INTRODUCTION TO SUPERSYMMETRY

LECTURE 1

BABAR QURESHI (LUMS, PHYSICS)

Symmetries

→ Relate different objects (states)

→ Represented by unitary operators

$$|\psi\rangle \longrightarrow |\psi'\rangle = \underbrace{U}_{\text{symmetry}}|\psi\rangle \quad , \quad \big(|\psi\rangle, |\varphi\rangle\big) = \big(U|\psi\rangle, U|\varphi\rangle\big)$$

$$U(\varepsilon) = 1 + i\varepsilon \underbrace{\hat{T}}_{\text{Generator}}$$

$$U(\theta) = \exp(i\theta \hat{T}) \qquad \big([L_i, L_j] = i\varepsilon_{ijk} L_j\big)$$

$$U(\theta_i) = \exp(i\theta_i \hat{T}_i) \quad ; \quad [T_i, T_j] = i f_{ijk} T_k$$

When do we have symmetry

$$U(\theta_i)\, e^{-it\hat{H}} |\psi\rangle = e^{-itH}\, U(\theta_i) |\psi\rangle$$

$\Rightarrow [T_i, H] = 0 \qquad [L_i, H] = 0$

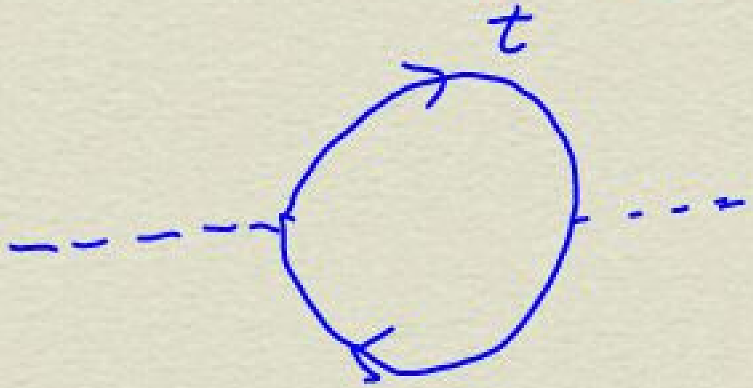
$\Rightarrow |\uparrow\rangle, |\downarrow\rangle$ have same dynamics

$\Rightarrow$ Symmetries lead to multiplets

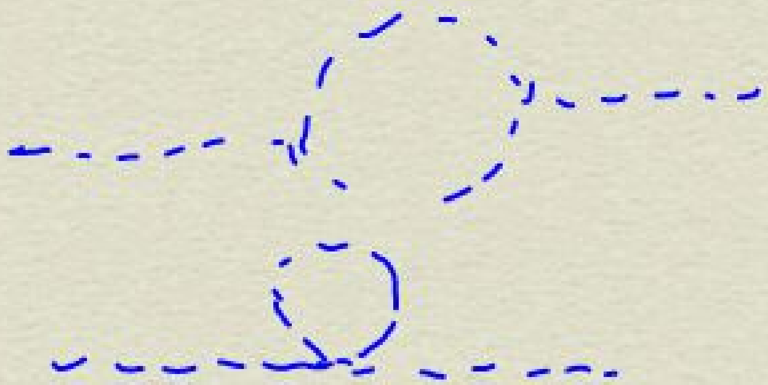
$\Rightarrow |N\rangle \;, |P\rangle \sim SU(2)$ Isospin.

In SUSY, relate

Fermions $\longleftrightarrow$ Bosons



$$\delta m_h^2 \sim - \frac{N_c |y_t|^2}{8\pi^2}\left[\Lambda^2 - 3m^2 \log\left(\frac{\Lambda^2+m^2}{m^2}\right)\right]$$



If couplings match and masses match

all contributions cancel.

Harmonic Oscillator

$$H = \frac{1}{2}(\hat{p}^2 + w^2 \hat{x}^2) \qquad m = 1$$

$$a = \frac{1}{\sqrt{2w}} (i\hat{P} + w\hat{x}) \quad , \quad a^{+} = \frac{1}{\sqrt{2w}} (-i\hat{P} + w\hat{x})$$

$$[a, a^{+}] = 1 \quad , \quad H = w(a^{+}a + \tfrac{1}{2}) = \frac{w}{2}(a^{+}a + aa^{+}) \qquad [a,a] = 0$$

$$[a^{+}, a^{+}] = 0$$

$$[H, a^{+}] = w a^{+} \quad , \quad [H, a] = -wa$$

$$|n\rangle = \frac{(a^{+})^{n}}{n!} |0\rangle \quad , \quad E_n = (n + \tfrac{1}{2})\hbar w \quad , \quad n = 0, 1, 2,$$

Fermionic Oscillator

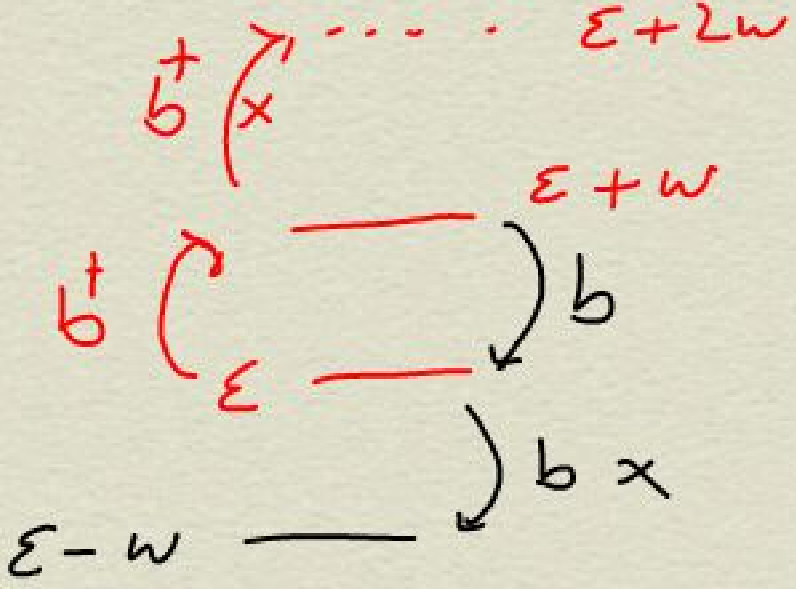
$\{b, b^\dagger\} = 1$, $\{b, b\} = 0$, $\{b^\dagger, b^\dagger\} = 0$ $\Rightarrow$ $b^2 = 0$ & $b^{\dagger 2} = 0$

$H_F = w(b^\dagger b - \frac{1}{2}) = w\,(b^\dagger b - b b^\dagger)$

$\boxed{bb^\dagger + b^\dagger b = 1}$

$[H_F, b^\dagger] = wb^\dagger$ $\quad w b^\dagger b b^\dagger - w \cancelto{0}{b^\dagger b^\dagger} b = w b^\dagger (1 - b^\dagger b)$
$= w b^\dagger$

$[H_F, b] = -wb$



$\Rightarrow H|\varepsilon\rangle = \varepsilon|\varepsilon\rangle \Rightarrow H\, b^\dagger|\varepsilon\rangle = b^\dagger H + w b^\dagger|\varepsilon\rangle$
$= (\varepsilon + w) b^\dagger |\varepsilon\rangle$

Similarly $H b|\varepsilon\rangle = (\varepsilon - w) b|\varepsilon\rangle$

$a|\varepsilon\rangle = 0 \Rightarrow |\varepsilon\rangle \equiv |0\rangle$ has $E = -\frac{w}{2}$

$|0_F\rangle$, $|1_F\rangle$ are the only two eigen states

→ Any two level system.

→ Describes a pure frequency mode of a fermion

(Pauli's exclusion rule!)

Combined System

$$H_S = H_B + H_F$$

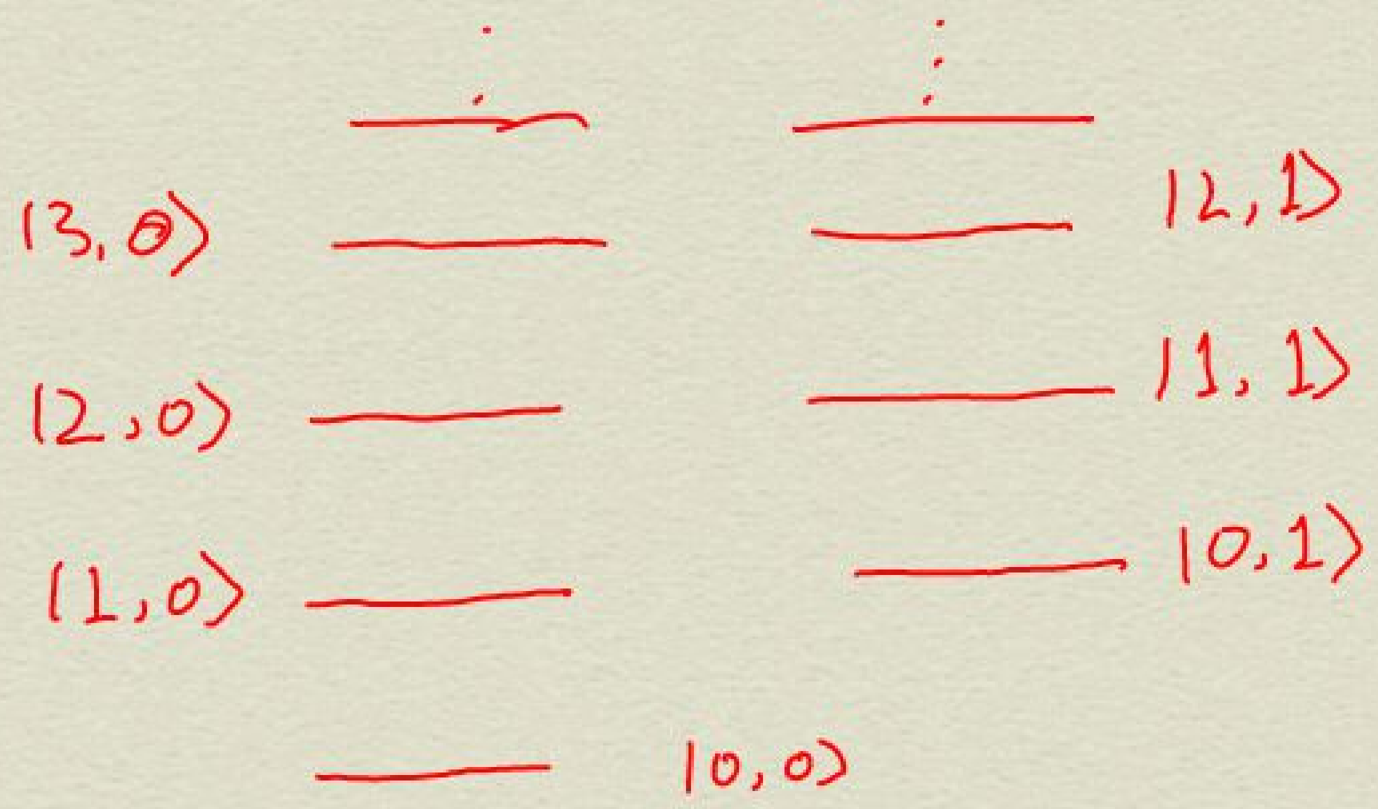
$$= w(a^\dagger a + b^\dagger b)$$

$$([a, b] = 0)$$

$$|n_B, n_F\rangle \equiv |n_B\rangle \otimes |n_F\rangle$$

$$|n_B, 0_F\rangle \quad , \quad |n_B, 1_F\rangle$$

Bosonic sector   Fermionic sector.



$\ldots \; E_2 = 2w$

$\ldots \; E_1 = w$

$E = 0$

$$H = H_B \oplus H_F$$

→ Ground State has zero energy

→ Doubling of spectra (except ground state).

→ Degenaracy between bosonic + fermionic sectors

→ Look for a symmetry

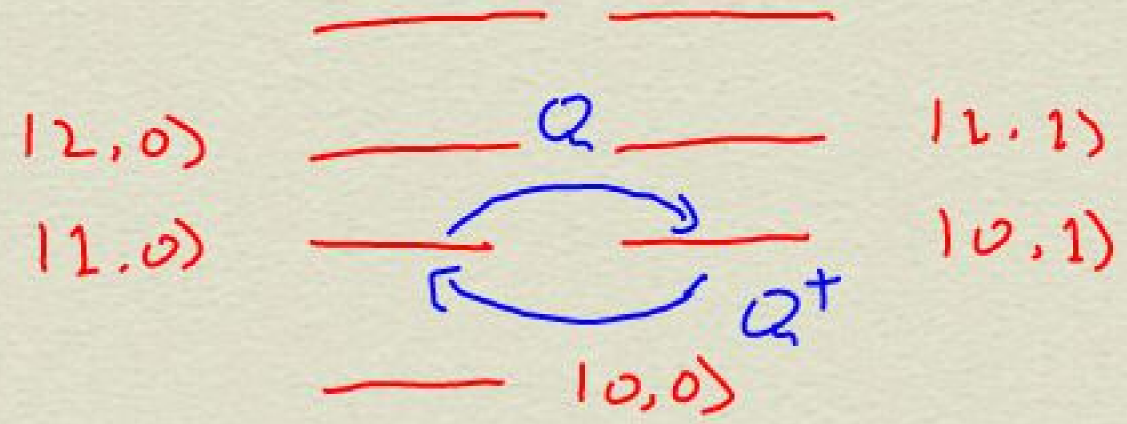
Supercharges

$$Q = ab^{\dagger} \quad ; \quad Q^{\dagger} = b^{\dagger}a$$

$$Q^2 = 0 \quad ; \quad ab^{\dagger}ab^{\dagger} = a^2 b^{\dagger 2} = 0$$

$$\{Q, Q\} = \{Q^{\dagger}, Q^{\dagger}\} = 0$$

anticommuting operators.



$$[Q, H] = [ab^{\dagger}, a^{\dagger}a + b^{\dagger}b] = [a, a^{\dagger}a]b^{\dagger} + a[b^{\dagger}, b^{\dagger}b]$$

$$= ab^{\dagger} - ab^{\dagger} = 0$$

$$\{Q, \bar{Q}\} = ab^{\dagger}ba^{\dagger} + ba^{\dagger}ab^{\dagger} = b^{\dagger}b\,(a^{\dagger}a + 1) + (1 - b^{\dagger}b)\,a^{\dagger}a$$

$$= b^{\dagger}b + a^{\dagger}a = \sim H$$

We have a (super) symmetry

Define $Q_1 = Q + Q^\dagger$

$\rightarrow$ Q_1 is Hermitian

$\rightarrow$ $\star$ Interchanges fermionic & bosonic states

$\rightarrow$ $[Q_1, H] = 0$

Since $Q\,|\text{Bosonic}\rangle = |\text{fermionic}\rangle$

& $Q\,|\text{fermionic}\rangle = 0$

$Q^\dagger\,|\text{fermionic}\rangle = |\text{Bosonic}$

$Q^\dagger\,|\text{Bosonic}\rangle = 0$

$\Rightarrow$ $Q_1\,|\text{Bosonic}\rangle = |\text{fermionic}\rangle$

& $Q_1\,|\text{Fermionic}\rangle = |\text{Bosonic}\rangle$ $\star$

Superalgebra

$$\{Q,Q\} = 0 = \{Q^\dagger, Q^\dagger\} \quad ; \quad [H,Q] = [H, Q^\dagger] = 0$$

$$\{Q, Q^\dagger\} \sim H$$

$\Rightarrow$ Spacetime symmetry, half of time translations.

$\Rightarrow$ $H \sim \{Q, Q^\dagger\}$ $\Rightarrow$

$$\langle \psi | H | \psi \rangle = \langle \psi | Q Q^\dagger | \psi \rangle + \langle \psi Q^\dagger Q | \psi \rangle$$
$$= \left(Q^\dagger | \psi \rangle, Q^\dagger | \psi \rangle \right) + \left(Q | \psi \rangle, Q | \psi \rangle \right)$$
$$\geq 0$$

$\Rightarrow$ Energy is positive definite

→ Lowest possible energy is zero.

Ground State

$E_g = 0$ iff $H|0\rangle = 0 = QQ^\dagger|0\rangle + Q^\dagger Q|0\rangle$

$\Rightarrow Q|0\rangle = 0 \,,\; Q^\dagger|0\rangle = 0.$

Supersymmetric ground state.

→ Doubling of spectrum :-

If $H|E\rangle = E|E\rangle$

then also $H|E'\rangle = H\,Q|E\rangle = Q\cancel{H}|E\rangle = E\,Q|E\rangle$ unless $Q|E\rangle = 0$

But $Q(Q|E\rangle) = 0 \Rightarrow |E\rangle$ + $Q|E\rangle$ are a pair with same energy fermion = Boson rule.

Lagrangian

→ Bosonic part: $L_B = \frac{1}{2}\dot{x}^2 - \frac{1}{2} w x^2$

→ How to get it from Quantum $H = w(a^+ a + \frac{1}{2})$.

→ Identify canonical variables

→ Dirac! $[\phi, \pi] = i \longleftrightarrow \{\phi, \pi\} = 1$ Canonical pair

$$a = \frac{1}{\sqrt{2w}}(wx + ip) \; ; \quad a^+ = \frac{1}{\sqrt{2w}}(wx - ip) \Rightarrow [x, p] = i$$

$$L = p\dot{x} - H = \frac{1}{2}\dot{x}^2 - \frac{1}{2} w x^2$$

For Fermionic variables

$$\{\phi, \pi\} = i \qquad \text{Canonical pair}$$

$$\hookrightarrow \quad \underbrace{\{\phi . \pi\}}_{\text{anti-Poisson}} = 1$$

$$\Rightarrow \quad \{b, ib^\dagger\} = i \quad \Rightarrow \quad b, ib^\dagger \quad \text{canonical pair}$$

$$L = i\psi^\dagger\dot{\psi} - \frac{\omega}{2}(\psi^\dagger\psi - \psi\psi^\dagger)$$

also $\{b^\dagger . b^\dagger\} = \{b . b\} = 0$

$\Rightarrow \psi, \psi^\dagger$ anticommute even classically

$$= \frac{i}{2}(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) - \frac{\omega}{2}(\psi^\dagger\psi - \psi\psi^\dagger).$$

$\psi^2 = \psi^{\dagger 2} = 0 \quad \psi\psi^\dagger = -\psi^\dagger\psi$

etc.

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 + \frac{i}{2}\left(\bar\psi\dot\psi - \dot{\bar\psi}\psi\right) - \omega\bar\psi\psi$$

→ Supersymmetry

$$\delta x = \epsilon\bar\psi - \bar\epsilon\psi \qquad \delta x = [\epsilon Q + \bar\epsilon\bar Q, x]$$

$$\delta\psi = \epsilon(i\dot x + \omega x) \qquad = [\epsilon Q + \bar\epsilon\bar Q, b]$$

$$\delta\bar\psi = \epsilon(-i\dot x + \omega x) \qquad = [\epsilon Q + \bar\epsilon\bar Q, b^\dagger]$$

$$\delta L = -i\epsilon\frac{d}{dt}\left[\bar\psi(i\dot x + \omega x)\right] - i\bar\epsilon\frac{d}{dt}\left[\psi(-i\dot x + \omega x)\right]$$

$\Rightarrow$ $\delta S = 0$ Above transformations are a symmetry.

$\Rightarrow$ By Noether's theorem.

$$\delta S = \int dt \, \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q = \int dt \, \frac{d}{dt} F \qquad \text{if symmetry}$$

$$\Rightarrow \int dt \left[\frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\delta q - \frac{d}{dt}F \right]$$

$$\Rightarrow \overset{0 \;\; \text{E.O.M.}}{\cancel{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}}} - \frac{d}{dt}F = 0 \quad \underset{0}{\Rightarrow} \quad F \text{ is conserved.}$$

Conserved Charges

$$Q = \bar{\psi}\,(i\dot{x} + \omega x)$$

$$Q^{\dagger} = \psi\,(-i\dot{x} + \omega x).$$

Generalization

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi$$

$$\delta x = \varepsilon\bar{\psi} - \bar{\varepsilon}\psi$$

$$\delta\psi = \varepsilon(i\dot{x} + h'(x))$$

$$\delta\bar{\psi} = \bar{\varepsilon}(-i\dot{x} + h'(x))$$

$$[\delta_1, \delta_2]\, x = 2i(\varepsilon_1\bar{\varepsilon}_2 - \varepsilon_2\bar{\varepsilon}_1)\,\dot{x}$$

$$\psi \qquad\qquad \dot{\psi}$$

$$H = \frac{p^2}{2} + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi})$$

TOMORROW

→ How to write general Lagrangians

(Superspace)

→ Real deal: Supersymmetry in 4-D Field theories

→ Representations & Lagrangians

→ Gauge theory