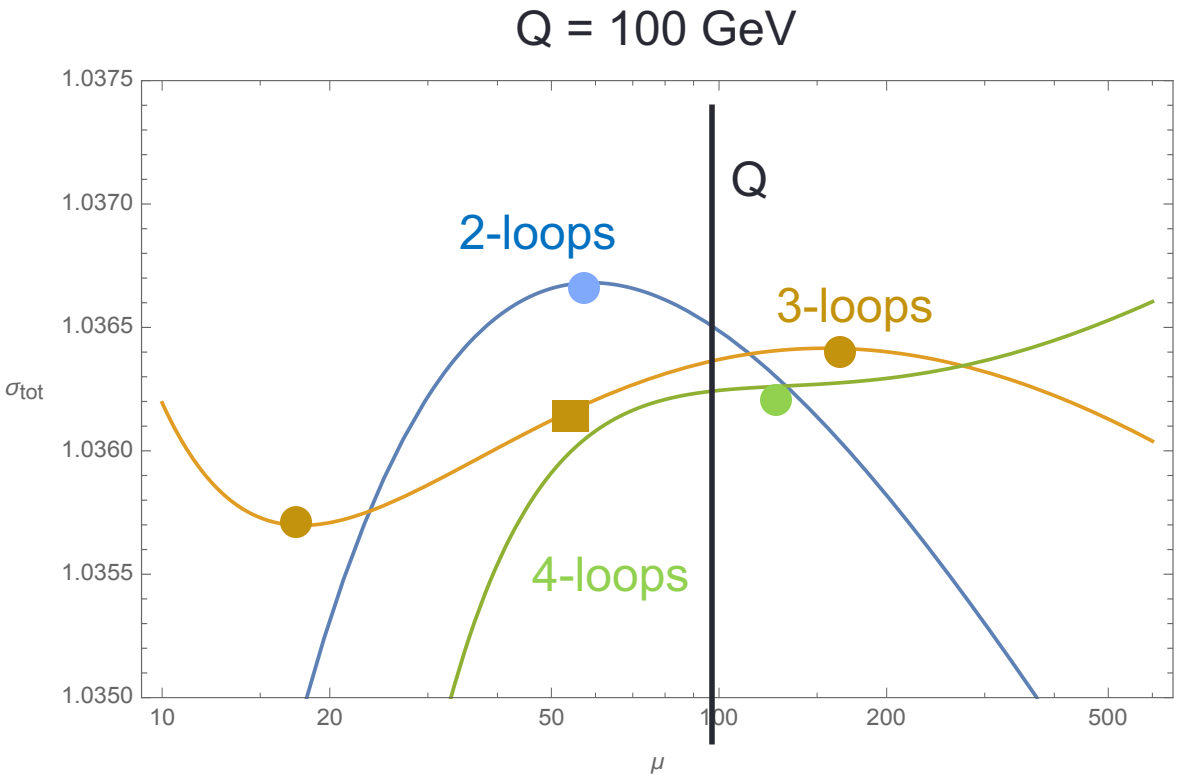

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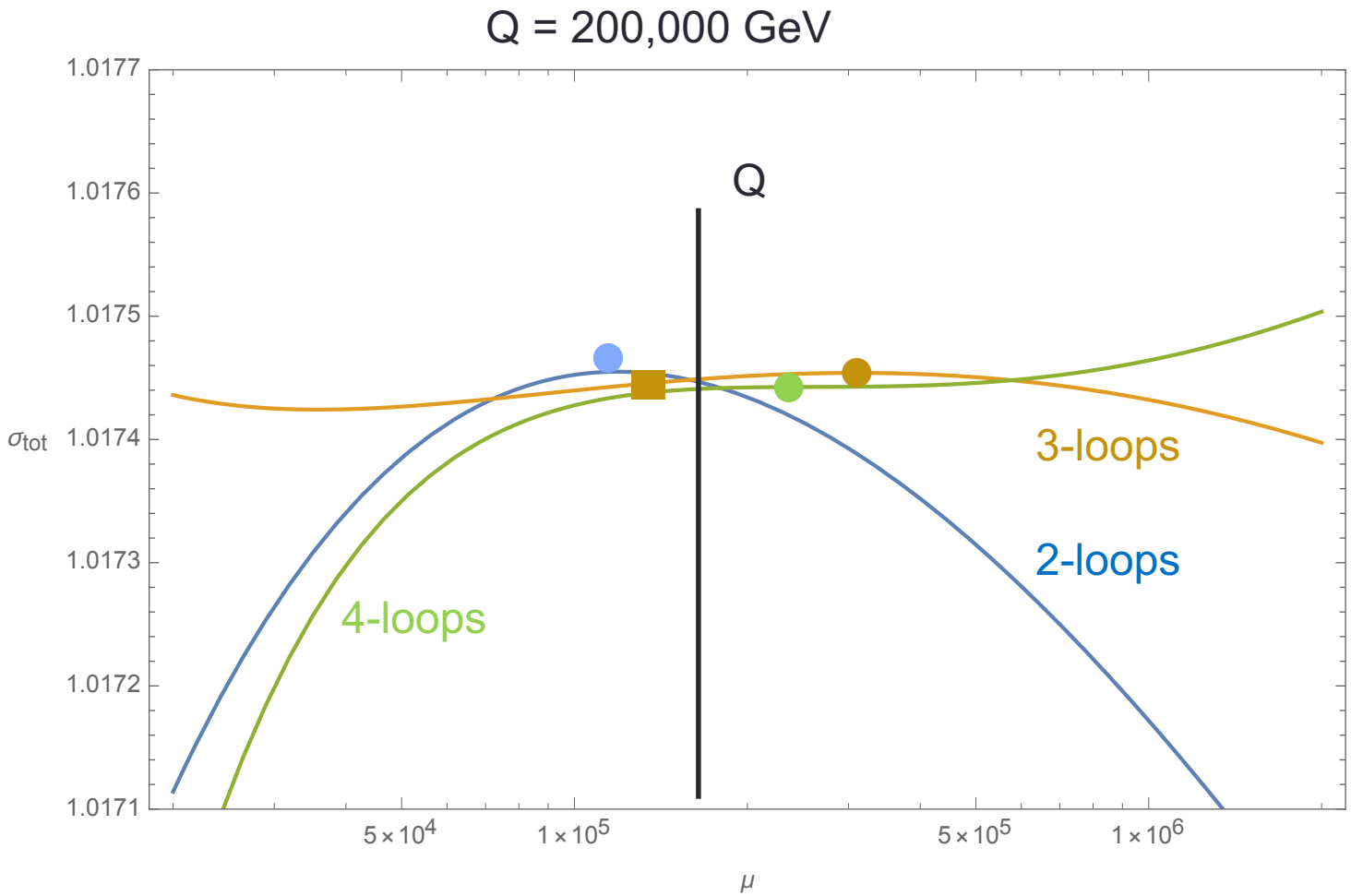
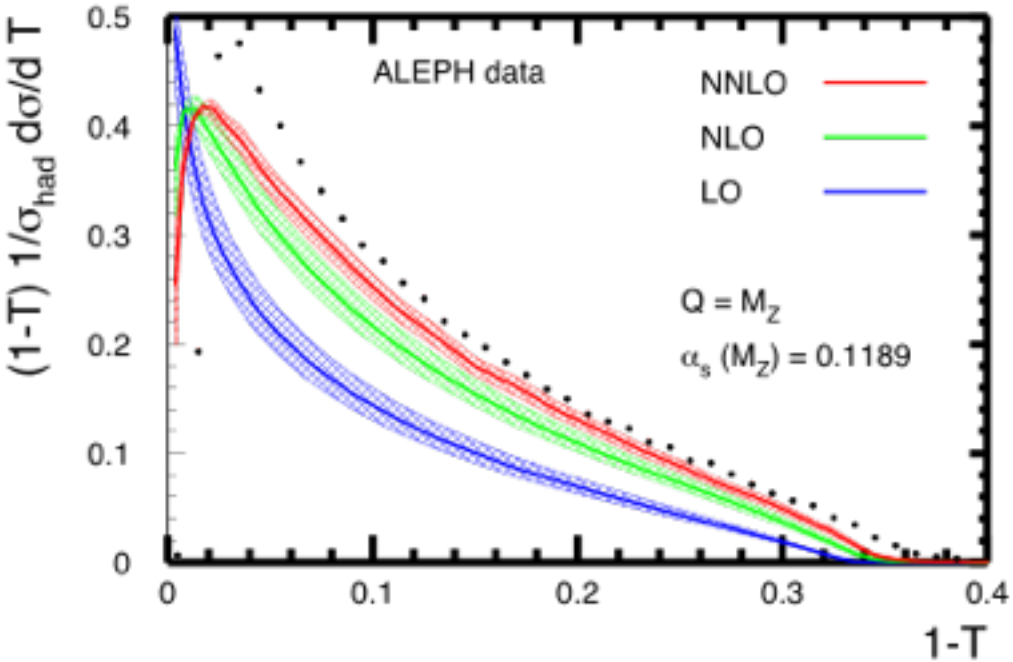


Thrust distribution: fixed order

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \delta(\tau) + C_F \frac{\alpha_s(\mu)}{2\pi}\left[\frac{-4\ln\tau - 3}{\tau} - 8 + 2\ln\tau + \cdots\right]$$

$$+\left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\left[\beta_0 \ln\frac{Q}{\mu} + \frac{C_F^2}{2}\frac{\ln^3\tau}{\tau} + \cdots\right]$$

[Gehrmann et al. 0711.4711, 2011]



- Choose $\mu = Q$
  - by dimensional analysis?
- $\mu = 2Q$, 15 Q, 0.1 Q equally good
- Why not $\mu = \tau Q$ or $\mu^2 = \tau Q^2$?

- Clearly underestimating errors!
- Poor convergence

Thrust distribution: SCET

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau} = H(Q^2,\mu)\int \mathrm{d}p_L^2\mathrm{d}p_R^2\mathrm{d}k\, J(p_L^2,\mu)\, J(p_R^2,\mu)\, S_T(k,\mu)\delta(\tau - \frac{p_L^2+p_R^2}{Q^2} - \frac{k}{Q})$$

Each function has one scale

Hard function: Q
(hard scale, like COM energy)

Jet function: p^2
(mass of the jet)

Soft scale
(out-of-jet energy)

- Natural scales read off from factorization formula

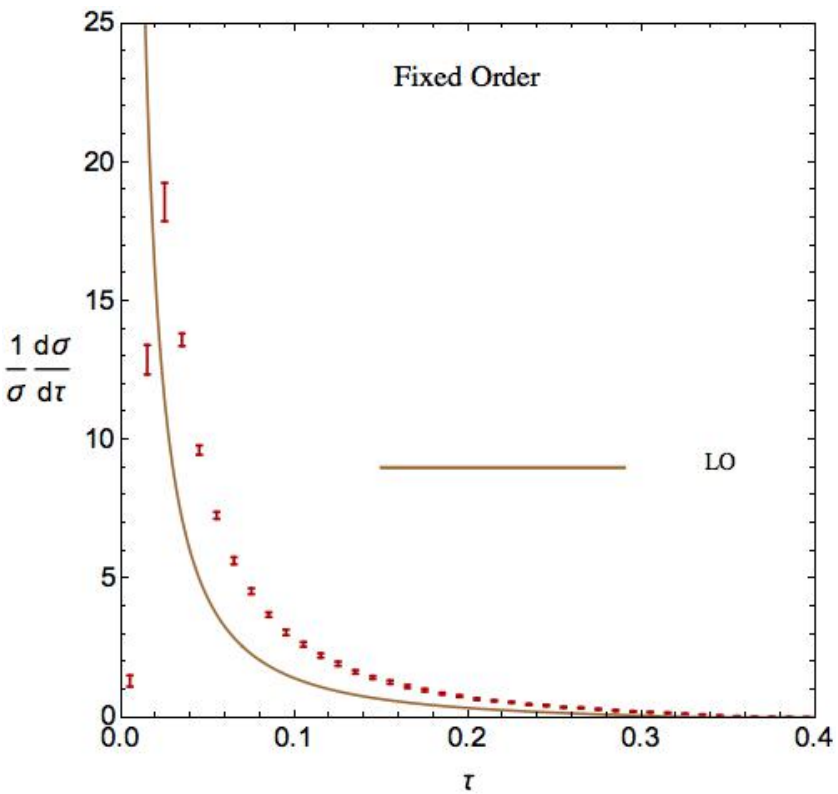
$$\mu_h = Q \qquad \mu_j = \sqrt{\tau}Q \qquad \mu_s = \tau Q$$

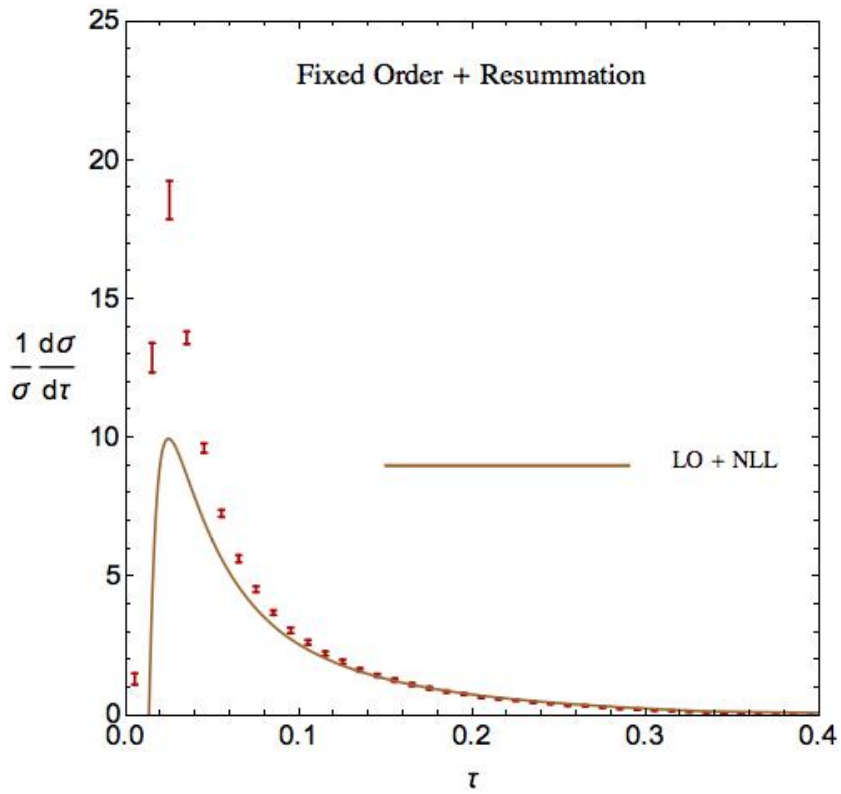
- Evolve each function from its scale to common scale μ using RGE
- Logs of μ linked to logs of τ

$$\ln\frac{\mu_h}{\mu} = \ln\frac{Q}{\mu} \qquad \ln\frac{\mu_j}{\mu} = \ln\frac{Q}{\mu} + \frac{1}{2}\ln\tau \qquad \ln\frac{\mu_s}{\mu} = \ln\frac{Q}{\mu} + \ln\tau$$

- Reduces problem to the fixed-order inclusive calculation case
- Single scale at fixed order is misleading: multiple scale problem

Resummation improves fit





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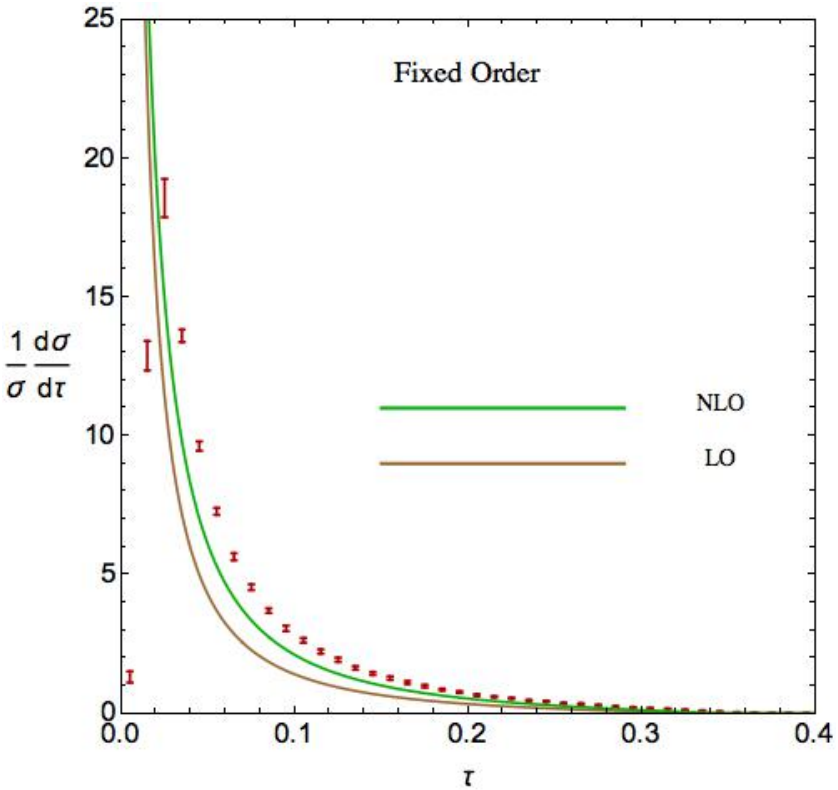

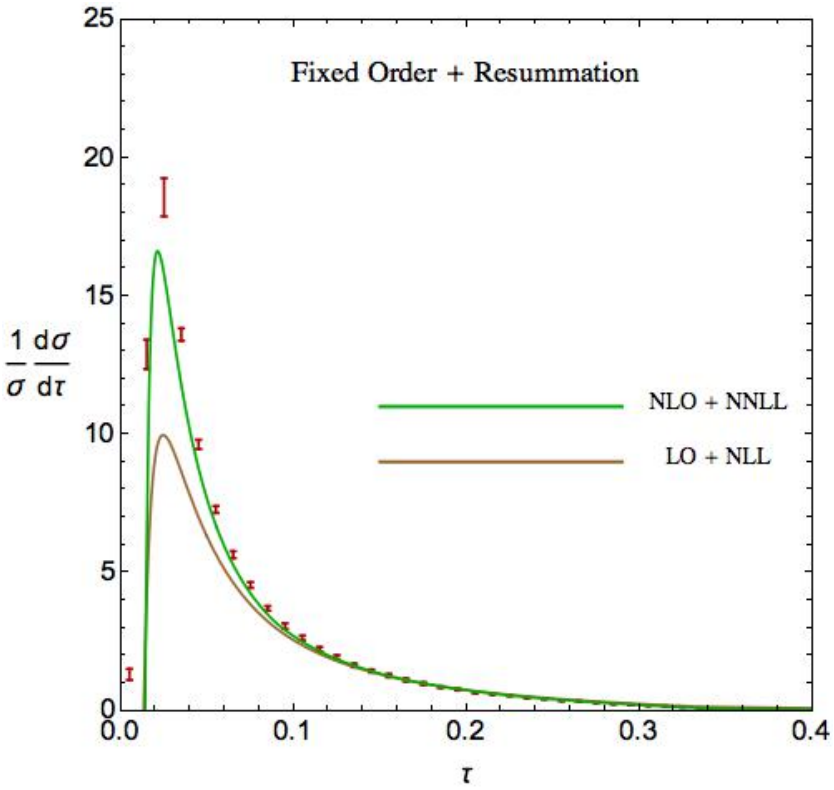

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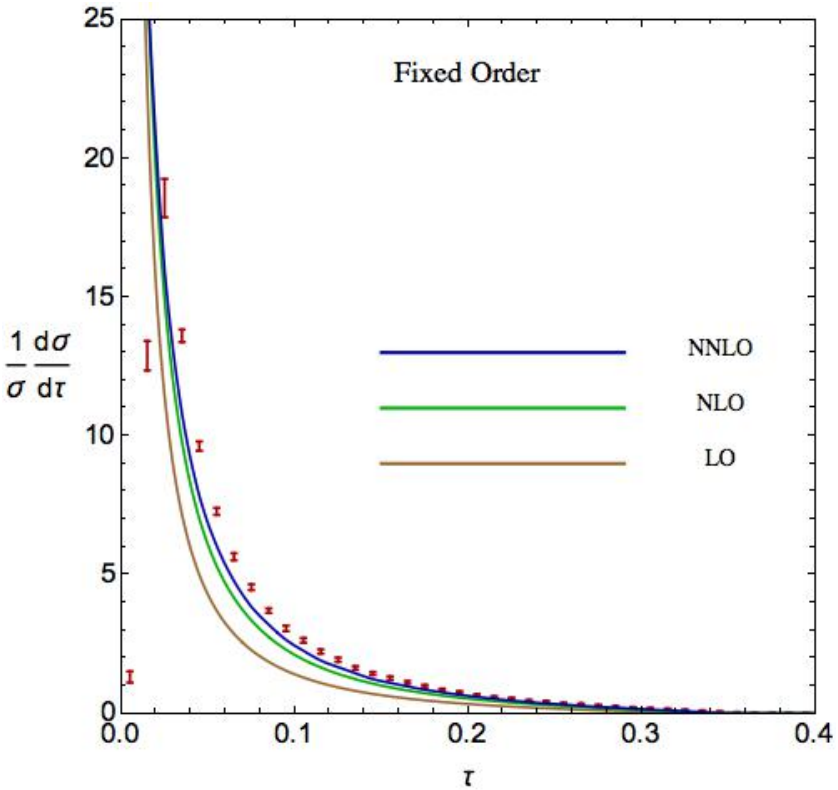
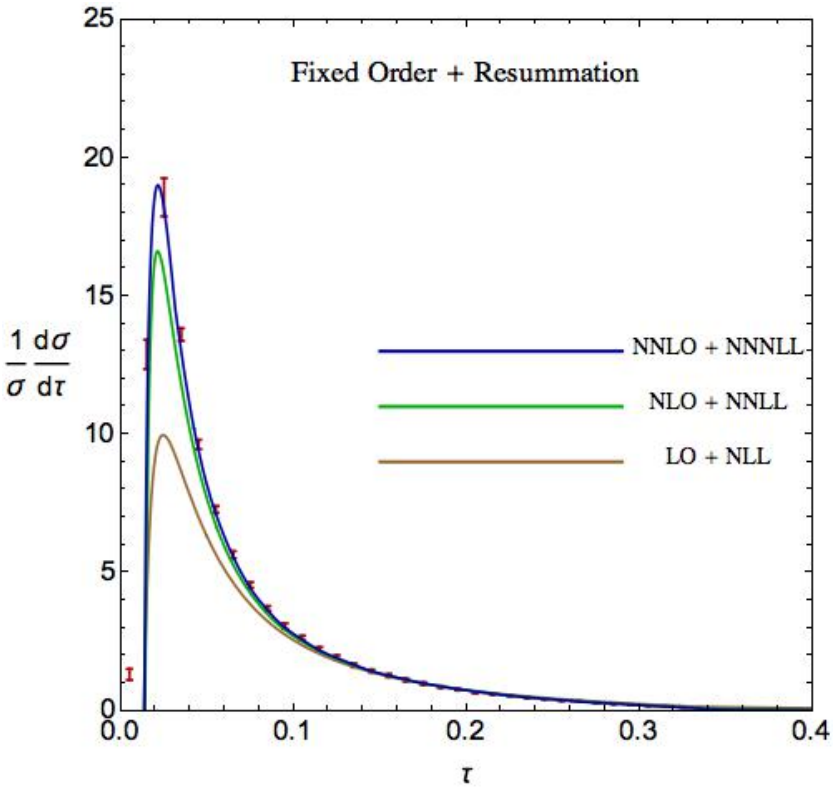
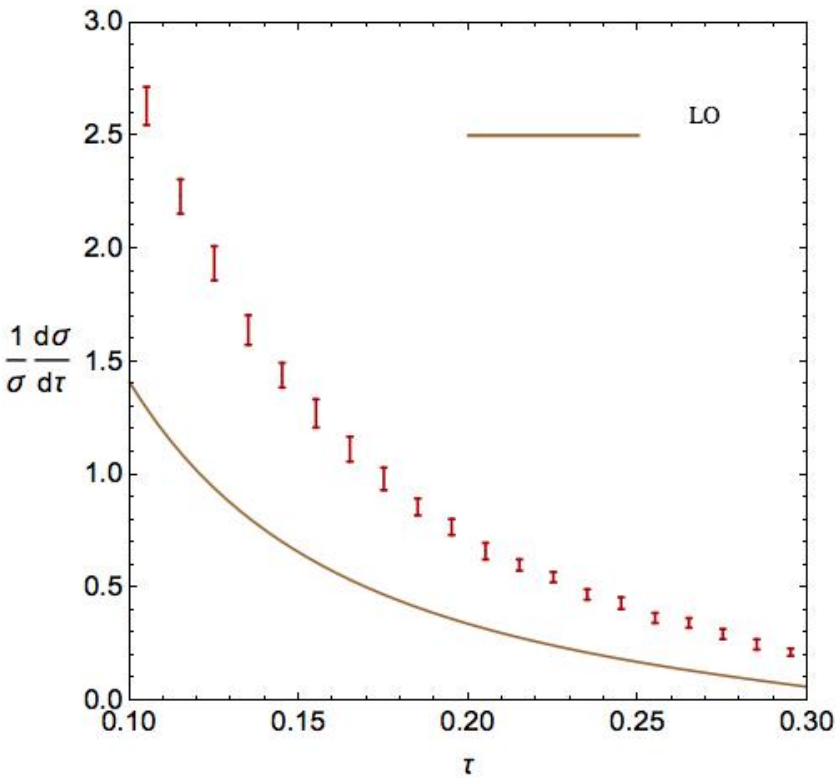
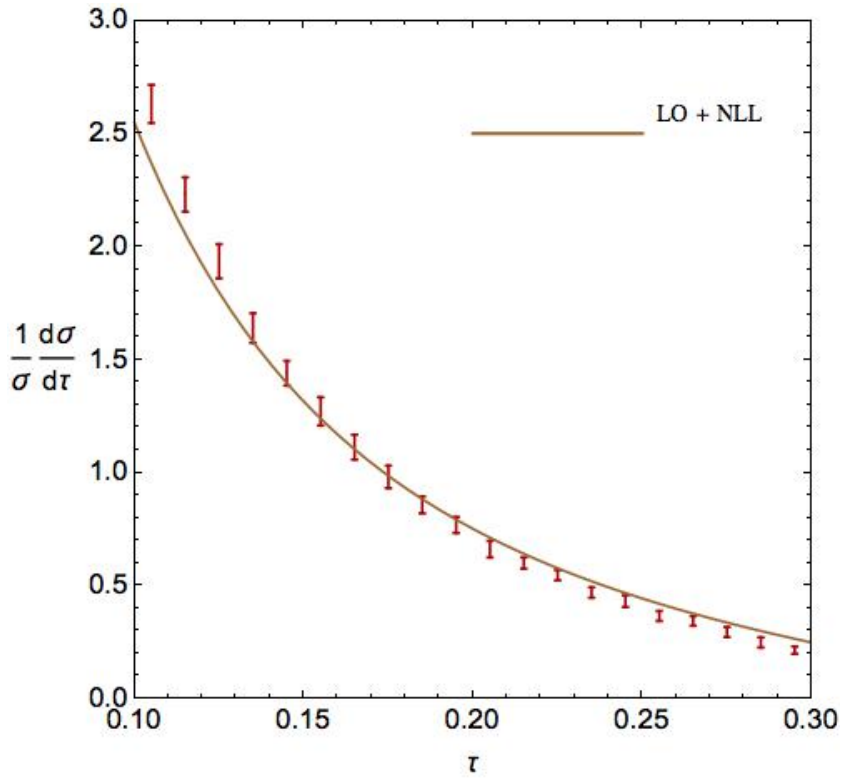




Natural scales improve convergence





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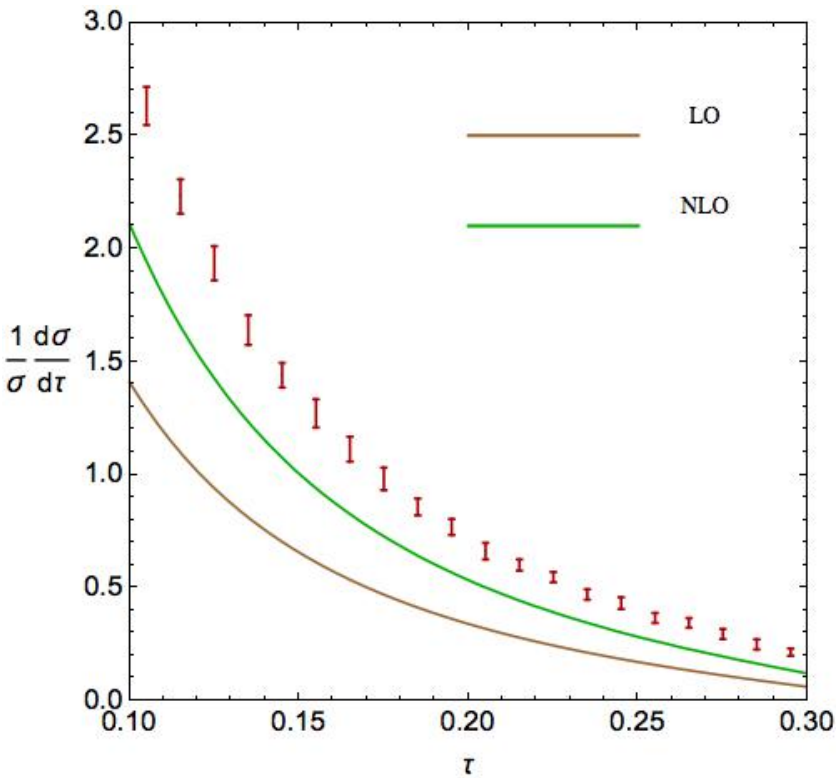

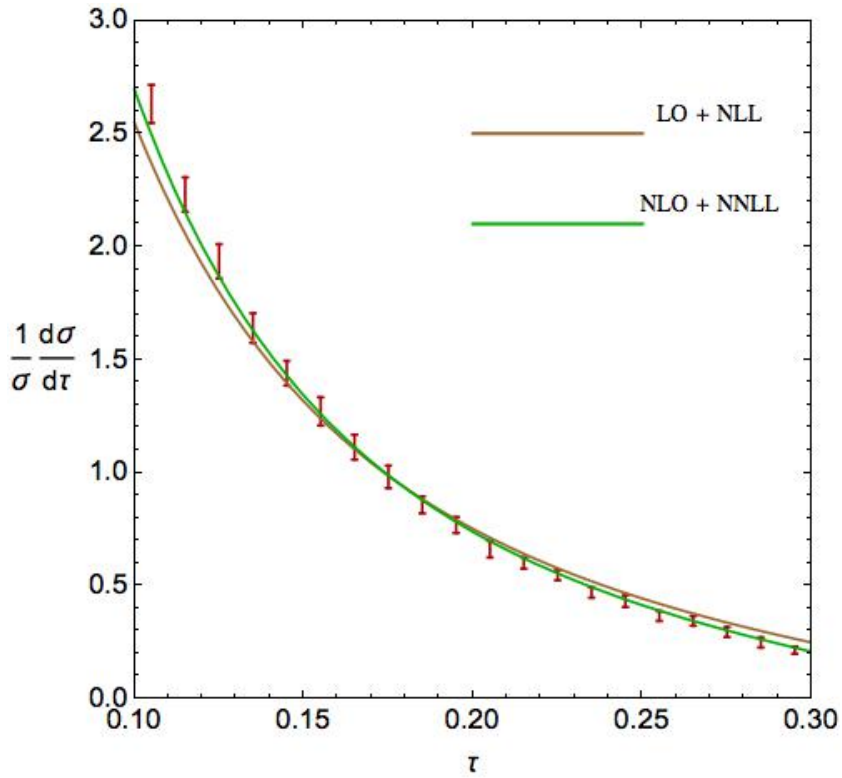

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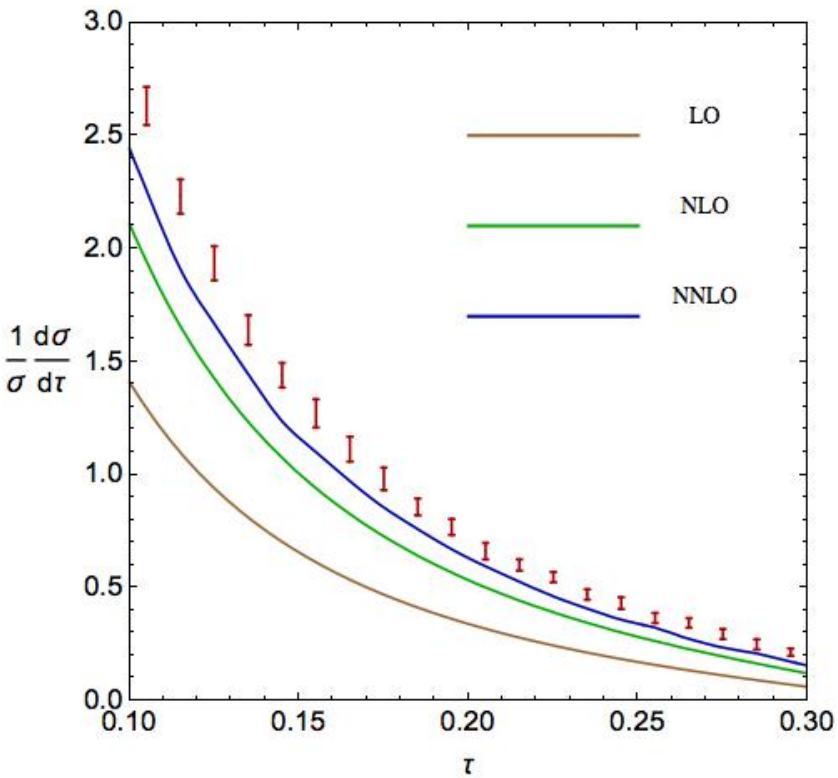
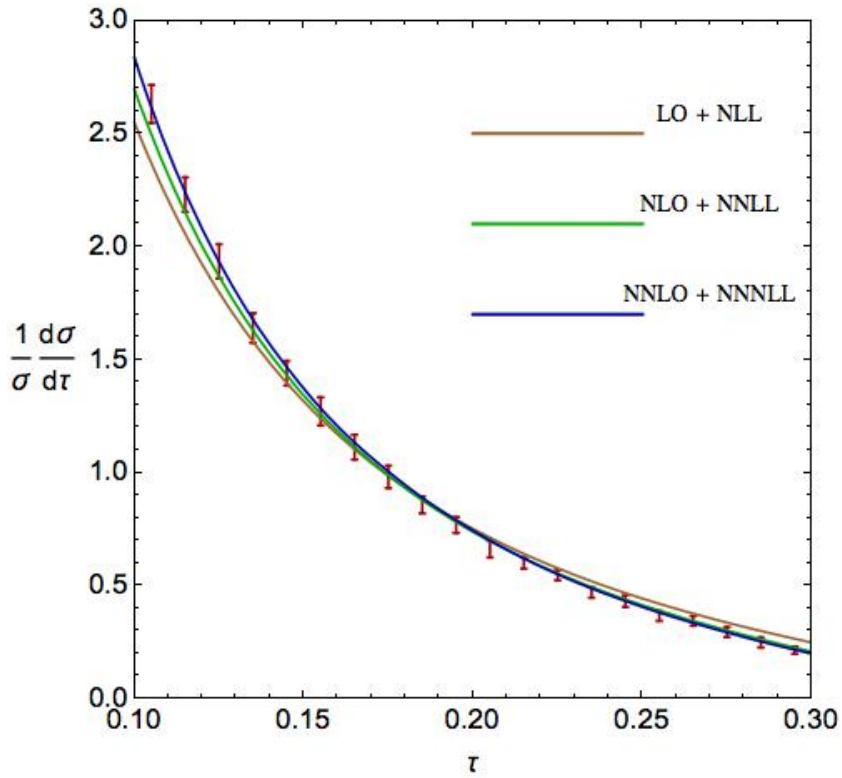
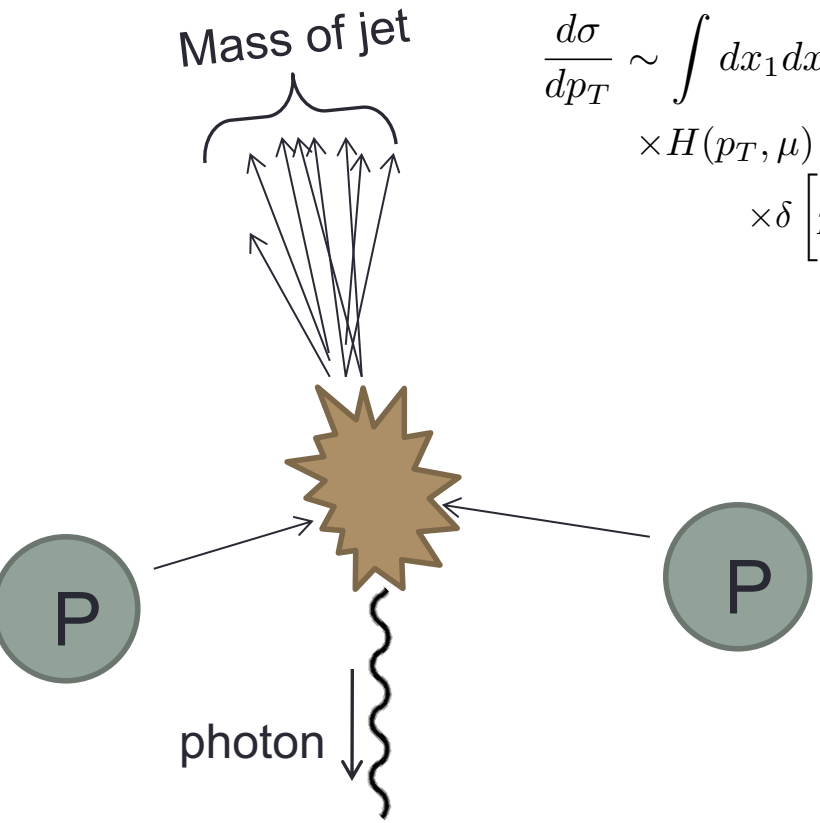




Hadron collisions more complicated



Threshold resummation:

$$\frac{d\sigma}{dp_T} \sim \int dx_1 dx_2 x_1 f_1(x_1,\mu) x_2 f_2(x_2,\mu)$$
$$\times H(p_T,\mu) \times \int dm^2 dk J(m^2,\mu) S(k,\mu)$$
$$\times \delta\left[p_T\sqrt{S} - m^2 - kp_T - (1-x_2)\frac{t}{s} - (1-x_2)\frac{u}{s}\right]$$

Want to choose

$$\mu_h = p_T \qquad \mu_j = m \qquad \mu_s = \frac{\mu_j^2}{\mu_h} = \frac{m}{p_T}$$

Similar to thrust scale choices:

$$\mu_h = Q \qquad \mu_j = \sqrt{\tau} Q \qquad \mu_s = \frac{\mu_j^2}{\mu_h} = \tau Q$$

… but m and k are integrated over!

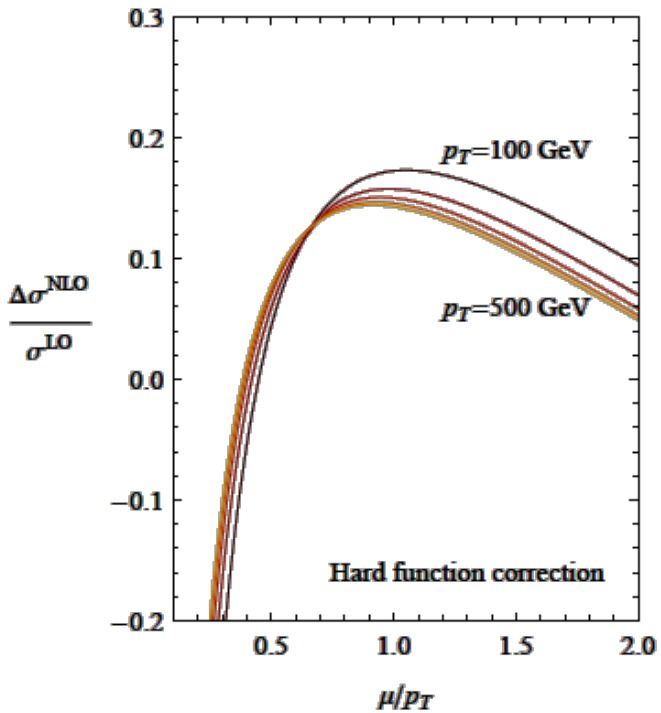
- Dangerous since $\int dm \alpha_s(m)$ integrates over Landau pole
- Must choose scales as functions of measured quantities

$$\mu_j \sim \langle m \rangle$$

How to decide scales

Hard Scale

- Include 1-loop hard function, tree-level jet and soft functions
- Integrate over pdfs and phase space at fixed p_T
- Vary scale and look for stationary point



Theory tells us that natural hard scale is

$$\mu_h = p_T$$

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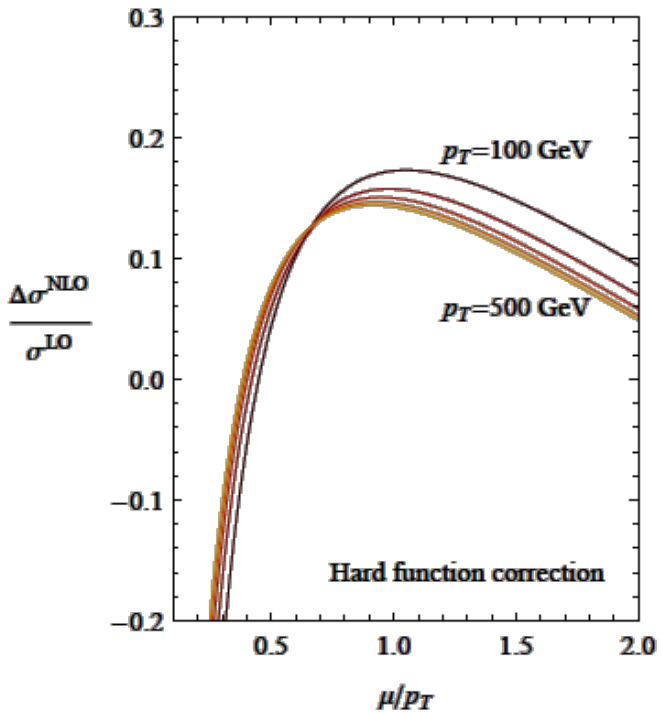
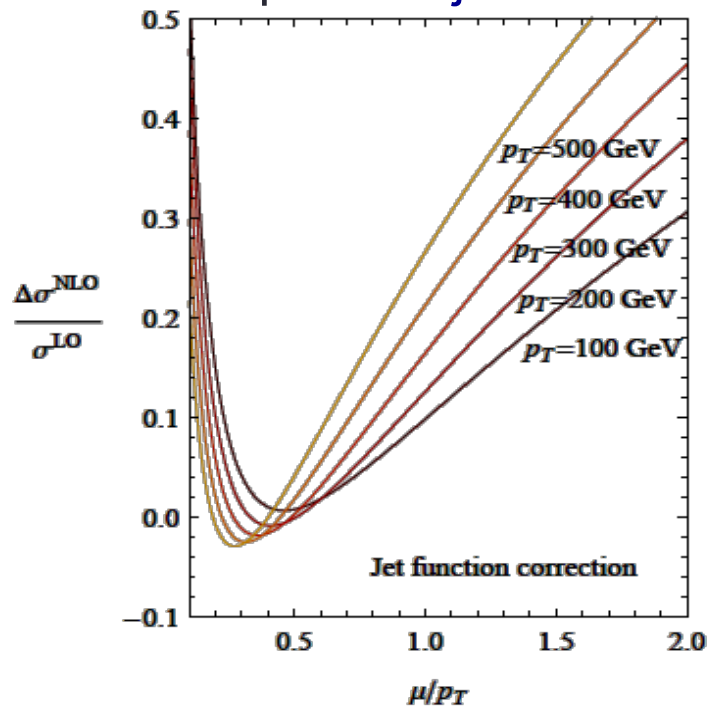


$\mu_h = p_T$

Repeat for jet scale



Lower scale indicated

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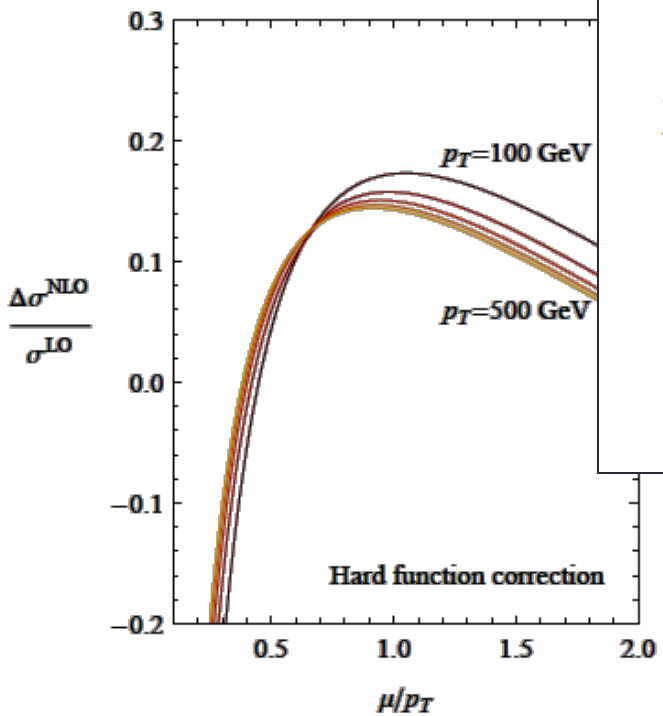

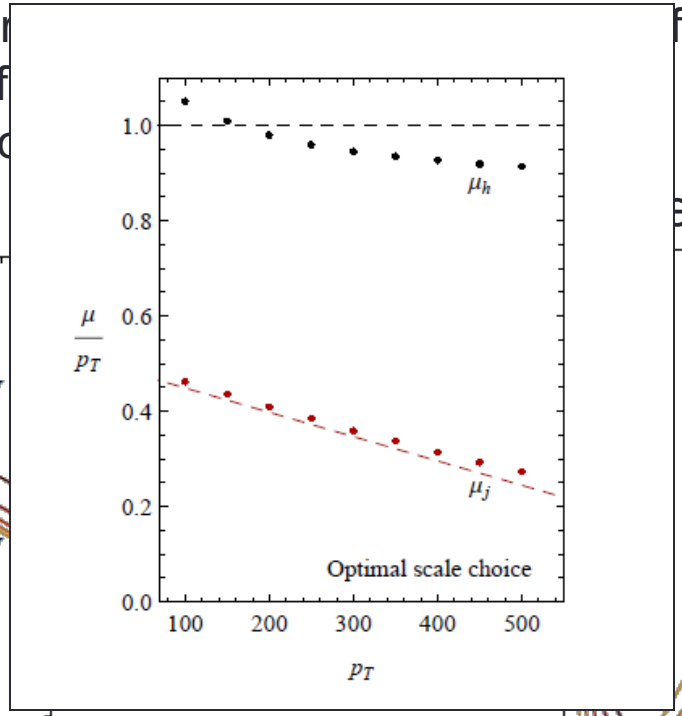

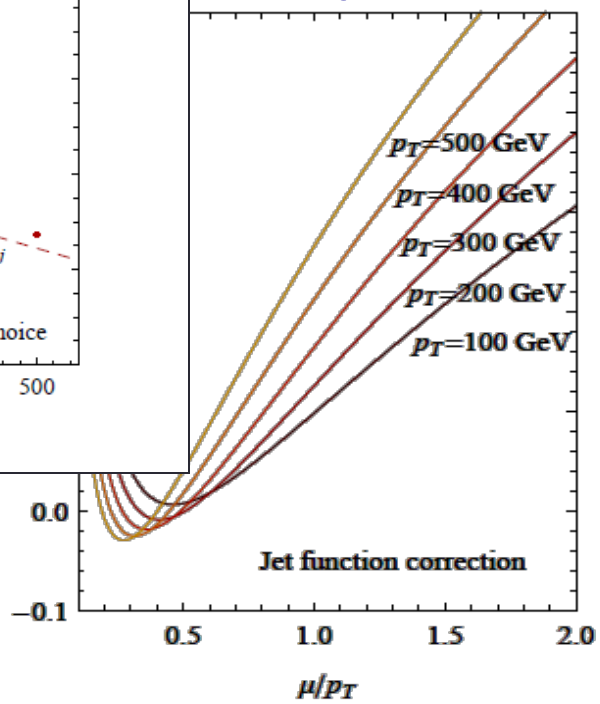

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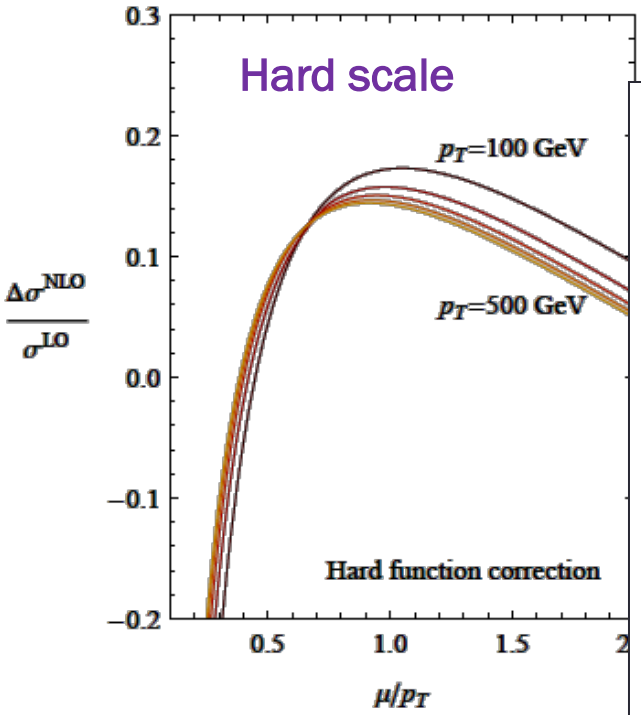
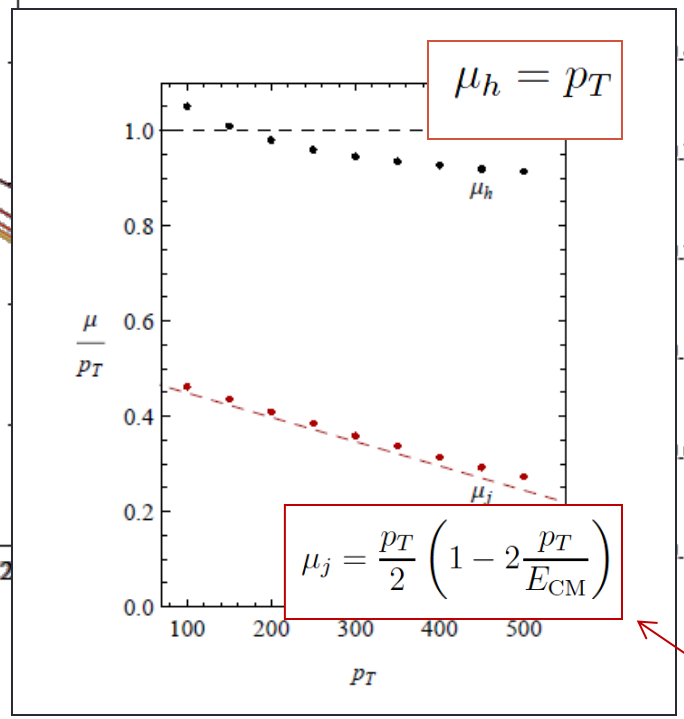
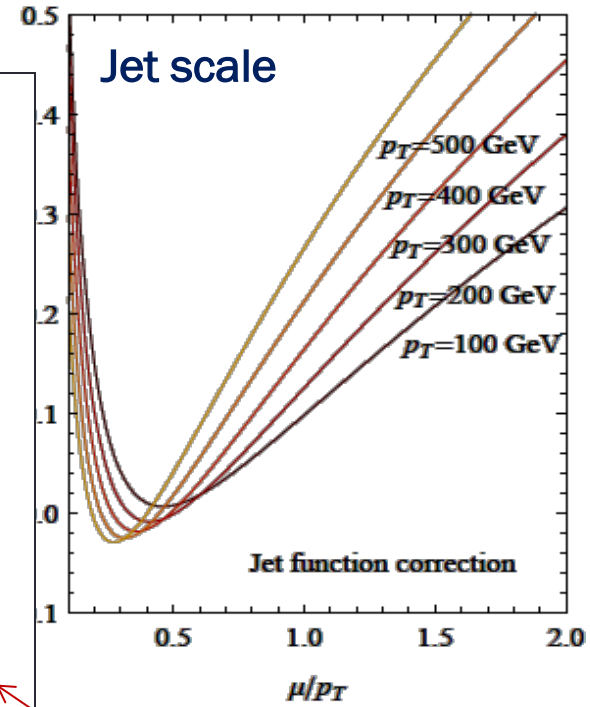
Natural scales







So we take:

$$\left.\begin{aligned} \mu_h &= p_T \\ \mu_j &= \frac{p_T}{2}\left(1 - 2\frac{p_T}{E_{\rm CM}}\right) \\ \mu_s &= \mu_j^2/\mu_h \end{aligned}\right\}$$

Always well above $\Lambda_{\rm QCD}$

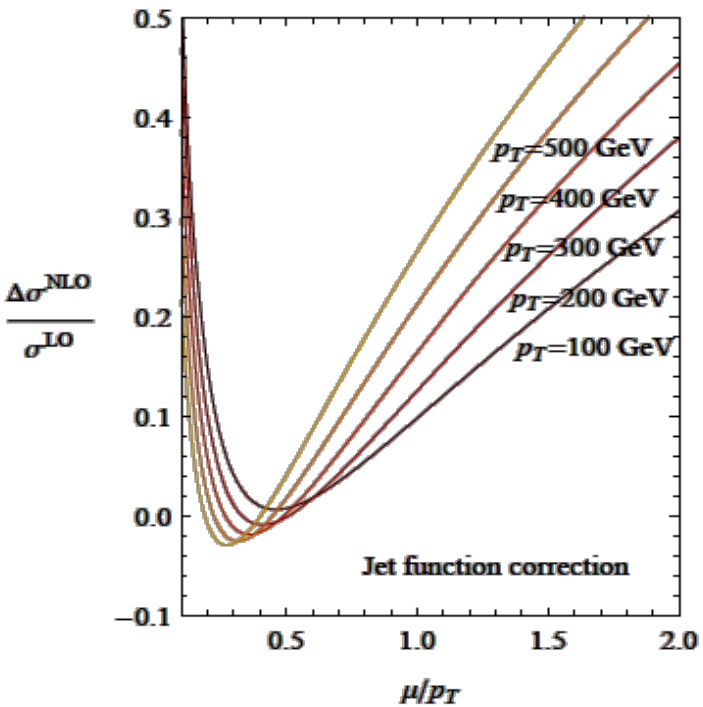
- avoids unphysical region

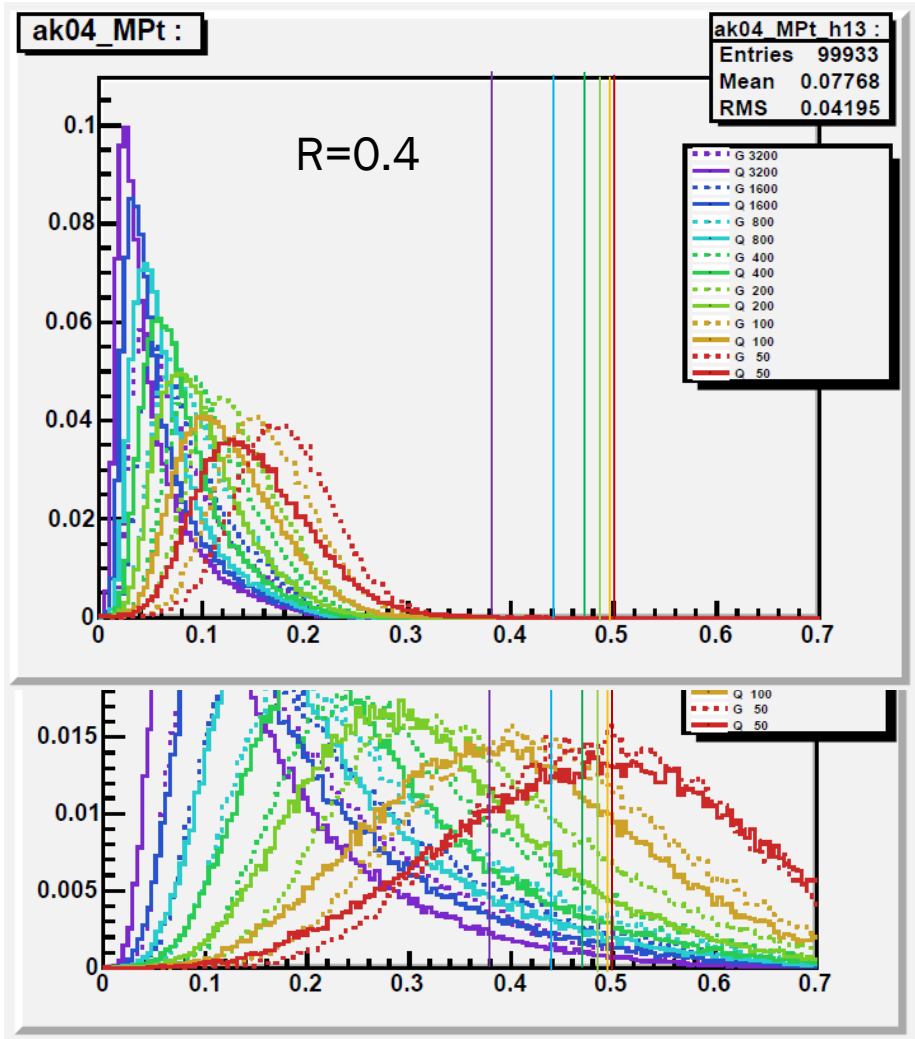
note that

$$\mu_J = \langle m_J \rangle \lesssim p_T^\gamma$$

Jet masses

Rule of thumb "m = 0.2 pt"

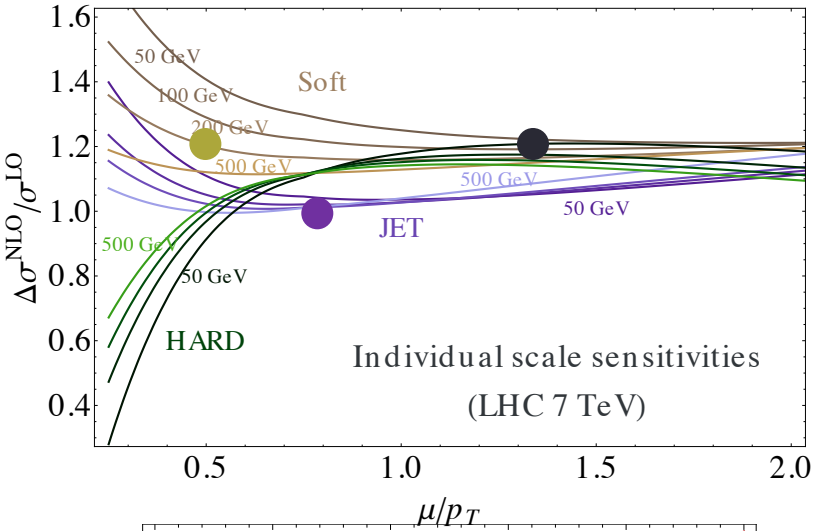




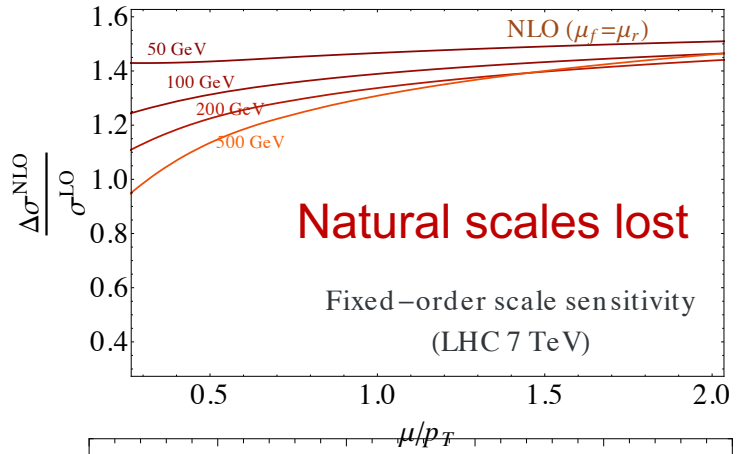
m_J really is close to the mass of the partonic jet

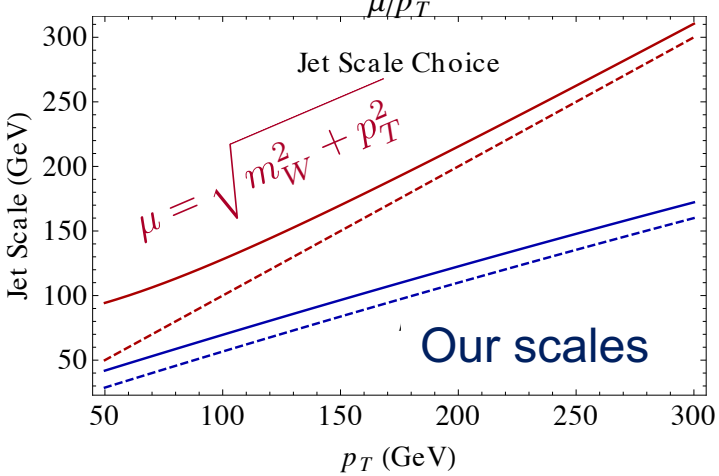
W boson p_T: 2 scales m_W and p_T

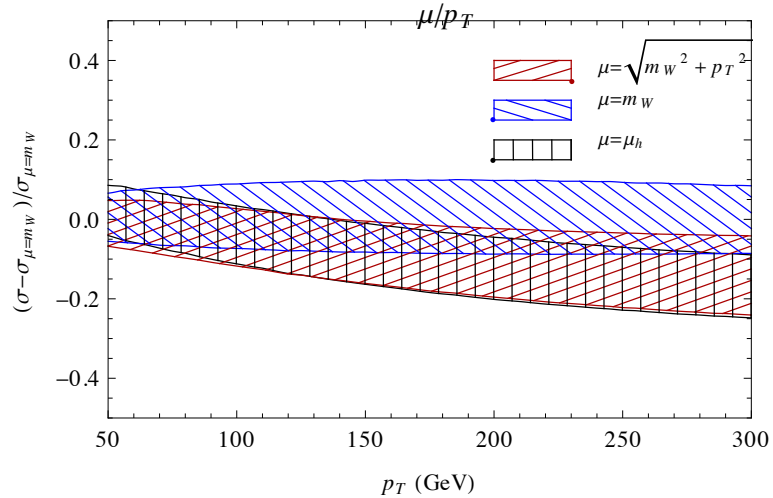
[Becher, Lorentzen and MDS arXiv:1206.6115]



Fixed order sets $\mu_h = \mu_j = \mu_s = \mu$

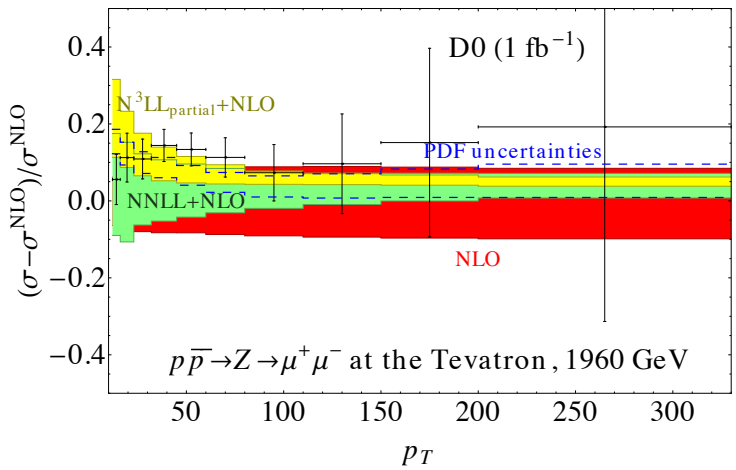




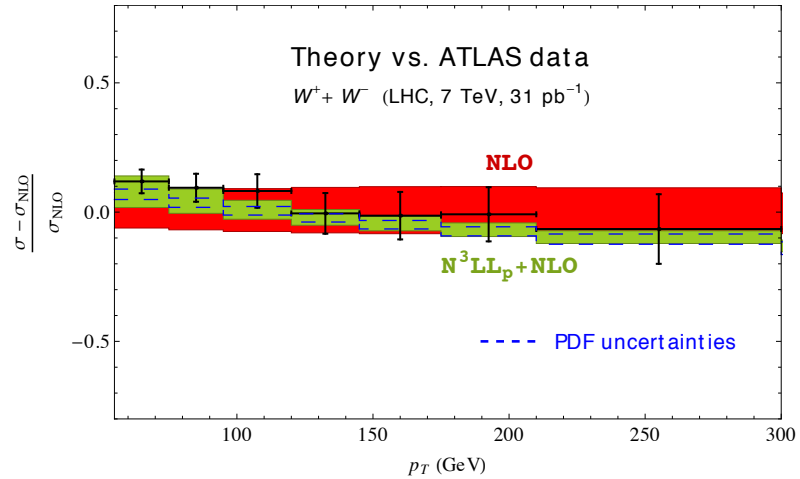


Multiple scales improves agreement with data

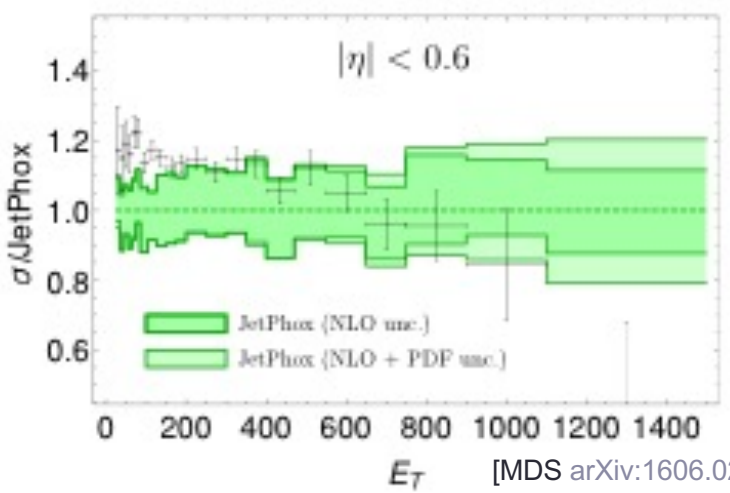
Z boson p_T [Becher, Lorentzen and MDS arXiv:1106.4310]



photon p_T (DZero)

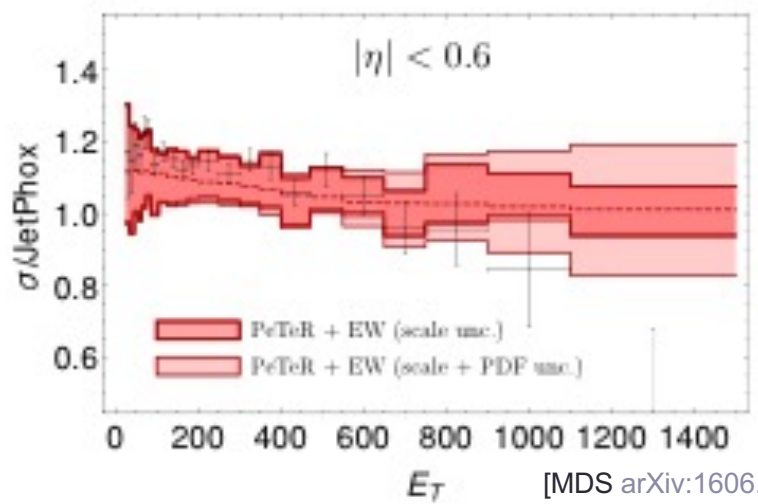


photon p_T ATLAS (NLO)



[MDS arXiv:1606.02313]

photon p_T ATLAS (NNNLL)



[MDS arXiv:1606.02313]

UNCERTAINTY

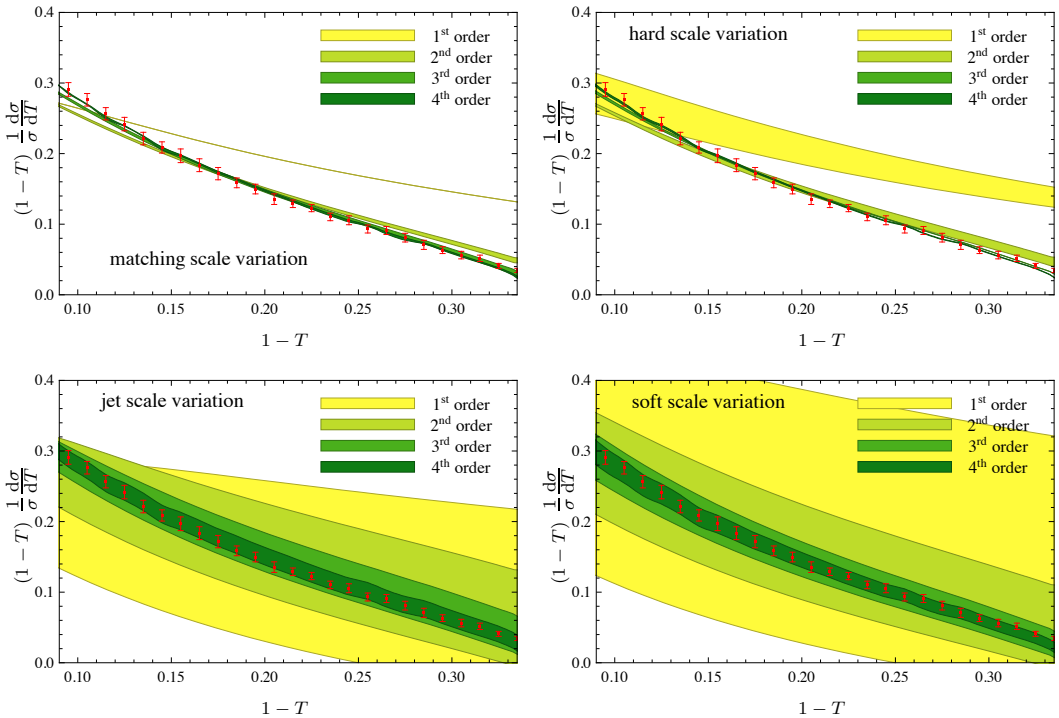
With multiple scales comes multiple variations

Thrust scale choices

$\mu_h = Q$

$\mu_j = \sqrt{\tau} Q$

$\mu_s = \tau Q$



- Soft scale is lower, α_s is larger, uncertainty is larger
- Scale variations are correlated: can't add in quadrature
- Should never have $\mu_j < \mu_s$ or $\mu_h < \mu_j$

Correlated variations

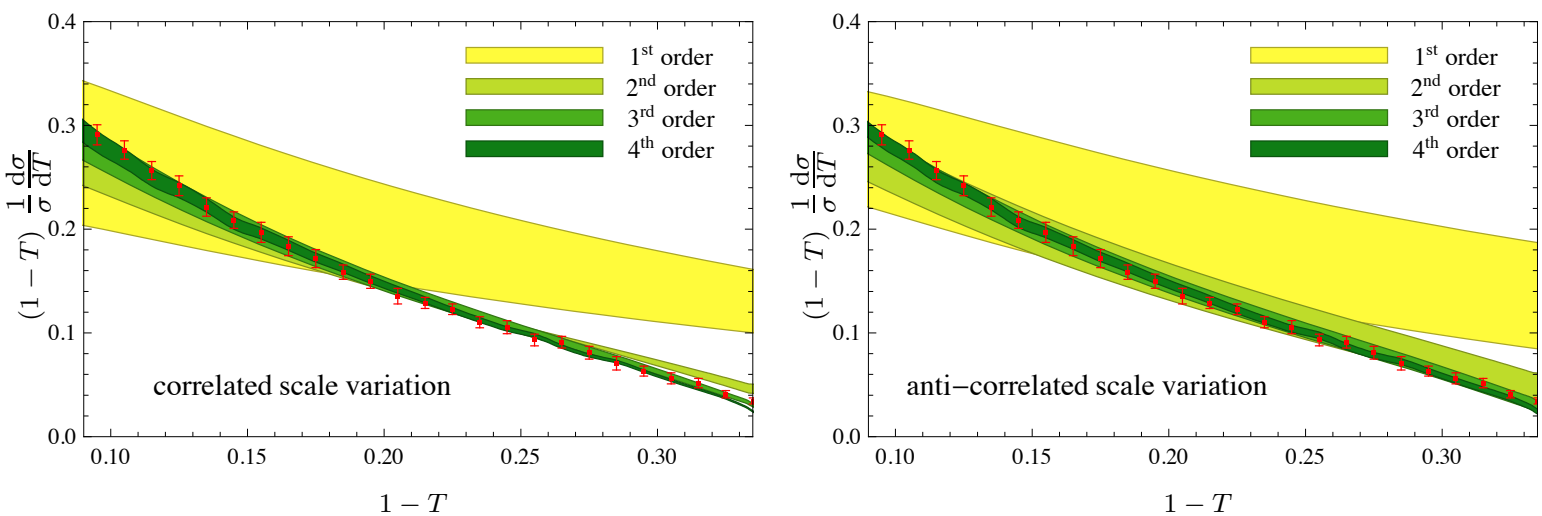
- Correlated: vary jet and soft scales holding $\frac{\mu_s}{\mu_j}$ fixed

$$\mu_j \to c\sqrt{\tau}Q, \quad \mu_s \to c\tau Q, \quad \frac{1}{2} < c < 2\,.$$

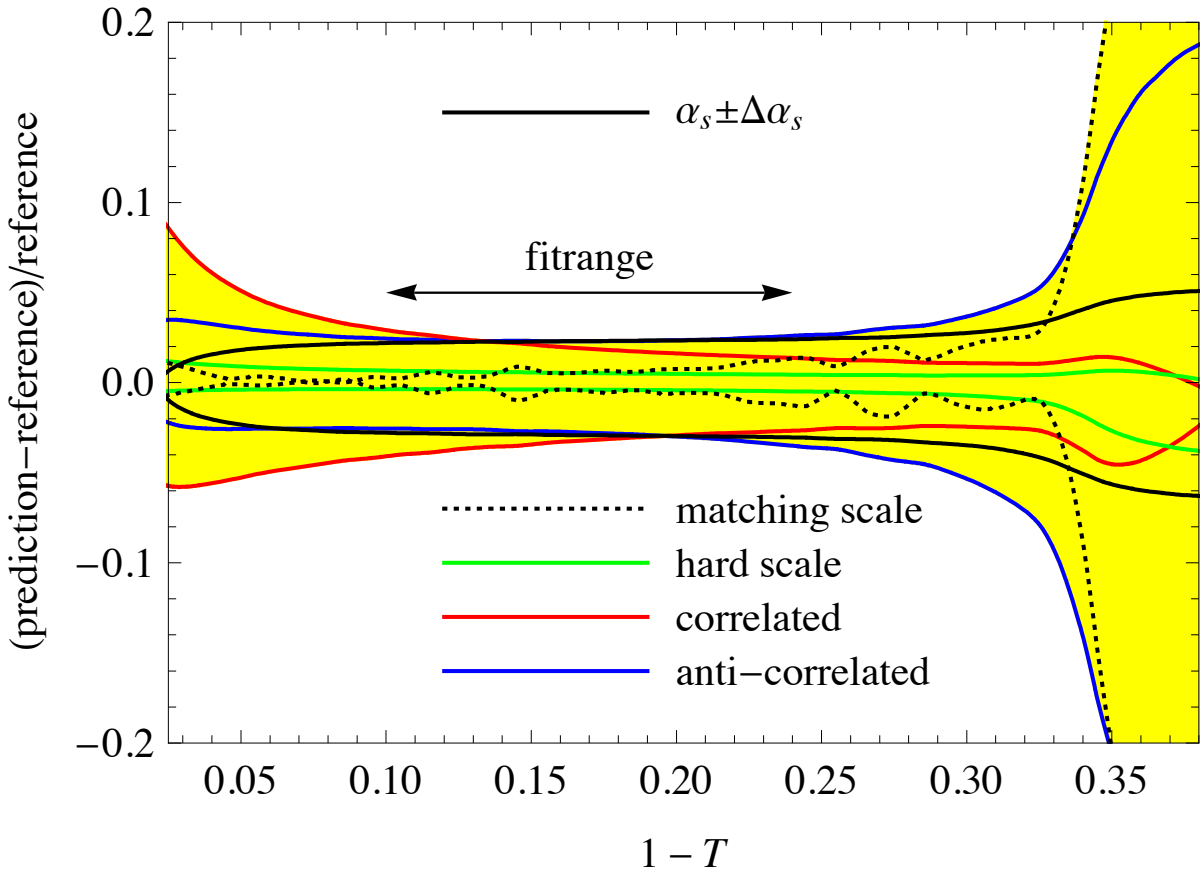
- Anti-correlated: Vary jet and soft scales holding $\frac{\mu_j^2}{Q\mu_s}$ fixed

$$\mu_j^2 \to aQ^2\tau \quad \mu_s \to aQ\tau, \quad \frac{1}{\sqrt{2}} < a < \sqrt{2}\,.$$

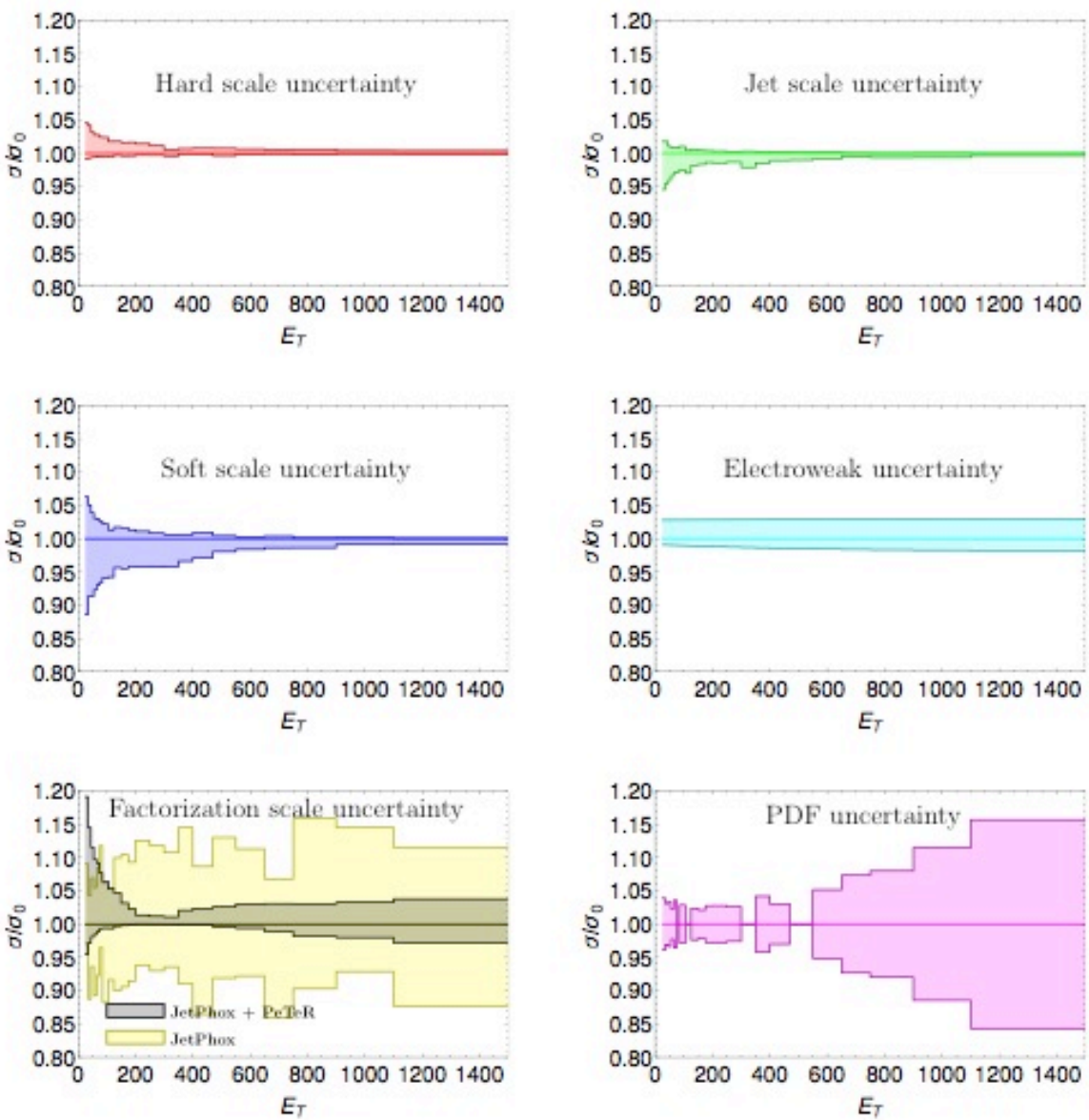
$\mu_j \to c\sqrt{\tau}Q, \quad \mu_s \to c\tau Q, \quad \frac{1}{2} < c < 2\,.$



Combine with uncertainty band



For hadron collisions, uncertainties smaller



Direct photon 13 TeV

- Adding in quadrature gives smaller uncertainty than NLO

Scale variation

Very problematic at **fixed order**:

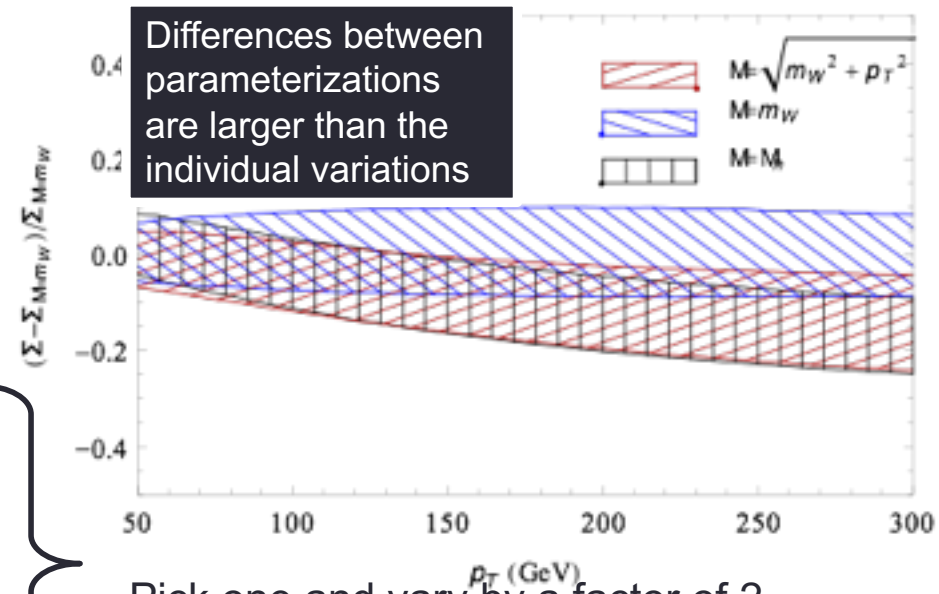
e.g. p_T spectrum of W

Many reasonable scale choices:

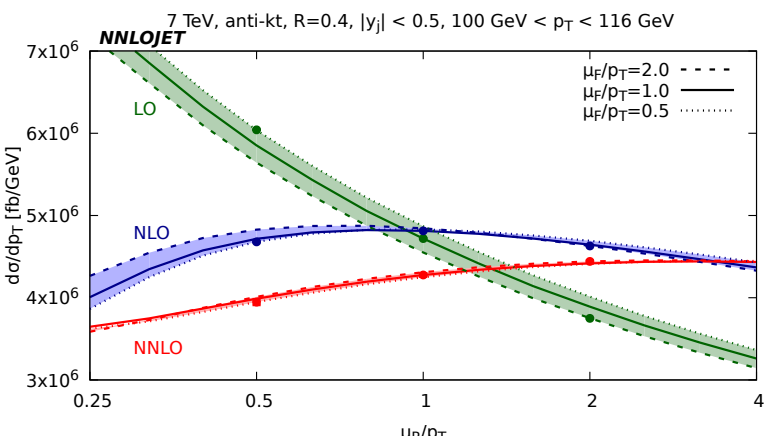
$$\mu = H_T$$

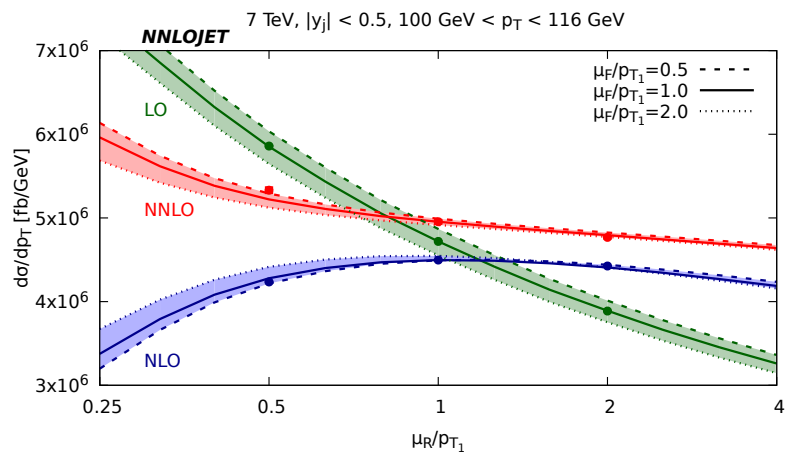
$$\mu = \sqrt{p_T^2 + m_W^2}$$

$$\mu = \max\{m_W, E_{\rm jet}\}$$



Pick one and vary by a factor of 2





- can underestimates uncertainties

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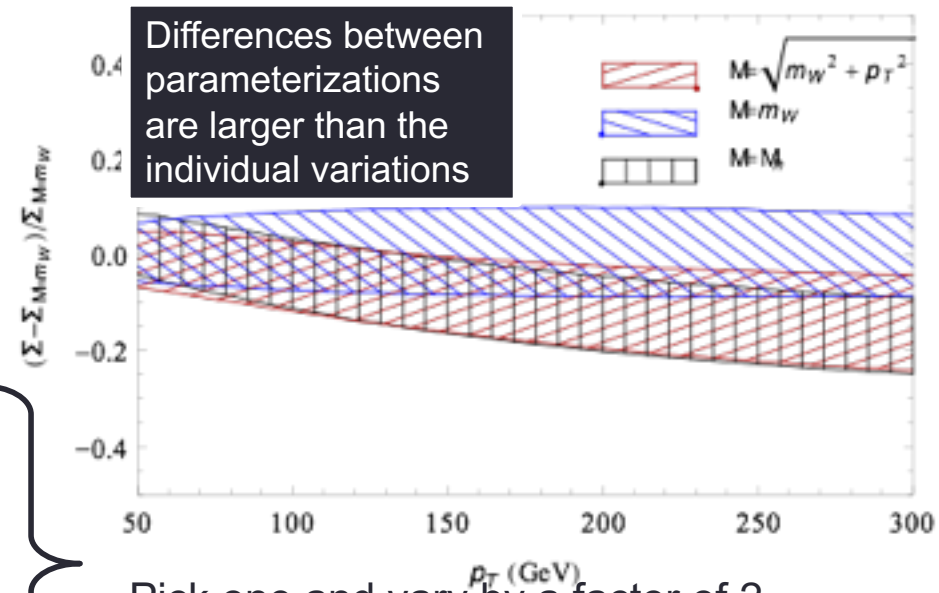

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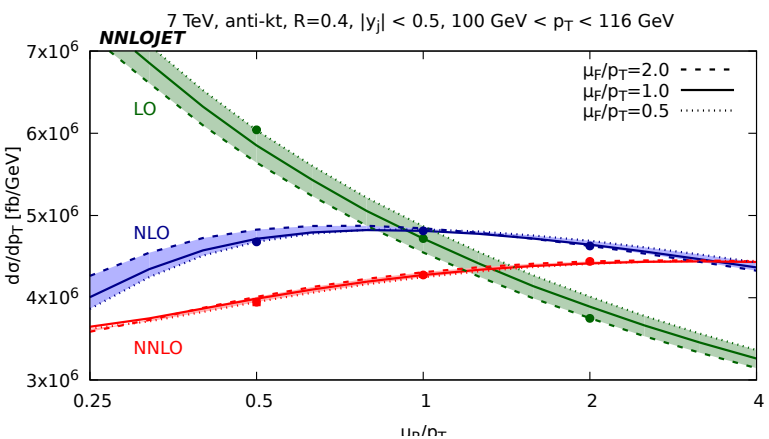


- can underestimates uncertainties

Also problematic with **resummation**

how to combine variations of different scales?

Additional problems

does not get color structures right

does not account for asymptotic nature of series

Alternatives?

Resummed formula depends on a bunch of anomalous dimensions

$$\gamma_J(\alpha) = \left(\frac{\alpha}{4\pi}\right)\gamma_0^J + \left(\frac{\alpha}{4\pi}\right)^2\gamma_1^J + \left(\frac{\alpha}{4\pi}\right)^3\gamma_2^J + \cdots,$$

$$\gamma_0^J = -3C_F\,,$$

$$\gamma_1^J = C_F^2(-\frac{3}{2} + 2\pi^2 - 24\zeta_3) + C_F C_A\left(-\frac{1769}{54} - \frac{11\pi^2}{9} + 40\zeta_3\right) + C_F T_F n_f\left(\frac{242}{27} + \frac{4\pi^2}{9}\right)$$

$$\gamma_2^J = -0.7255 n_f^2 + 85.35 n_f - 203.8\,.$$

$$\gamma_J(\alpha) = \left(\frac{\alpha}{4\pi}\right)\gamma_0^J + \left(\frac{\alpha}{4\pi}\right)^2\gamma_1^J + \left(\frac{\alpha}{4\pi}\right)^3\gamma_2^J + \cdots,$$

$$\Gamma_{\rm cusp}(\alpha) = \left(\frac{\alpha}{4\pi}\right)\Gamma_0 + \left(\frac{\alpha}{4\pi}\right)^2\Gamma_1 + \left(\frac{\alpha}{4\pi}\right)^3\Gamma_2 + \cdots$$

$$\Gamma_0 = 4C_F\,,$$

$$\Gamma_1 = 4C_F\left[C_A\left(\frac{67}{9} - \frac{\pi^2}{3}\right) - \frac{20}{9}C_F T_F n_f\right]$$

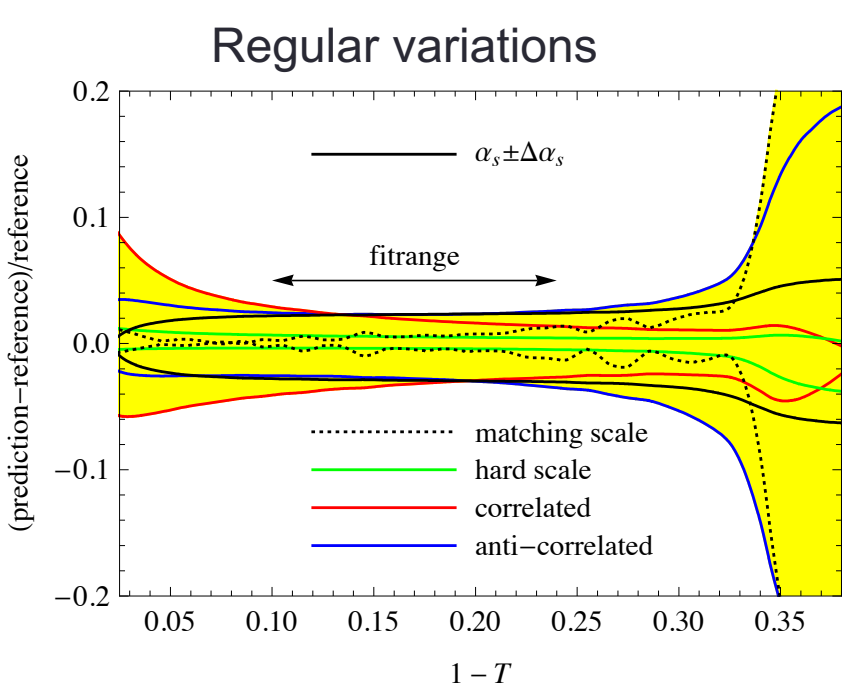
$$\Gamma_2 = -0.7901 n_f^2 - 183.2 n_f + 1175\,,$$

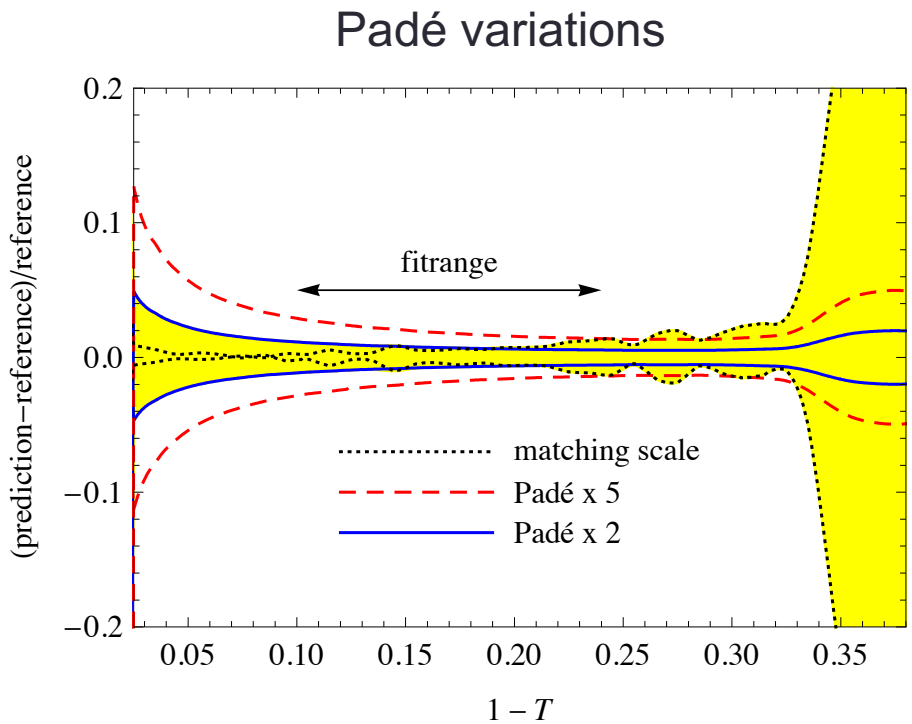
- Each anomalous dimension is a series
- Series generally asymptotic
- Estimate higher order terms with
  Padé approximation

$$\Gamma_{n+1} \approx c\frac{\Gamma_n^2}{\Gamma_{n-1}} \qquad \gamma_{n+1}^j \approx c\frac{(\gamma_n^j)^2}{\gamma_{n-1}^j}$$

- vary c by a factor of 2 or 5

Padé variation





- Reasonable estimate of what higher order calculation will give you
- Not perfect, but I think it's a good idea needing refinement
- Can be used at fixed order too…

Fixed order estimates

Can we estimate higher order non-logarithmic terms?

$$\frac{\sigma_{\rm had}}{\sigma_0} = 1 + \left(\frac{\alpha_s}{4\pi}\right)(3C_F) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[C_F C_A \left(\frac{123}{2} - 44\zeta_3\right) + C_F n_f(-11 + 8\zeta_3) - C_F^2 \frac{3}{2} + 3C_F \beta_0 \ln \frac{\mu}{Q}\right]$$

$$+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 \left[C_F^3 a_1 + C_F^2 C_A a_2 + C_F C_A^2 a_3 + C_F^2 n_f a_4 + C_F n_f^2 a_5 + C_A C_F n_f a_6\right.$$

$$\left. + (3C_F)\frac{1}{2}\beta_0^2 \ln^2 \frac{Q}{\mu} + (3C_F)\beta_1 \ln \frac{Q}{\mu} + [C_F C_A(\frac{123}{2} - 44\zeta_3) + \cdots]\beta_0 \ln \frac{Q}{\mu}\right]$$

Fixed order estimates

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$$+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 \left[C_F^3 a_1 + C_F^2 C_A a_2 + C_F C_A^2 a_3 + C_F^2 n_f a_4 + C_F n_f^2 a_5 + C_A C_F n_f a_6\right.$$

$$\left. + (3C_F)\frac{1}{2}\beta_0^2 \ln^2\frac{Q}{\mu} + (3C_F)\beta_1 \ln\frac{Q}{\mu} + [C_F C_A(\frac{123}{2} - 44\zeta_3) + \cdots]\beta_0 \ln\frac{Q}{\mu}\right]$$

- Different color structures known
- Logs all known
- Coefficients can be studied/estimated to look for patterns
  - Statistical distributions of coefficients -> confidence levels?
- Study graphs at next order
  - New channels opening up
  - Now scales appearing (particle masses, new logarithms)

theoretical uncertainty → Scale sensitivity

theoretical uncertainty → Estimte higher order effects

Conclusions

Two problems:

1. What scales should we choose?

- Solution clear for single scale problems
  - Even single-scale problems can have multiple scales
- Factorized distribution separates scales
  - Effectively reduces to single-scale case

2. How do we estimate theory error?

- Multiple scales makes problem worse
  - How do we combine separate variations?
- Scale variation is poor proxy for theoretical estimates
  - Does not even get color structures right
  - Different parameterizations don't agree
- Think about what can appear at higher order
  - More work, but may be more realistic