Comments on the spectrum of conformal field theories in d>2

Based on:
A. Belin, JdB, J. Kruthoff, B. Michel, E. Shaghoulian, M. Shyani, arXiv:1610.06186
JdB, J. Järkelä, K. Eski-Vakkuri, to appear

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The String Theory Universe

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AdS/CFT: there exist conformal field theories with a remarkable spectrum:

- Few light degrees of freedom
- No light degrees of freedom with spin>2
- Exactly solvable in the large N limit (no BH)
- Spectrum at high energies is random, complex, chaotic

A special role is played by orbifolds (d=2) and gauge theories (d>2). D=2 is special because of modular invariance.

Many questions such as:

- Which features of the spectrum are universal for generic CFTs and which for CFTs with a weakly coupled dual?
- What is the largest possible gap in d=2?

Will consider “large N” theories, i.e. theories with a parameter N which scales as

$$N \sim \left(\frac{\ell_{AdS}}{\ell_P}\right)^a$$

Also assume that density of states has a finite large N limit

$$\lim_{N\to\infty} \rho_N(E) < \infty$$

This is for example not true for N free bosons.

In d=2 Hartman, Keller and Stoica showed that if

$$\rho(\Delta) \leq e^{2\pi\Delta} \qquad \text{(sparseness)} \qquad E = \Delta - \frac{c}{12}$$

then

$$\rho(\Delta) \sim e^{2\pi\sqrt{\frac{c}{3}(\Delta - \frac{c}{12})}}, \qquad \Delta \geq \frac{c}{6}$$

This is much stronger than the Cardy formula, which holds only for $\Delta \gg c$.

This also guarantees that the partition function has the right form

$$\log Z(\beta) \sim \frac{c}{12}\beta, \qquad \beta > 2\pi \qquad \text{Thermal AdS}$$

$$\log Z(\beta) \sim \frac{c}{12}\frac{4\pi^2}{\beta}, \quad \beta < 2\pi \qquad \text{BTZ}$$

Thus, sparseness is sufficient to capture some important features of AdS/CFT.

Free S_N orbifold theories obey the sparseness condition but in general have light higher spin fields. To get to a standard AdS/CFT duality without higher spins need to turn on a further coupling constant.

Does any of this generalize to higher d?

Consider d dimensional CFT's on T^{d-1}

Peculiar theories: for example no state-operator correspondence.

What was important in d=2 was modular invariance and the Casimir energy $-c/24$

Question: are higher-dimensional CFT's compactified on tori automatically modular invariant? (will assume yes)

Casimir energy in d>2

Consider a torus

$$L_1 \times L_2 \times L_3 \times \ldots$$

The Casimir energy is a function $E_{\rm vac}(L_1, L_2, \ldots)$

Scale invariance: $E_{\rm vac}(\lambda L_1, \lambda L_2, \ldots) = \lambda^{-1} E_{\rm vac}(L_1, L_2, \ldots)$

Extensivity: $\lim_{L_k \to \infty} L_k^{-1} E_{\rm vac}(\lambda L_1, \lambda L_2, \ldots) = \text{finite}$

For example, free boson:

$$E_{\text{vac}}(L_1, L_2) = \sum_{n,m\in\mathbb{Z}} \left(\left(\frac{n}{L_1}\right)^2 + \left(\frac{m}{L_2}\right)^2 \right)^{1/2}$$

Structure:

$$E_{\text{vac}}(L_1, L_2) = -\frac{\epsilon_{\text{vac}} L_2}{L_1^2}(1 + f(L_1/L_2))$$

$$f(0) = 0 \qquad\qquad f(y \to \infty) = -1 + y^3$$

(Modular invariance: f is positive and monotonic)

Cardy formula for d>2 (E. Shaghoulian)

Consider the theory on $S^1_\beta \times S^1_{L_1} \times S^1_{L_2}$

Take $\beta \to 0, \qquad L_1 \to \infty$

First perspective: β is temperature

$$Z \sim \exp\left(\tilde{c}\frac{L_1 L_2}{\beta^2}\right)$$

(extensivity)

Second case: L_1 is temperature

$$Z \sim \exp\left(-L_1 E_{\text{vac}}(\beta, L_2)\right)$$

$$Z \sim \exp\left(\epsilon_{\text{vac}} L_1 \frac{L_2}{\beta^2}\right)$$

(Projection onto the ground state)

$$\Longrightarrow \tilde{c} = \epsilon_{vac}$$

Extract density of states

$$\log \rho(E) = \frac{d}{(d-1)^{\frac{d-1}{d}}} (\epsilon_{\text{vac}} L_1 L_2 \ldots)^{\frac{1}{d}} E^{\frac{d-1}{d}}$$

General statement for large E. What is needed in order to have agreement with AdS/CFT?

Relevant solutions

$$ds^2_{\rm pp} = r^2 dx_0^2 + \frac{dr^2}{r^2} + r^2 d\phi_i d\phi^i \,,$$

$$ds^2_{\rm bb} = r^2 \left(1 - (r_h/r)^d\right) dx_0^2 + \frac{dr^2}{r^2\left(1 - (r_h/r)^d\right)} + r^2 d\phi_i d\phi^i \,,$$

$$ds^2_{\rm sol,k} = r^2 dx_0^2 + \frac{dr^2}{r^2\left(1 - (r_{0,k}/r)^d\right)} + r^2 \left(1 - (r_{0,k}/r)^d\right) d\phi_k^2 + r^2 d\phi_j d\phi^j \,,$$

$$r_h = \frac{4\pi}{d\beta}\,, \qquad r_{0,k} = \frac{4\pi}{dL_k}$$

Free energies:

$$V_{d-1} = L_1 \dots L_{d-1}$$

$$F_{\rm bb} = -\frac{r_h^d V_{d-1}}{16\pi G}\,, \qquad F_{\rm sol,k} = -\frac{r_{0,k}^d V_{d-1}}{16\pi G}\,, \qquad F_{\rm pp} = 0$$

Relevant solutions — singular — Time pinches off

Space pinches off

$$ds^2_{\rm pp} = r^2 dx_0^2 + \frac{dr^2}{r^2} + r^2 d\phi_i d\phi^i\,,$$

$$ds^2_{\rm bb} = r^2\left(1-(r_h/r)^d\right)dx_0^2 + \frac{dr^2}{r^2\left(1-(r_h/r)^d\right)} + r^2 d\phi_i d\phi^i\,,$$

$$ds^2_{\rm sol,k} = r^2 dx_0^2 + \frac{dr^2}{r^2\left(1-(r_{0,k}/r)^d\right)} + r^2\left(1-(r_{0,k}/r)^d\right)d\phi_k^2 + r^2 d\phi_j d\phi^j\,,$$

$$r_h = \frac{4\pi}{d\beta}\,, \qquad r_{0,k} = \frac{4\pi}{dL_k}$$

Free energies:

$$V_{d-1} = L_1 \dots L_{d-1}$$

$$F_{\rm bb} = -\frac{r_h^d V_{d-1}}{16\pi G}\,, \qquad F_{\rm sol,k} = -\frac{r_{0,k}^d V_{d-1}}{16\pi G}\,, \qquad F_{\rm pp} = 0$$

We get a series of sharp (quantum) phase transitions:

$$\beta < L_1, L_2 \qquad F = -\epsilon_{\rm vac} \frac{L_1 L_2}{\beta^3}$$

$$L_1 < \beta, L_2 \qquad F = -\epsilon_{\rm vac} \frac{\beta L_2}{L_1^3}$$

$$L_2 < \beta, L_1 \qquad F = -\epsilon_{\rm vac} \frac{\beta L_1}{L_2^3}$$

Suppose L_1 is large and we view L_1 as time.

Then

$$Z = e^{-L_1 E_{\rm vac}(\beta, L_2)} \sum_E \rho(E) e^{-L_1 (E - E_{\rm vac}(\beta, L_2))}$$

Recall

$$E_{\rm vac}(\beta, L_2) = -\frac{\epsilon_{\rm vac} L_2}{\beta^2}(1 + f(\beta/L_2))$$

This can only agree with ads/cft if

$$f = 0, \qquad \beta < L_2$$

new condition (trivial in d=2)

and

$$\rho(E) < e^{L_1 (E - E_{\rm vac}(\beta, L_2))}$$

sparseness

Upshot: phase structure is the same as that of ads/cft if and only if the partition function is vacuum dominated in all but the shortest channel.

Is sparseness enough to guarantee this? Yes but

$$\rho(E) < e^{L_k(E-E_{\mathrm{vac}}(L_1,L_2,\ldots))}$$

must hold for all k and all E. Stronger than in d=2.

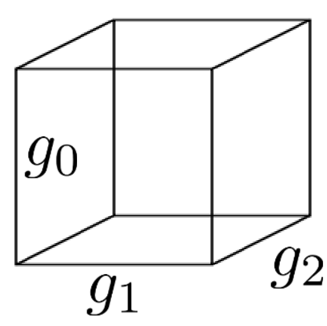
Orbifolds in d>2

By analogy with the 2d case, can take a seed theory C and try to build a theory

$$\frac{C^{\otimes N}}{S_N}$$

On a torus, modular invariance will then require to sum over twisted sectors

$$Z = \frac{1}{N!} \sum_{\substack{g_0, g_1, g_2 \in S_N \\ g_i g_j = g_j g_i \forall i,j}}$$



Sparseness condition a la 2d still holds $\rho(\Delta) \leq e^{2\pi\Delta}$

But this is not sufficient anymore to get the same thermodynamics as in ads/cft.

To get f=0 in the Casimir energy we need f=0 already in the seed theory C…

- Do these orbifold theories really exist as proper quantum field theories?
- What replaces the state-operator correspondence?
- Are there rules to combine local, line and surface operators and restrict their correlators?
- For any quantum field theory T with a global symmetry G, does T/G exist?

A different story: spectrum of conformal field theory on the hyperbolic plane.

Due to Casini-Huerta-Myers this is directly related to the entanglement spectrum of a CFT for a spherical region. The full spectrum carries much more information than just the entanglement entropy.

The spectrum of a conformal field theory on the hyperbolic plane can in principle be obtained through an inverse Laplace transformation of the Renyi's S_n with respect to n.

For example, in d=2:

$$\rho(E) = \theta(E - E_c) I_0(2\sqrt{E_c(E - E_c)})$$

$$E_c = \frac{c}{6} \log\left(\frac{\ell}{\epsilon}\right)$$

This is compatible with the Cardy formula though the central charge gets replaced by a divergent quantity.

In higher dimensions, we find

$$\rho(E) \sim \exp\left(cE_c^{\frac{1}{d}}(E - E_c)^{\frac{d-1}{d}}\right)$$

Related to the "variance" in entanglement entropy

Related to the entanglement entropy

Interestingly, $\Delta S_{EE}^2 \sim S_{EE}$

Some remarks/questions:

- Can get the variance from the expansion of the Renyi near n=1.
- Does this variance have other interesting applications?
- Is there a simple reason why variance~entanglement? Law of large numbers?
- Can the Casimir energy be computed in a different way from first principles?
- Generalizations to other spatial manifolds?
- Does the equation extend to low energies?