QUANTUM FIELD THEORY

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CONTENTS

- Basic Concepts
- Classical field theory:
- Lagrangian and Hamiltonian formalisms.
- Canonical quantization.
- Quantization of free scalar fields,
- Covariant commutation relations and propagators of scalar fields.
- Prospects for Curved Spacetime.

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3), \\ \mathbf{J} &= (J_1, J_2, J_3), \\ \mathbf{A} &= (A_1, A_2, A_3), \end{aligned}$$

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i = \delta_{ij} A_i B_j = \delta^{ij} A_i B_j$$

$$\begin{aligned} x^\mu &= (ct, \mathbf{x}), \\ x_\mu &= (ct, -\mathbf{x}) \end{aligned}$$

Minkowski space.

$(+, -, -, -)$

$$\begin{aligned} x_\mu &= \eta_{\mu\nu} x^\nu, \\ x^\mu &= \eta^{\mu\nu} x_\nu. \end{aligned}$$

$$\begin{aligned} A \cdot B &= A^\mu B_\mu = A_\mu B^\mu \\ &= \eta^{\mu\nu} A_\mu B_\nu = \eta_{\mu\nu} A^\mu B^\nu \\ &= A^0 B^0 - \mathbf{A} \cdot \mathbf{B}. \end{aligned}$$

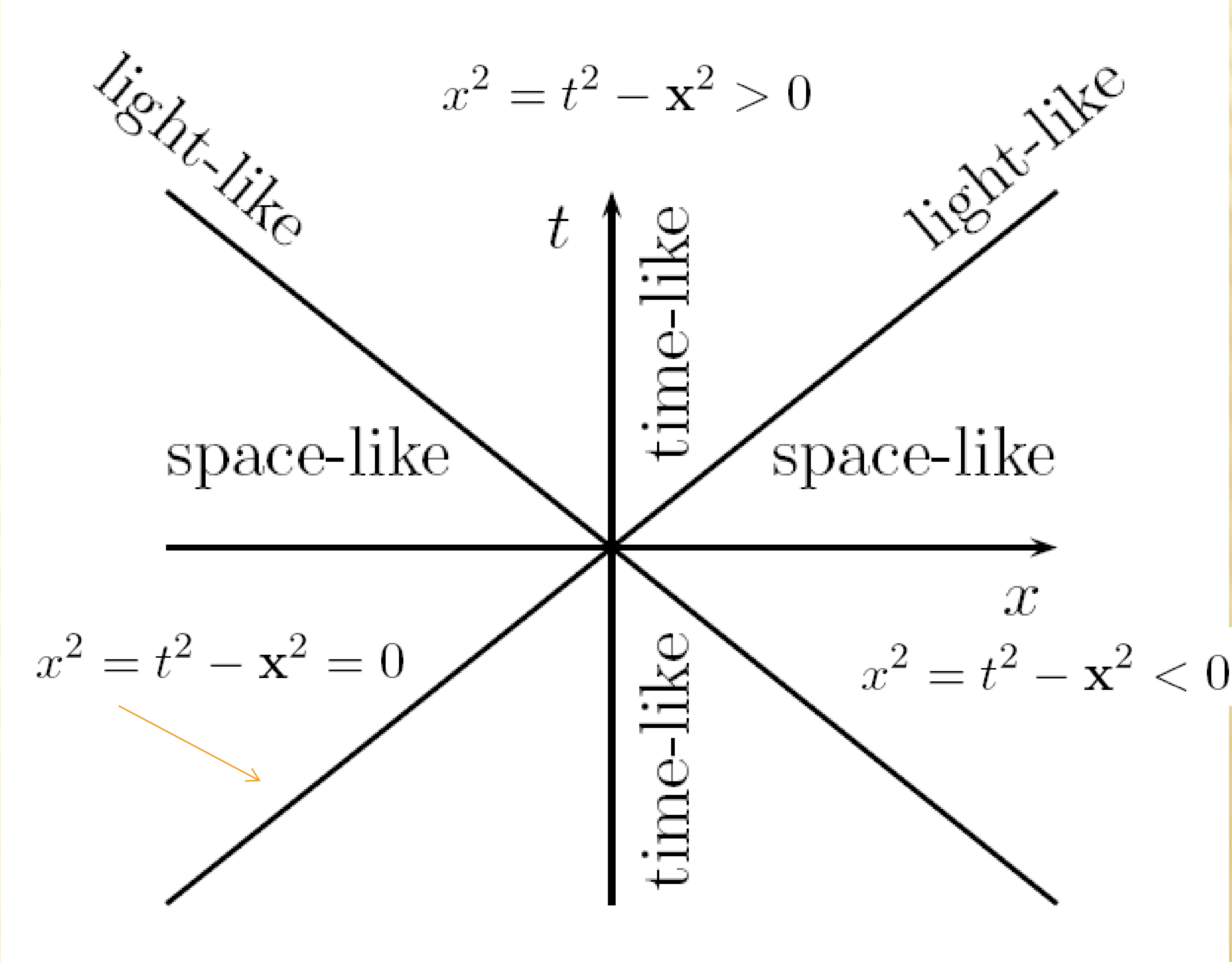
$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Curved Spacetime

$$g_{\mu\nu}$$

$$x^2 = x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = c^2 t^2 - \mathbf{x}^2.$$

$$\mathrm{d}s^2 = c^2 \mathrm{d}\tau^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu.$$



$x^2 = t^2 - \mathbf{x}^2 > 0$
light-like
light-like
t
time-like
space-like
space-like
x
$x^2 = t^2 - \mathbf{x}^2 = 0$
$x^2 = t^2 - \mathbf{x}^2 < 0$
time-like

$$\begin{aligned}\partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right)\\ \partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right).\end{aligned}$$

$$\Box = \partial^2 = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$

$$\begin{aligned}p^\mu &= i\hbar\partial^\mu = i\hbar\frac{\partial}{\partial x_\mu} = \left(i\hbar\frac{\partial}{\partial t}, -i\hbar\boldsymbol{\nabla}\right)\\ p_\mu &= i\hbar\partial_\mu = i\hbar\frac{\partial}{\partial x^\mu} = \left(i\hbar\frac{\partial}{\partial t}, i\hbar\boldsymbol{\nabla}\right).\end{aligned}$$

Quantum field theory is a theoretical framework that combines *quantum mechanics* and *special relativity*. Generally speaking, quantum mechanics is a theory that describes the behavior of small systems, such as atoms and individual electrons. Special relativity is the study of high energy physics, that is, the motion of particles and systems at velocities near the speed of light (but without gravity).

physical observables are *mathematical operators* in the theory.

Hamiltonian (i.e., the energy) of a *simple harmonic oscillator*

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$

In quantum field theory,

- Position x and momentum p are not operators—they are just numbers like in classical physics.
- The fields $\varphi(x,t)$ and their conjugate momentum fields $\pi(x,t)$ are operators.
- Canonical commutation relations are imposed on the fields.

$$[x_i, p_j] = i\delta_{ij}$$

$$[x_i, x_j] = [p_i, p_j] = 0$$

Klein-Gordon field theory

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi + m^2 \varphi = 0$$

free field solution of the Klein-Gordon equation

$$\varphi(x,t) \sim e^{-i(Et - \vec{p}\cdot\vec{x})}$$

$$E \to k_0 = \omega_k, \vec{p} \to \vec{k}$$

$$\varphi(x) \sim e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

Fields ⇨ Operators

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\varphi(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \varphi^*(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

$$\varphi(\vec{k}) \to \hat{a}(\vec{k})$$

$$\varphi^*(\vec{k}) \to \hat{a}^\dagger(\vec{k})$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

conjugate momentum to the field is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \partial_0 \varphi$$

$$\partial_0 \hat{\varphi}(x) = \partial_0 \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k}) \partial_0 (e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}) + \hat{a}^\dagger(\vec{k}) \partial_0 (e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}) \right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})(-i\omega_k) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k})(+i\omega_k) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

$$= -i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

the conjugate momentum to the field

$$\hat{\pi}(x) = -i\int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right]$$

$$[\hat{\varphi}(x), \hat{\pi}(y)] = i\delta(\vec{x} - \vec{y})$$

$$[\hat{\varphi}(x), \hat{\varphi}(y)] = 0$$

$$[\hat{\pi}(x), \hat{\pi}(y)] = 0$$

equal time commutation relations.

Exercise:1

Suppose that a real scalar field is given by

$$\varphi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p})e^{ipx} + a^\dagger(\vec{p})e^{-ipx}\right]$$

Compute

$$[\varphi(x), \pi(y)]$$

where $x^0 = y^0$.

States in Quantum Field Theory

$$\hat{a}(\vec{k})|0\rangle = 0$$

$$|\vec{k}\rangle = \hat{a}^{\dagger}(\vec{k})|0\rangle$$

This describes a one-particle state

the two-particle state $|\vec{k}_1, \vec{k}_2\rangle$ is created by

$$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^{\dagger}(\vec{k}_1)\hat{a}^{\dagger}(\vec{k}_2)|0\rangle$$

By extension, we can create an n-particle state using

$$|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle = \hat{a}^{\dagger}(\vec{k}_1)\hat{a}^{\dagger}(\vec{k}_2) \ldots \hat{a}^{\dagger}(\vec{k}_n)|0\rangle$$

Each creation operator $\hat{a}^{\dagger}(\vec{k}_i)$ creates a single particle with momentum $\hbar\vec{k}_i$ and energy $\hbar\omega_{k_i}$

$$\omega_{k_i} = \sqrt{\vec{k}_i^{\,2} + m^2}$$

Positive and Negative Frequency Decomposition

$$\hat{\varphi}^{+}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

$$\hat{\varphi}^{+}(x)\left|0\right\rangle = 0$$

$$\hat{\varphi}^{-}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}^{\dagger}(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

$$\begin{aligned}\hat{\varphi}^{-}(x)\left|0\right\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \hat{a}^{\dagger}(\vec{k})\left|0\right\rangle \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \left|\vec{k}\right\rangle\end{aligned}$$

Number Operators

$$\hat{N}(\vec{k}) = \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

The eigenvalues of the number operator are called *occupation numbers*.

$$n(\vec{k}) = 0, 1, 2, \ldots$$

which tell us how many particles there are of momentum $\vec{k}$ for a given state.

state $\Rightarrow$ $\left|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\right\rangle = \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)\ldots\hat{a}^\dagger(\vec{k}_n)|0\rangle$ $\Rightarrow$ consists of n particles

$$\left|\vec{k}_1, \vec{k}_1, \vec{k}_2\right\rangle = \frac{\hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_1)}{\sqrt{2}}\hat{a}^\dagger(\vec{k}_2)|0\rangle \Rightarrow \left|\vec{k}_1, \vec{k}_1, \vec{k}_2\right\rangle = \left|n(\vec{k}_1)n(\vec{k}_2)\right\rangle$$

where $n(\vec{k}_1) = 2,\ n(\vec{k}_2) = 1$

From the vacuum state

$$\left|n(\vec{k}_1)n(\vec{k}_2)\right\rangle = \frac{\hat{a}^\dagger(\vec{k}_1)^{n(\vec{k}_1)}}{\sqrt{n(k_1)!}} \frac{\hat{a}^\dagger(\vec{k}_2)^{n(\vec{k}_2)}}{\sqrt{n(k_2)!}} \left|0\right\rangle$$

In general

$$\left|n(\vec{k}_1)n(\vec{k}_2)\ldots n(\vec{k}_m)\right\rangle = \prod_j \frac{\hat{a}^\dagger(\vec{k}_j)^{n(\vec{k}_j)}}{\sqrt{n(k_j)!}} \left|0\right\rangle$$

number density of particles in a given state

$$\hat{N} = \int d^3k\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

Exercise: Find $\hat{N}\left|\vec{k}'\right\rangle$

Normalization of the States

$$\langle 0 | 0 \rangle = 1$$

Compute the normalization of the state $|\vec{k}\rangle$ by considering the inner product $\langle \vec{k} | \vec{k}' \rangle$

$$\begin{aligned}
\langle \vec{k} | \vec{k}' \rangle &= \langle 0 | \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}') | 0 \rangle \\
&= \langle 0 | \hat{a}^\dagger(\vec{k}') \hat{a}(\vec{k}) + \delta(\vec{k} - \vec{k}') | 0 \rangle \\
&= \langle 0 | \hat{a}^\dagger(\vec{k}') \hat{a}(\vec{k}) | 0 \rangle + \langle 0 | \delta(\vec{k} - \vec{k}') | 0 \rangle \\
&= \delta(\vec{k} - \vec{k}') \langle 0 | 0 \rangle \\
&= \delta(\vec{k} - \vec{k}')
\end{aligned}$$

$$\Rightarrow \langle \vec{k} | \vec{k}' \rangle = \delta(\vec{k} - \vec{k}')$$

ENERGY AND MOMENTUM

The Hamiltonian density

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L}$$

$$H = \int \mathcal{H}\, d^3x$$

Starting with the operator expansion of the field

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

$$\hat{H} = \int d^3k\, \omega_k \left[\hat{N}(\vec{k}) + \frac{1}{2} \right]$$

The momentum in the field $\Rightarrow$ $\hat{P} = \int d^3k\, \vec{k} \left[\hat{N}(\vec{k}) + \frac{1}{2} \right]$

Exercise: For the real scalar field, find the energy of the vacuum.

$\langle 0 | \hat{H} | 0 \rangle \Rightarrow = \frac{\omega_k}{2} \int d^3k$ But $\int d^3k \to \infty$

The *renormalized Hamiltonian*

$$\hat{H}_R = \hat{H} - \int d^3k$$

$$= \int d^3k\, \omega_k \hat{N}(\vec{k}) = \int d^3k\, \omega_k \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k})$$

Exercise

Find the energy of the state $|\vec{k}\rangle$ using the renormalized Hamiltonian

Normal and Time-Ordered Products

$$: \hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}) : = \hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})$$

$$\hat{\varphi}^{+}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

$$\hat{\varphi}^{-}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}^{\dagger}(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

$$: \varphi(x)\varphi(y) : = \varphi^{+}(x)\varphi^{+}(y) + \varphi^{-}(x)\varphi^{+}(y) + \varphi^{-}(y)\varphi^{+}(x) + \varphi^{-}(x)\varphi^{-}(y)$$

A *time-ordered product* is a mathematical representation of the physical fact that a particle has to be created before it gets destroyed. Time ordering is accomplished using the time-ordering operator which acts on the product

$$T[\varphi(t_1)\psi(t_2)] = \begin{cases} \varphi(t_1)\psi(t_2) \text{ if } t_1 > t_2 \\ \psi(t_2)\varphi(t_1) \text{ if } t_2 > t_1 \end{cases}$$

The Complex Scalar Field

complex scalar field represents particles with charge q and antiparticles with charge $-q$

$$\hat{a}^\dagger(\vec{k}) \qquad \hat{a}(\vec{k}) \quad \text{(particles)}$$

$$\hat{b}^\dagger(\vec{k}) \qquad \hat{b}(\vec{k}) \quad \text{(antiparticles)}$$

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{b}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

$$\hat{\varphi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{b}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

$$\left[\hat{a}(\vec{k}),\hat{a}^{\dagger}(\vec{k}')\right]=\delta(\vec{k}-\vec{k}')$$

$$\left[\hat{b}(\vec{k}),\hat{b}^{\dagger}(\vec{k}')\right]=\delta(\vec{k}-\vec{k}')$$

$$\hat{\pi}(x)=\partial_0\hat{\varphi}(x)$$

$$=\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(-i\omega_k)\hat{a}(\vec{k})e^{-i(\omega_k x^0-\vec{k}\cdot\vec{x})}+(i\omega_k)\hat{b}^{\dagger}(\vec{k})e^{i(\omega_k x^0-\vec{k}\cdot\vec{x})}$$

$$=-i\int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\left[\hat{a}(\vec{k})e^{-i(\omega_k x^0-\vec{k}\cdot\vec{x})}+\hat{b}^{\dagger}(\vec{k})e^{i(\omega_k x^0-\vec{k}\cdot\vec{x})}\right]$$

$$\hat{N}_{\hat{a}} = \int d^3k\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

$$\hat{N}_{\hat{b}} = \int d^3k\, \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})$$

$$\hat{H} = \int d^3k\, \omega_k \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right]$$

$$\hat{P} = \int d^3k\, \vec{k} \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right]$$

A complex field corresponds to a charged field. Particles and antiparticles have opposite charge. The total charge is found by subtracting the charge due to antiparticles from the charge due to particles. The charge operator

$$\hat{Q} = \int d^3k \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) \right]$$
$$= \hat{N}_{\hat{a}} - \hat{N}_{\hat{b}}$$

$$\left[\hat{\varphi}(x),\hat{\pi}(y)\right]=\left[\hat{\varphi}^{\dagger}(x),\hat{\pi}^{\dagger}(y)\right]=i\delta(\vec{x}-\vec{y})$$

$$\left[\hat{\varphi}(x),\hat{\varphi}^{\dagger}(y)\right]=i\Delta(x-y)$$ $\Rightarrow$ *propagator,*

$$\left[\hat{\varphi}(x),\hat{\varphi}^{\dagger}(y)\right]=\left[\hat{\varphi}(x),\hat{\varphi}(y)\right]=\left[\hat{\varphi}^{\dagger}(x),\hat{\varphi}^{\dagger}(y)\right]=0$$

Summary

The Klein-Gordon equation results from a straightforward substitution of the quantum mechanical operators for energy and momentum into the Einstein relation for energy, momentum, and mass from special relativity. This leads to inconsistencies such as negative probabilities and negative energy states. We can get around the inconsistencies by reinterpreting the equation. Rather than viewing it as a single particle wave equation, we instead apply it to a field that includes creation and annihilation operators similar to the harmonic oscillator of quantum mechanics. There is one difference, however, in that the creation and annihilation operators now create and destroy particles, rather than changing the energy level of an individual particle.

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$$
\begin{aligned}
\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= -(1-2M/r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2\mathrm{d}\Omega^2 \,, \\
\mathrm{d}\Omega^2 &= \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \,,
\end{aligned}
$$

Schwarzschild Solution

$$
A = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}\,; \qquad B = 1/A
$$

Reissner-Nordström solution (1916, 1918).

$$
r = r_{\pm} = M \pm \sqrt{M^2 - Q^2}
$$

In partial derivative form above equation changes into

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial\nu\psi) = 0$$

Where RN metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Also

$$\sqrt{g} = r^2\sin\theta$$

As for the scalar field the equation is Klein Gordon i.e

$$(\Box + m^2)\Phi = 0$$

A scalar field Φ propagates in RN spacetime The equation which govern through the evolution of massless scalar field is,

$$\Box\,\Phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Phi = 0$$