BEC-BCS Crossover and Inhomogeneous Phases in Dense Matter

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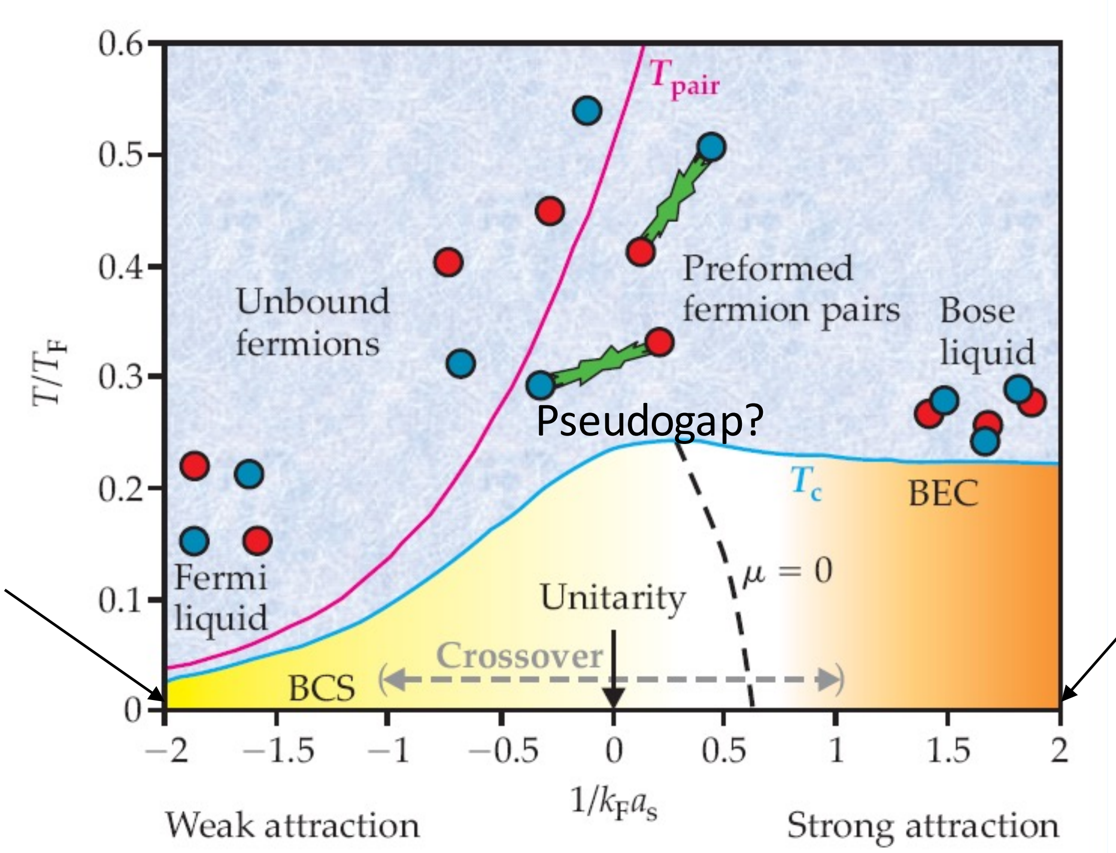
Talk given at workshop "quarkyonic, from theory to experiment", Central China Normal University

2016. 10. 24

Outline

- BEC-BCS crossover in cold Fermi gases
- BEC-BCS crossover in dense QCD-like theories
- Crystalline (LOFF) color superconductivity
- Summary and outlook

Phase diagram of BEC-BCS crossover (3D & s-wave)



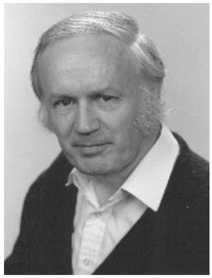
High T:
virial expansion

Low-density expansion for fermions

Low-density expansion for composite bosons

From C. A. R. Sa de Melo, Physics Today (2008)

1980



BCS

$$\Psi_0 = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}})|0\rangle$$

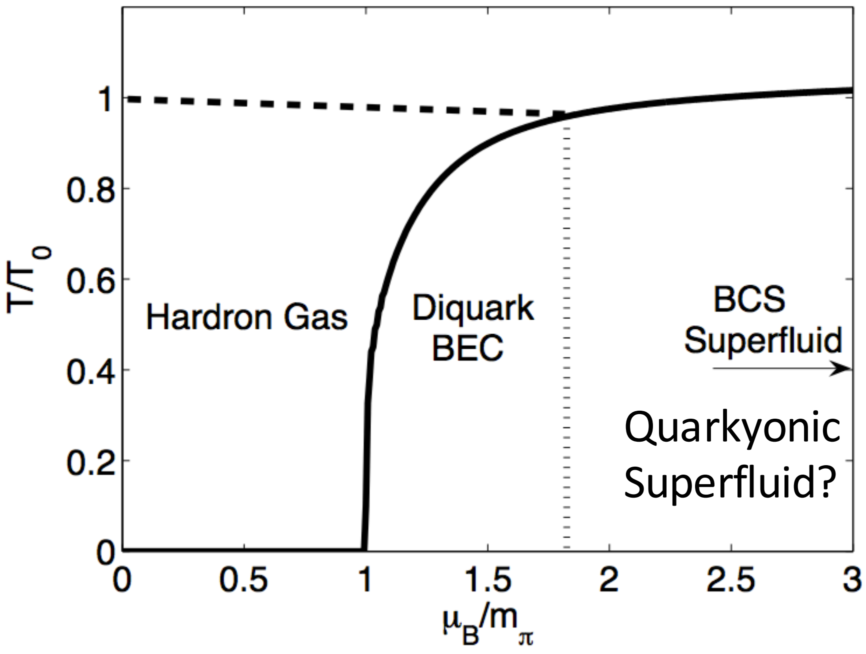
smooth crossover

$$v_k = \frac{N_B^{1/2}\phi_k}{\left(1 + N_B\phi_k^2\right)^{1/2}}$$

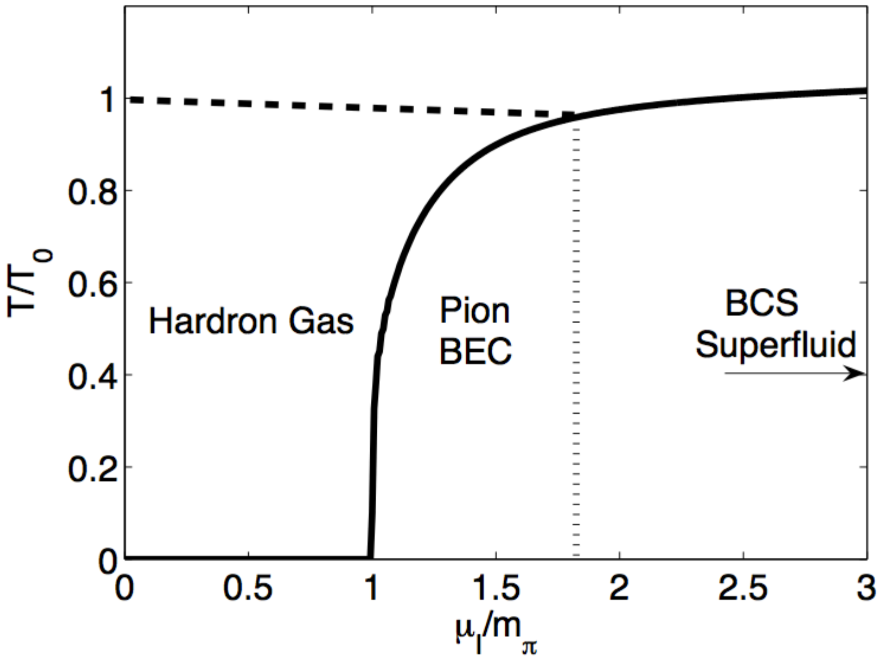
BEC

$$\Psi_0 = \exp\left(N_B^{1/2} \sum_{\mathbf{k}} \phi_{\mathbf{k}} c^{+}_{\mathbf{k}} c^{+}_{-\mathbf{k}}\right) \ |0\rangle$$

QCD at finite density: Evolution from hadron matter to quark matter via BEC-BCS crossover?



Two-color QCD at finite baryon density



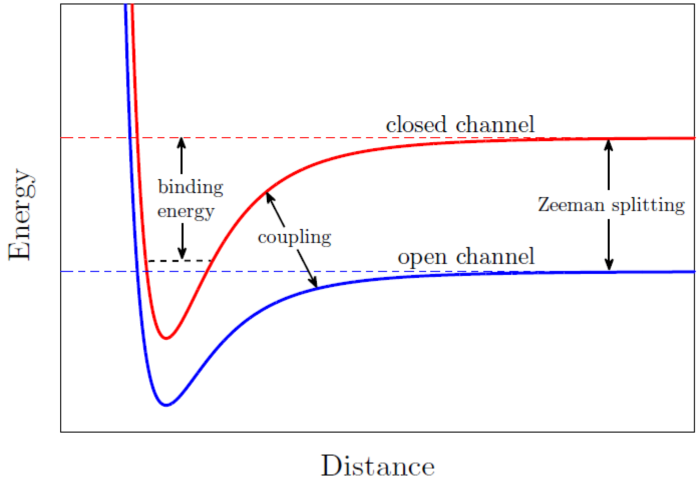
QCD at finite isospin density

H. Abuki, T. Hatsuda, K. Itakura, PRD 2002;
G. Sun, LH, P. Zhuang, PRD 2007;
and many others

Lattice simulation works!

Realizing strongly interacting Fermi gases in cold atoms: Magnetic field tuned (s-wave) Feshbach resonance

Alkali atoms (^{6}Li, ^{40}K) in a magnetic field:
Coupled multi-channel scattering



Low-energy scattering in open channel:
Scattering energy E << Zeeman splitting

$$a = a_{\mathrm{bg}}\left(1 - \frac{B_\Delta}{B - B_0}\right)$$

$$r_{\mathrm{e}} = -\frac{2}{m a_{\mathrm{bg}} \gamma B_\Delta}$$

Unitary Fermi Gas: A strongly coupled Fermi gas at infinite a
Expansion dynamics similar to QGP: another nearly perfect liquid
J. E. Thomas et al., Science (2002); Quark Matter 2009

Dilute Neutron Matter: A nearly unitary Fermi gas
A. Gezerlis & J. Carlson, PRC 2010

Broad Feshbach resonance: large scattering length & negligible effective range

⇒ **Universal many-body physics: A terrific gift from heaven for theorists!**

Physical properties depend on two parameters: $\frac{1}{k_{\mathrm{F}}a}$ & $\frac{T}{T_{\mathrm{F}}}$

Low-energy effective theory – contact interaction (one-channel model)

$$H = \int d\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \psi^{\dagger}_{\sigma}(\mathbf{r})\left(-\frac{\nabla^2}{2m} - \mu\right)\psi_{\sigma}(\mathbf{r}) - U \int d\mathbf{r}\, \psi^{\dagger}_{\uparrow}(\mathbf{r})\psi^{\dagger}_{\downarrow}(\mathbf{r})\psi_{\downarrow}(\mathbf{r})\psi_{\uparrow}(\mathbf{r})$$

Renormalization of bare coupling U in terms of physical scattering length a:
Computing the scattering amplitude & Matching known result

Small $k_{\mathrm{F}}a$ & low T: perturbation theory (low-density expansion)

Huang, Lee & Yang (1957) Galitskii (1958) Bishop (1973)

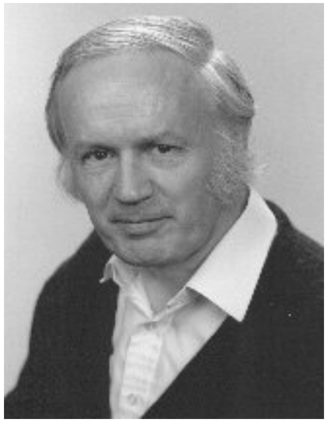
High T & arbitrary coupling: virial expansion – n-body problems are hard!

second virial expansion: Beth & Uhlenbeck (1937) Mueller & Ho (2004)

third virial expansion: Hu, Liu & Drummond (2009, 2010)

Large $k_{\mathrm{F}}a$ & low T: Less is known! QMC, T-matrix, epsilon expansion…

Theory of the BEC-BCS crossover



At T=0, BCS gap equation + number equation is qualitatively correct to describe the BEC-BCS crossover.

BCS-Leggett mean-field theory (1980)

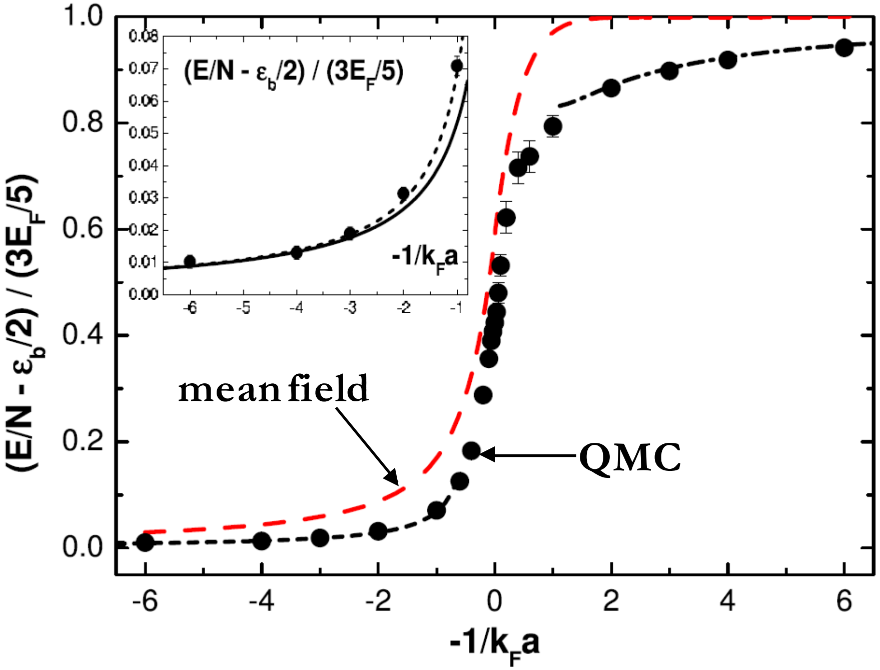
Above Tc, we must include the pair fluctuations to get correct superfluid transition temperature!

Nozieres–Schmitt-Rink (NSR) theory (1985)

How to connect the above two different approaches?

Experiments & QMC: Pair fluctuations are important even at T=0, especially for 2D.

3D



mean field

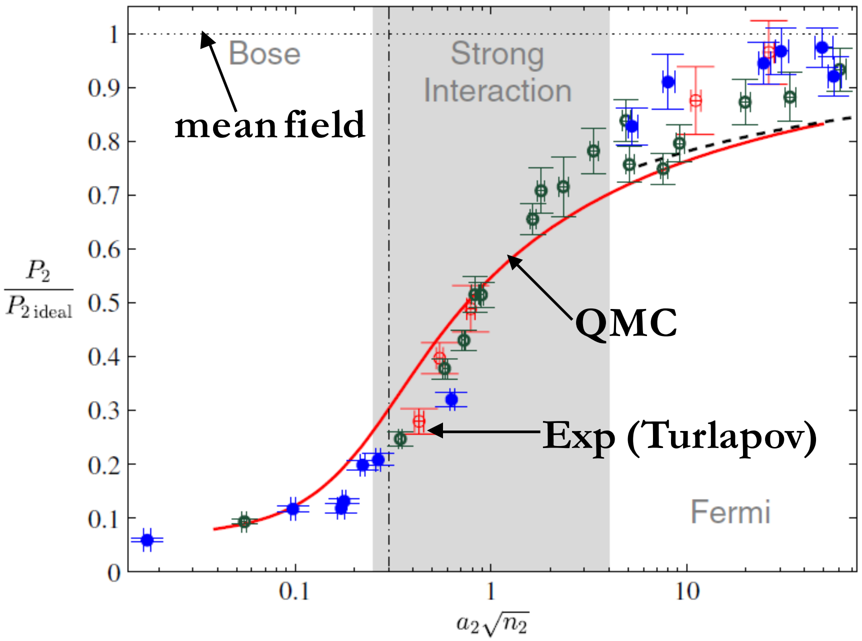
QMC

From Giorgini, Pitaevskii & Stringari, Rev. Mod. Phys. (2008)

QMC: Astrakharchik, Boronat, Casulleras, and Giorgini (2004)

See also Carlson *et al.* (2003),
Lobo *et al.* (2006),
Forbes *et al.* (2011), …

2D



mean field

QMC

Exp (Turlapov)

From Makhalov, Martiyanov & Turlapov, PRL (2014)

2D mean-field theory: Randeria *et al.* (1989)

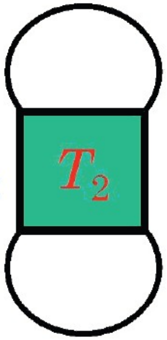
Experiments: Turlapov *et al.* (2014)
Thomas *et al.* (2015)

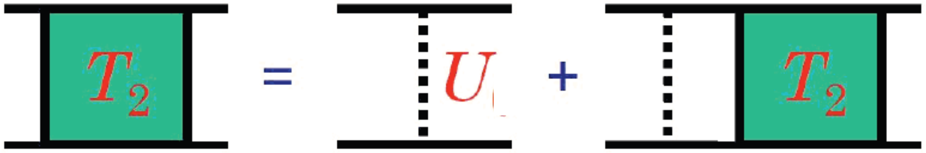
QMC: Bertaina & Giorgini (2011)
Shi, Chiesa & Zhang (2015)
Anderson & Drut (2015)

Beyond-mean-field theory: T-matrix approaches

Above Tc: $\Omega = \Omega^{(1)} + \Omega^{(2)} + \Omega^{(3)} + \cdots$

$\Omega^{(2)} =$





Different T-matrix approaches: Different choices of fermion Green's function

Below Tc: No clear classification of n-body contributions!

Because of condensation: fermion Green's function is a 2x2 matrix

G_0G_0: Ohashi & Griffin (2003) Pieri, Pisani & Strinati (2004)
Hu, Liu & Drummond (2006) Diener, Sensarma & Randeria (2008)
gapless approximation, but different treatments of number equation!

G_0G: Chen & Levin *et al.* (2004) recover mean field at T=0
pseudogap theory: extended mean-field theory at finite T

GG: Haussmann, Rantner, Cerrito, & Zwerger (2007)
conseving/Luttinger-Ward approximation: but still need truncation

Toward an analytical & quantitative theory at T<Tc

Considerations: a quantitative theory should recover known limits.

(1)BCS limit: Fermi liquid correction – Huang & Yang (1957) Galitskii (1958)

(2)BEC limit: weakly interacting Bose condensate – Bogoliubov (1947)
Lee, Huang & Yang (1957)

composite boson scattering length – Petrov, Salomon & Shlyapnikov (2004)

(3)High-T limit: second virial expansion – Beth & Uhlenbeck (1937)

Who is the winner under these terms?

Surprisingly, it is a G_0G_0 theory with a careful treatment of the number equation!

Gaussian Pair Fluctuation (GPF) Theory

Diagrammatic version: Hu, Liu & Drummond, EPL 2006

Functional path integral version: Diener, Sensarma & Randeria, PRA 2008

2D: mean-field theory fails to describe the strong coupling (BEC) limit
GPF is a must! LH, Lü, Cao, Hu & Liu, PRA 2015

NJL model for meson-quark crossover: P. Zhuang et al., NPA 1994

Gaussian Pair Fluctuation (GPF) Theory: Generalized NSR

Path integral formulation: My favorite!

Sa de Melo, Randeria & Engelbrent (1993, 1997)

$$\mathcal{Z} = \int [d\psi][d\bar{\psi}] \exp\{-\mathcal{S}[\psi,\bar{\psi}]\} \qquad \mathcal{S}[\psi,\bar{\psi}] = \int dx\, [\bar{\psi}\partial_\tau\psi + H(\psi,\bar{\psi})]$$

$$\xrightarrow{\text{Strotonovich-Hubbard}} \quad \mathcal{Z} = \int [d\Phi][d\Phi^*] \exp\Big\{-\mathcal{S}_{\rm eff}[\Phi,\Phi^*]\Big\}$$

$$\mathcal{S}_{\rm eff}[\Phi,\Phi^*] = \frac{1}{U}\int dx\, |\Phi(x)|^2 - \mathrm{Tr}\ln[\mathbf{G}^{-1}(x,x')]$$

$$\mathbf{G}^{-1}(x,x') = \begin{pmatrix} -\partial_\tau - \mathcal{H}_0 & \Phi(x) \\ \Phi^*(x) & -\partial_\tau + \mathcal{H}_0 \end{pmatrix} \delta(x-x')$$

Mean field + fluctuations: $\Phi(x) = \Delta + \phi(x)$

$$\mathcal{S}_{\rm eff}[\Phi,\Phi^*] = \mathcal{S}_{\rm MF} + \mathcal{S}_{\rm GF}[\phi,\phi^*] + \cdots$$

mean field

Gaussian pair fluctuations

Goldstone mode fluctuation at T<Tc

Fermi-liquid correction at T=0 and weak coupling

BCS-Leggett mean-field theory

$$S_{\text{eff}}[\Phi, \Phi^*] \simeq S_{\text{MF}}$$

T=0: $$\Omega_{\text{MF}} = \frac{\Delta^2}{U} + \frac{1}{V}\sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}})$$

⇨ **Gap Equation:** $$\frac{1}{U} = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}}$$

Number Equation: $$n = \frac{1}{V}\sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

BEC limit of mean-field theory at T=0: 3D vs 2D

Ginzburg-Landau expansion near the vacuum-BEC transition $\mu_c = -\frac{\varepsilon_\mathrm{B}}{2}$

$$\Omega_\mathrm{GL}[\Delta] = \int dx \left[\Delta^* \left(a\frac{\partial}{\partial\tau} - b\frac{\boldsymbol{\nabla}^2}{4m} - c\right)\Delta + \frac{d}{2}|\Delta|^4\right]$$

In terms of molecule field: Gross-Pitaevskii free energy

$$\Omega_\mathrm{GP}[\varphi] = \int dx \left[\varphi^* \left(\frac{\partial}{\partial\tau} - \frac{\boldsymbol{\nabla}^2}{2m_\mathrm{B}} - \mu_\mathrm{B}\right)\varphi + \frac{g_\mathrm{B}}{2}|\varphi|^4\right]$$

$$\mu_\mathrm{B} = 2\mu + \varepsilon_\mathrm{B}$$
$$m_\mathrm{B} = 2m$$

Mean-field description of composite boson interaction
= Born approximation for four-body scattering

3D: The composite boson coupling is qualitatively correct.

$g_\mathrm{B} = \frac{4\pi a_\mathrm{B}}{m_\mathrm{B}}$ with $a_\mathrm{B} = 2a$ exact four-body result $a_\mathrm{B} = 0.6a$

2D: The composite boson coupling is qualitatively wrong!

$g_\mathrm{B} = \frac{4\pi}{m}$ ➡ **NOT weakly interacting 2D Bose condensate**
Logarithmic dependence on energy missing!

Note: four-body correlations are included even in T=0 mean field!

Gaussian fluctuations: Collective modes

$$\mathcal{S}_{\mathrm{GF}}[\phi,\phi^*] = \frac{1}{2}\sum_{Q}\begin{pmatrix} \phi^*(Q) & \phi(-Q) \end{pmatrix}\mathbf{M}(Q)\begin{pmatrix} \phi(Q) \\ \phi^*(-Q) \end{pmatrix}$$

$$\mathbf{M}(Q) = \begin{pmatrix} \mathbf{M}_{11}(Q) & \mathbf{M}_{12}(Q) \\ \mathbf{M}_{21}(Q) & \mathbf{M}_{22}(Q) \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{-+}(Q) & \mathbf{M}_{--}(Q) \\ \mathbf{M}_{++}(Q) & \mathbf{M}_{+-}(Q) \end{pmatrix}$$

$$\mathbf{M}_{11}(iq_l,\mathbf{q}) = \mathbf{M}_{22}(-iq_l,\mathbf{q}) = \frac{1}{U} + \frac{1}{\beta V}\sum_{K}[\mathcal{G}_{11}(K+Q)\mathcal{G}_{22}(K)]$$

$$\mathbf{M}_{12}(iq_l,\mathbf{q}) = \mathbf{M}_{21}(iq_l,\mathbf{q}) = \frac{1}{\beta V}\sum_{K}[\mathcal{G}_{12}(K+Q)\mathcal{G}_{12}(K)]\,.$$

⇨ Goldstone mode in the superfluid state

Gaussian Pair Fluctuation (GPF) theory

$$S_{\text{eff}}[\Phi, \Phi^*] \simeq S_{\text{MF}} + S_{\text{GF}}[\phi, \phi^*]$$

$$\Omega = \Omega_{\text{MF}} + \Omega_{\text{GF}}$$

$$\begin{aligned}\Omega_{\text{GF}} \;=\;& \frac{1}{2}\frac{1}{\beta}\sum_{q_l}\frac{1}{V}\sum_{\mathbf{q}}\Bigg\{\ln\left[\mathbf{M}_{11}(iq_l, \mathbf{q})\right]e^{iq_l 0^+} + \ln\left[\mathbf{M}_{22}(iq_l, \mathbf{q})\right]e^{-iq_l 0^+} \\ & + \ln\left[1 - \frac{\mathbf{M}_{12}^2(iq_l, \mathbf{q})}{\mathbf{M}_{11}(iq_l, \mathbf{q})\mathbf{M}_{22}(iq_l, \mathbf{q})}\right]\Bigg\}.\end{aligned}$$

The crucial element of the GPF theory is that the relation between the order parameter Δ and the chemical potential μ, $\Delta = \Delta(\mu)$, is determined by the extreme of the mean-field grand potential Ω_{MF} rather than the full grand potential Ω_{GPF}. We therefore determine $\Delta(\mu)$ from the following extreme condition

$$\frac{\partial \Omega_{\text{MF}}(\mu, \Delta)}{\partial \Delta} = 0 \Rightarrow \frac{1}{U} = \frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2E_{\mathbf{k}}}.$$

Gaussian Pair Fluctuation (GPF) Theory: Equation of State

$$\Omega(\mu) = \Omega_{\mathrm{MF}}(\mu) + \Omega_{\mathrm{GF}}(\mu)$$

$$\Longrightarrow \quad n(\mu) = n_{\mathrm{MF}}(\mu) + n_{\mathrm{GF}}(\mu)$$

$$n_{\mathrm{GF}}(\mu) = -\frac{d\Omega_{\mathrm{GF}}(\mu)}{d\mu} \qquad \frac{d\Omega_{\mathrm{GF}}(\mu)}{d\mu} = \frac{\partial\Omega_{\mathrm{GF}}}{\partial\mu} + \frac{\partial\Omega_{\mathrm{GF}}}{\partial\Delta}\frac{d\Delta}{d\mu}$$

thermodynamic consistency
correct four-body contribution

Note: the order parameter as a function of the chemical potential, $\Delta(\mu)$, is determined by minimizing the mean-field grand potential!

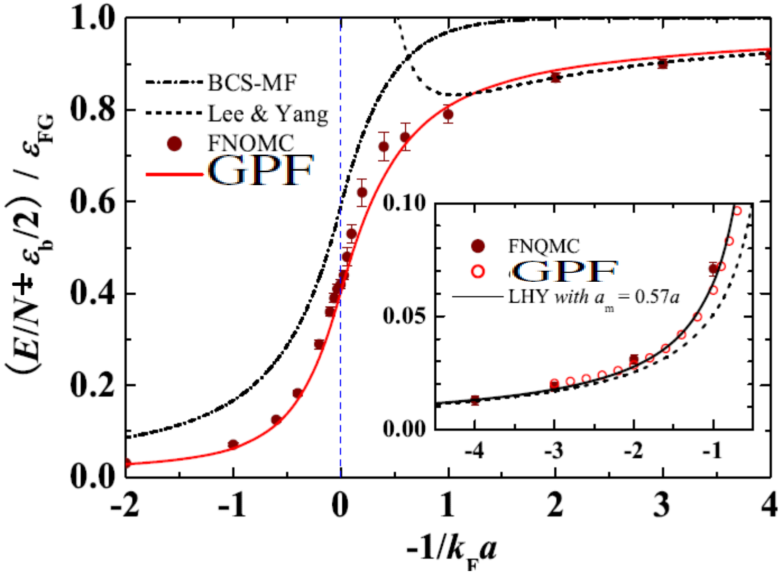
Why: (1) Guarantee Goldstone's theorem: gapless approximation

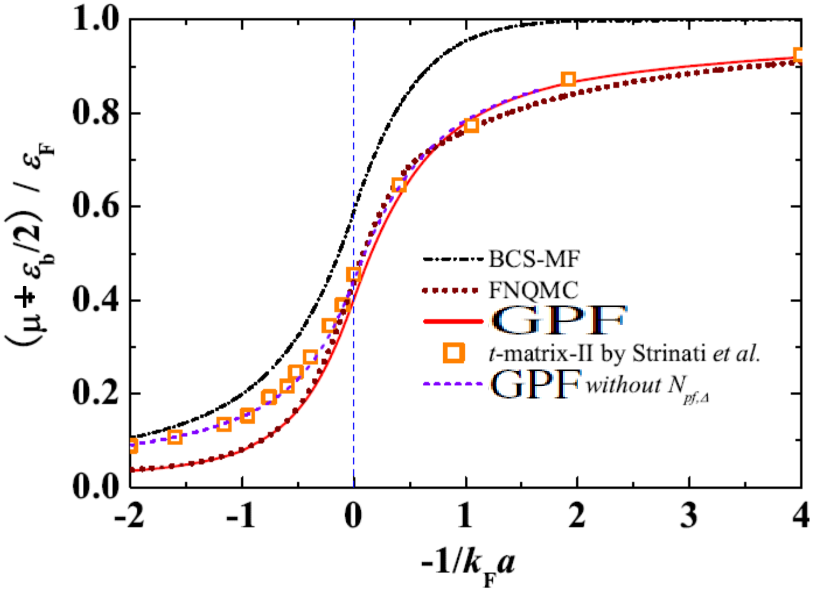
(2) Maintain the Silver Blaze property: vacuum state should have vanishing pressure and density! $\mu < -\frac{\varepsilon_{\mathrm{B}}}{2}$

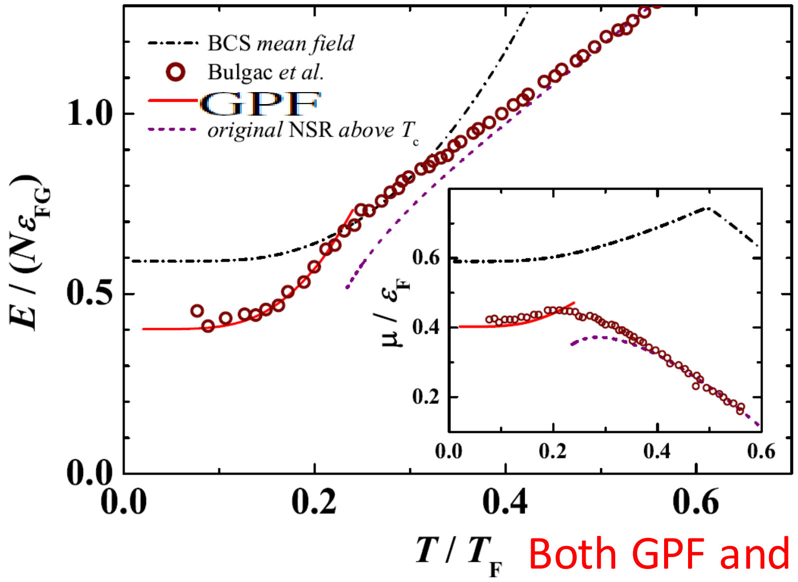
analogy in QCD: critical baryon & isospin chemical potential

Gaussian Pair Fluctuation (GPF) Theory: Results for 3D

Hu, Liu & Drummond, EPL (2006) & Nat. Phys (2007)







Both GPF and NSR have problems near Tc!

Composite boson scattering length:

$$a_B = 0.57a$$

Exact result: $a_B = 0.6a$

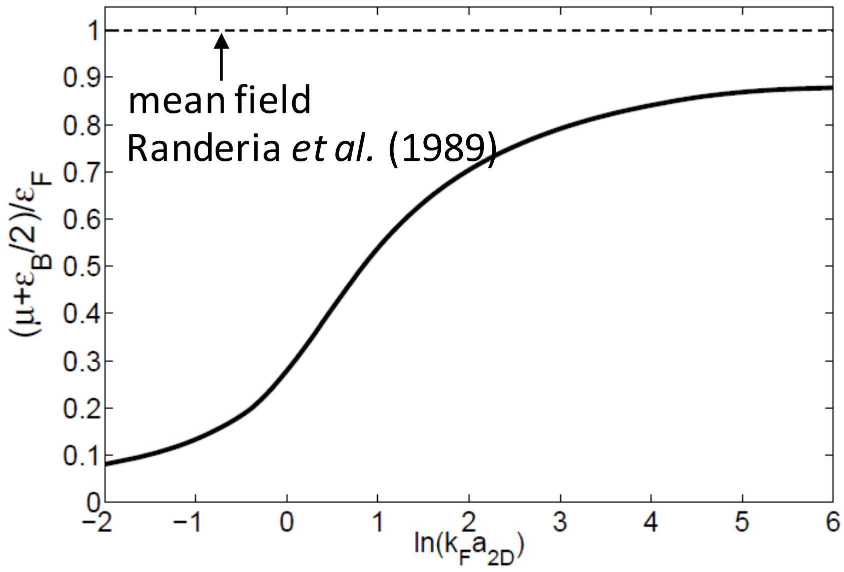
Bertsch parameter: $\xi = 0.401$

Latest QMC: $\xi = 0.37 - 0.38$

Stefano *et al.* (2011); Carlson et al. (2011)

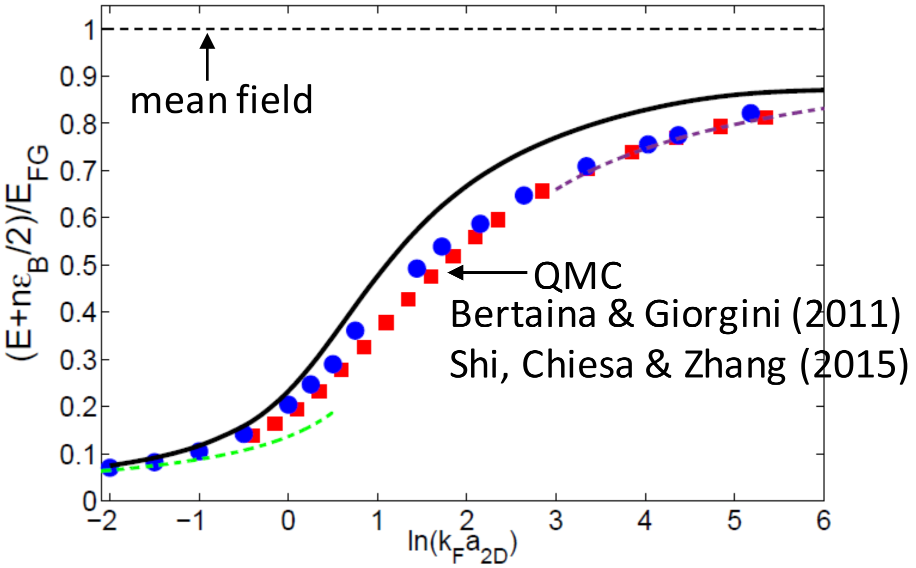
Gaussian Pair Fluctuation (GPF) Theory: Results for 2D

LH, Lü, Cao, Hu & Liu, PRA (2015)



mean field

Randeria *et al.* (1989)

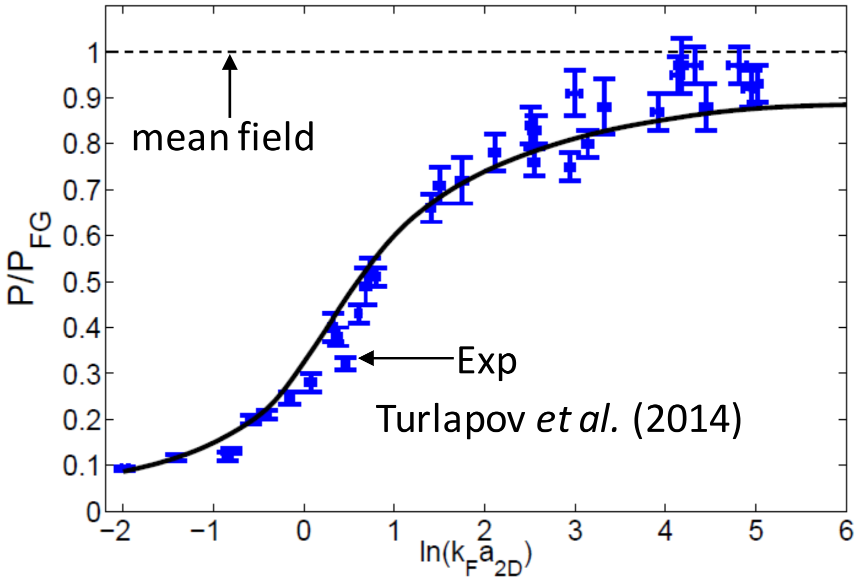


mean field

QMC

Bertaina & Giorgini (2011)

Shi, Chiesa & Zhang (2015)



mean field

Exp

Turlapov *et al.* (2014)

Unlike 3D, in 2D we must consider (quantum) pair fluctuations to get correct BEC limit!

More to do: finite T (BKT transition) imbalanced superfluidity…

Composite boson scattering length in 2D

Compare the grand canonical EOS with known EOS of weakly interacting 2D Bose condensate near the vacuum-BEC transition.

Bogoliubov EOS of 2D Bose gas: $\Omega(\mu_{\mathrm{B}}) = -\frac{m_{\mathrm{B}}\mu_{\mathrm{B}}^2}{8\pi}\left[\ln\left(\frac{4}{\mu_{\mathrm{B}} m_{\mathrm{B}} a_{\mathrm{B}}^2 e^{2\gamma+1}}\right) + \frac{1}{2}\right]$

Mean-field theory in 2D: $\Omega_{\mathrm{MF}}(\mu_{\mathrm{B}}) = -\frac{m_{\mathrm{B}}}{16\pi}\mu_{\mathrm{B}}^2$ **gives nothing!**

GPF theory in 2D: $\Omega(\mu_{\mathrm{B}}) = -\frac{m_{\mathrm{B}}\mu_{\mathrm{B}}^2}{8\pi}\left[f(\zeta) + \frac{1}{2}\right]$ $\zeta = \mu_{\mathrm{B}}/\varepsilon_{\mathrm{B}}$

The function $f(\zeta)$ is complicated but, fortunately, can be handled analytically in the limit $\zeta \rightarrow 0$. In this limit, we can show that it behaves asymptotically as

$$f(\zeta) \sim -\ln\zeta + \lambda \qquad \lambda \simeq -0.54$$

➡ $a_{\mathrm{B}} \simeq 0.56 a_{2\mathrm{D}}$ agrees with Petrov's four-body result!

Petrov, Baranov & Shlyapnikov, PRA (2003)

LH, Lü, Cao, Hu & Liu, PRA (2015)

See also Salasnich & Tiogo, PRA (2015): pole approx. + dimensional reg.

QCD at high baryon density: color superconductivity

At sufficiently high baryon density and low temperature, QCD matter is a degenerate Fermi gas of quarks (u, d, s)

⇒ *Dense quark matter (in compact stars?)*

Attractive interactions in certain diquark channels (QCD interaction)

⇒ *BCS instability of quark Fermi seas*

⇒ *Superconductivity of quarks*
or "Color Superconductivity"

M. Alford, K. Rajagopal, T. Schaefer, and A. Schmitt,
Rev. Mod. Phys. 80, 1455 (2008)

What occurs if we lower the density?

- At very high density: weakly coupled CSC
- There should be a transition or crossover to the nuclear matter
- In between: A strongly coupled color superconductor?

  BEC-BCS crossover?

  However, real QCD at finite baryon density is hard: diquarks are not colorless object, confinement effect is important but hard…
- Consider some QCD-like theories:

  QCD at finite isospin density (Son & Stephanov PRL 2001)

  Two-color QCD at finite density (Kogut et al., NPB2000)

  Lattice simulation works!

BCS-BEC crossover in dense QCD-like theories?

- **Two-color QCD:**
  Baryons are scalar diquarks; these diquarks are colorless bosons, with their masses degenerate with pions

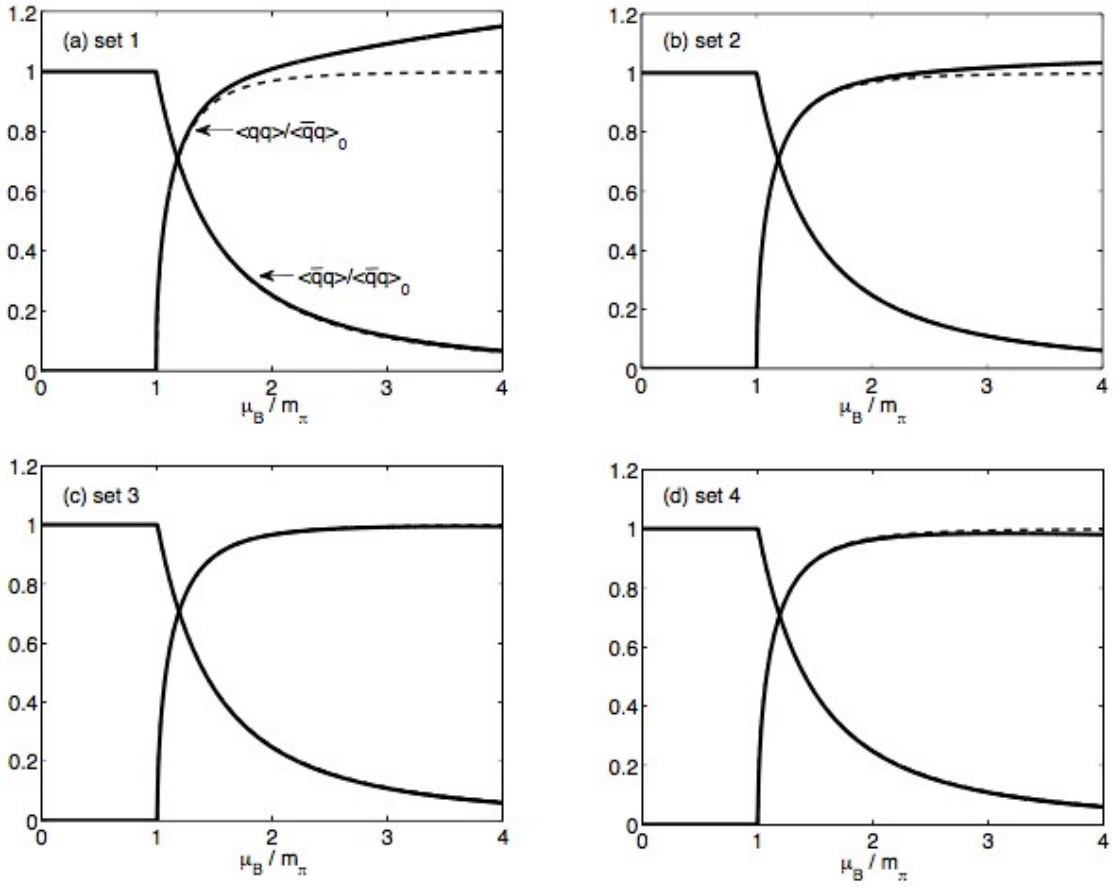
- **Two-color QCD at finite baryon density:**
  High density: weakly-coupled BCS superfluid
  Low density: weakly interacting Bose condensate of diquarks

- **QCD at finite isospin density:**
  High density: weakly-coupled BCS superfluid
  Low density: weakly interacting Bose condensate of pions

Theory for BCS-BEC crossover: Nambu-Jona-Lasino model, Quark-Meson model,…

NJL model description of two-color QCD

$$\mathcal{L}_{\mathrm{NJL}} = \bar{q}(i\gamma^{\mu}\partial_{\mu} - m_0)q + G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2 + (\bar{q}i\gamma_5\tau_2 t_2 q_c)(\bar{q}_c i\gamma_5\tau_2 t_2 q)\right]$$



C. Ratti & W. Weise, PRD 2004;
G. Sun, LH, P. Zhuang, PRD 2007;
T. Brauner, K. Fukushima, & Y. Hidaka, PRD 2009;
LH, PRD 2010

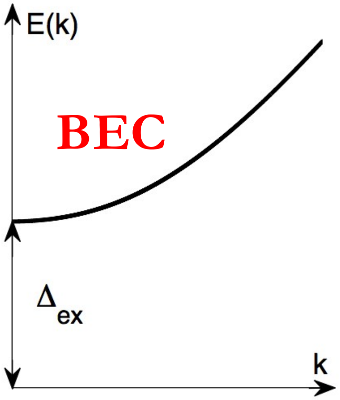
Near $\mu_B = m_\pi$: (low density limit)

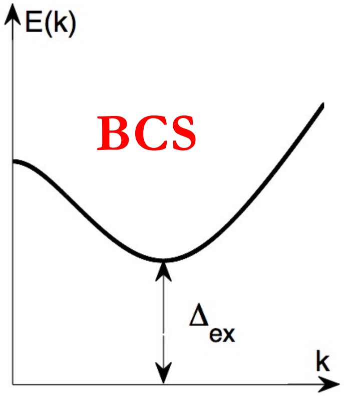
Weakly repulsive Bose condensate of diquarks

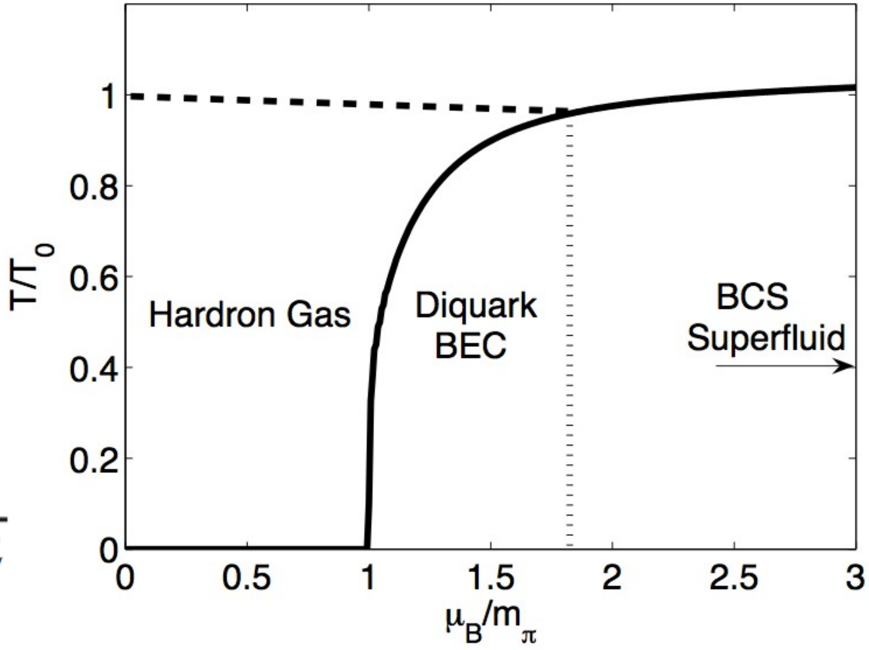
Diquark-diquark scattering length: $a_{dd} = \dfrac{m_\pi}{16\pi f_\pi^2}$

LH, PRD 2010

Higher density: BEC-BCS crossover

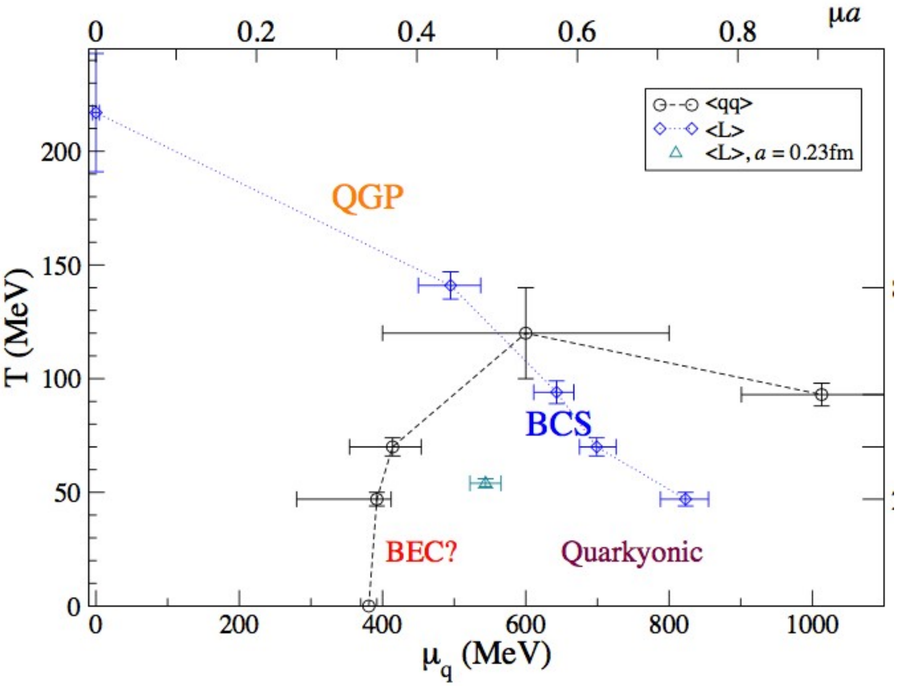






$$E_{\mathbf{k}}^{\pm} = \sqrt{(E_{\mathbf{k}} \pm \mu_B/2)^2 + |\Delta|^2}$$

Results from lattice QCD simulations



Two-color QCD phase diagram
at finite baryon density

S. Hands et al., PRD 2014

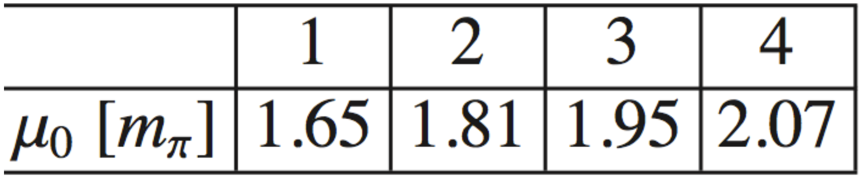
Crossover chemical potential

Latest lattice result:

about $1.76\, m_\pi$

A. A. Nikolaev et al., 1605.04090

NJL model prediction:

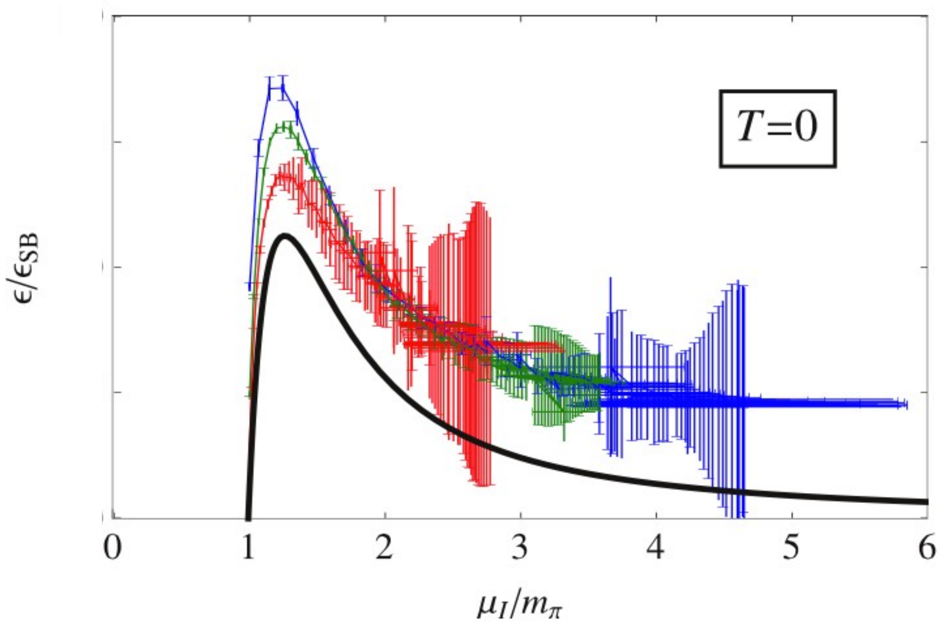


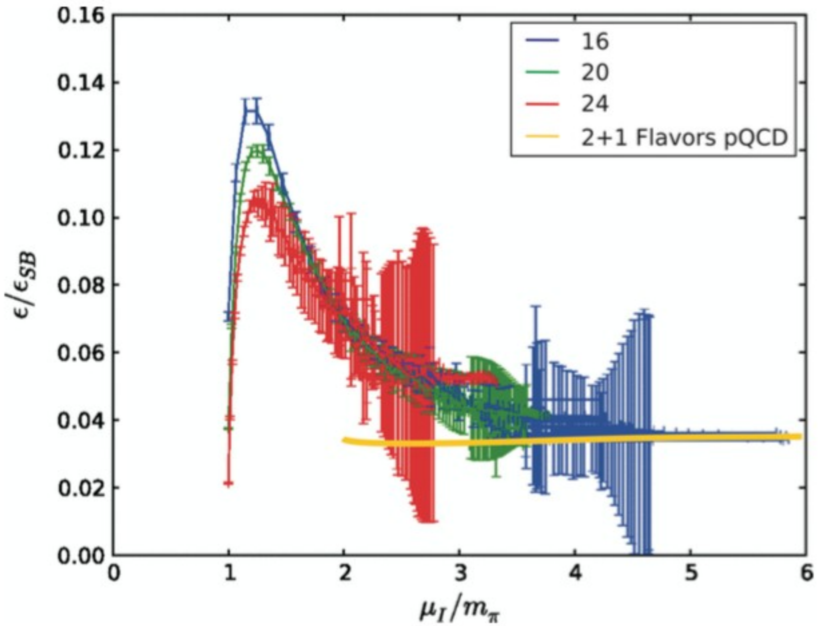
	1	2	3	4
μ_0 [m_π]	1.65	1.81	1.95	2.07

LH, PRD 2010

Isospin density: Theory vs Lattice QCD

Lattice result: W. Detmold et al., PRD 2012





NJL model: can describe the peak near the vacuum-BEC transition

T. Xia, LH &P. Zhuang, PRD 2013

Not good at high density because of the cutoff effect

pQCD: result agrees with pQCD at high isospin density

T. Graf et al., PRD 2016

Not good at low and at moderate density

Quark-Meson model?

Crystalline (LOFF) color superconductivity

In realistic case, quark cooper pairing occurs under stress because:
(1)Strange quark has a much larger mass than light quarks;
(2)Beta equilibrium and electric charge neutrality.

→ *Cooper pairing with mismatched Fermi surfaces*

QCD realization of the long-sought Lakin-Ovchinnikov-Fulde -Ferrell (LOFF) state where the order parameter forms crystal structure.

A. I. Larkin and Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 47, 1136 (1964).
P. Fulde and R. A. Ferrell, Phys. Rev. 135, A550 (1964).

→ *Crystalline color superconductivity*

M. Alford, J. Bowers, and K. Rajagopal, Phys. Rev. D63, 074016 (2001).
J. A. Bowers, K. Rajagopal, Phys. Rev. D66, 065002 (2002).
R. Casalbuoni and G. Nardulli, Rev. Mod. Phys. 76, 263 (2004).
R. Anglani, R. Casalbuoni, M. Ciminale, R. Gatto, N. Ippolito,
M. Mannarelli, and M. Ruggieri, Rev. Mod. Phys. 86, 509 (2014).

Ginzburg-Landau approach to LOFF

General crystal structure of the LOFF order parameter:

$$\Delta(\mathbf{r}) = \sum_{k=1}^{P} \Delta e^{2iq\hat{\mathbf{n}}_k\cdot\mathbf{r}}$$

Ginzburg-Landau potential near LOFF-N transition:

$$\frac{\delta\Omega(\Delta)}{\mathcal{N}_F} = P\alpha\Delta^2 + \frac{1}{2}\beta\Delta^4 + \frac{1}{3}\gamma\Delta^6 + O(\Delta^8)$$

α universal for all crystal structures, others structure dependent

$$\alpha = -1 + \frac{\delta\mu}{2q}\ln\frac{q+\delta\mu}{q-\delta\mu} - \frac{1}{2}\ln\frac{\Delta_0^2}{4(q^2-\delta\mu^2)}$$

Ginzburg-Landau approach to LOFF

Evaluate the 4th- and 6th-order GL coefficients at weak coupling for 23 crystal structures near the conventional second-order FF-N transition point

$$\frac{\delta\mu_2}{\Delta_0} = 0.7544, \qquad \frac{q}{\delta\mu_2} = 1.1997$$

Pioneer work: J. A. Bowers, K. Rajagopal, Phys. Rev. D66, 065002 (2002).

preferred structures (?)

BCC $\Delta(\mathbf{r}) = 2\Delta\left[\cos(2qx) + \cos(2qy) + \cos(2qz)\right]$ **P=6**

FCC $\Delta(\mathbf{r}) = 8\Delta\cos\left(\frac{2qx}{\sqrt{3}}\right)\cos\left(\frac{2qy}{\sqrt{3}}\right)\cos\left(\frac{2qz}{\sqrt{3}}\right)$ **P=8**

Ginzburg-Landau approach to LOFF

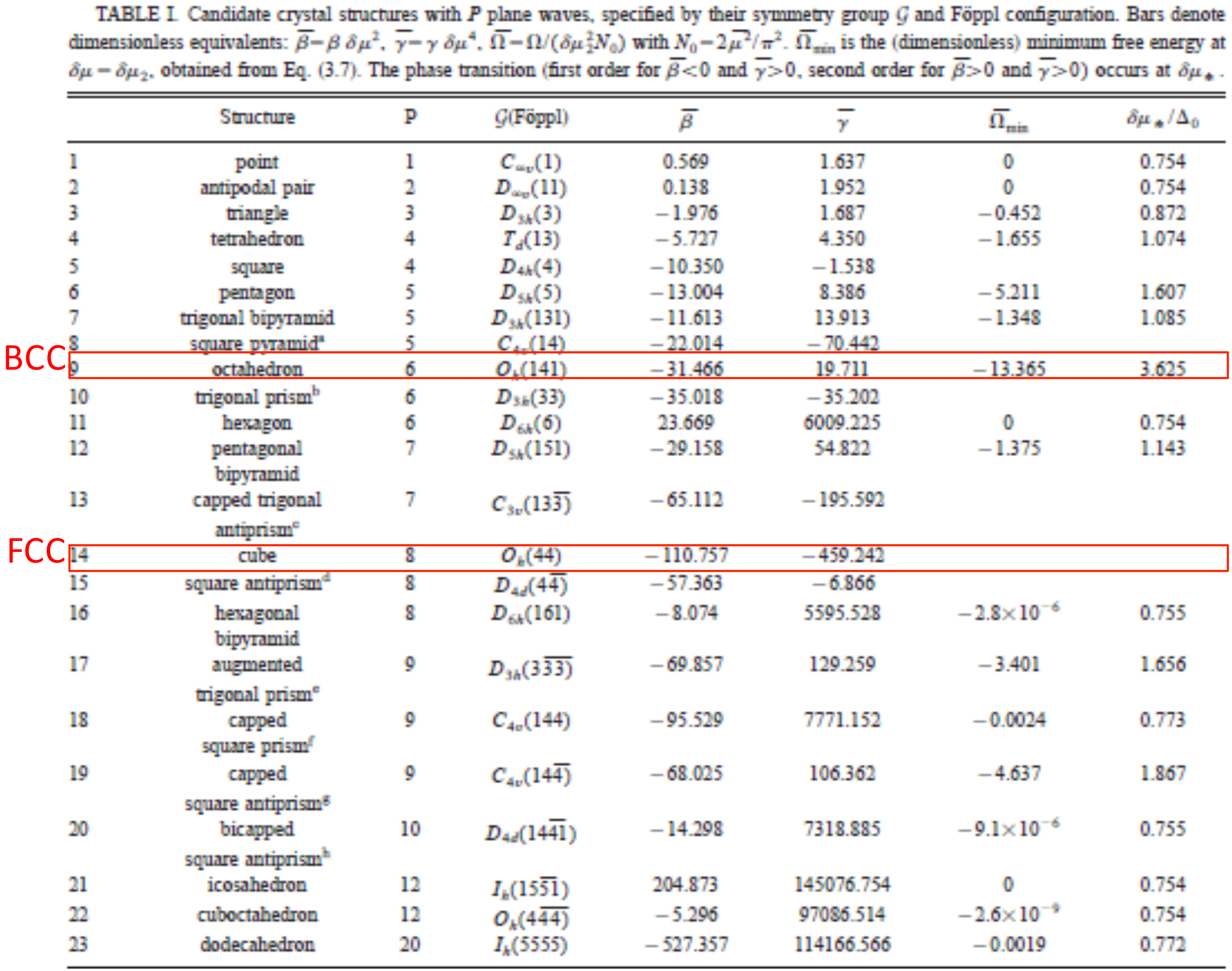


TABLE I. Candidate crystal structures with P plane waves, specified by their symmetry group $\mathcal{G}$ and Föppl configuration. Bars denote dimensionless equivalents: $\overline{\beta}=\beta\,\delta\mu^2$, $\overline{\gamma}=\gamma\,\delta\mu^4$, $\overline{\Omega}=\Omega/(\delta\mu_2^2 N_0)$ with $N_0=2\overline{\mu}^2/\pi^2$. $\overline{\Omega}_{\rm min}$ is the (dimensionless) minimum free energy at $\delta\mu=\delta\mu_2$, obtained from Eq. (3.7). The phase transition (first order for $\overline{\beta}<0$ and $\overline{\gamma}>0$, second order for $\overline{\beta}>0$ and $\overline{\gamma}>0$) occurs at $\delta\mu_*$.

	Structure	P	$\mathcal{G}$(Föppl)	$\overline{\beta}$	$\overline{\gamma}$	$\overline{\Omega}_{\rm min}$	$\delta\mu_*/\Delta_0$
1	point	1	$C_{\infty v}(1)$	0.569	1.637	0	0.754
2	antipodal pair	2	$D_{\infty h}(11)$	0.138	1.952	0	0.754
3	triangle	3	$D_{3h}(3)$	-1.976	1.687	-0.452	0.872
4	tetrahedron	4	$T_d(13)$	-5.727	4.350	-1.655	1.074
5	square	4	$D_{4h}(4)$	-10.350	-1.538		
6	pentagon	5	$D_{5h}(5)$	-13.004	8.386	-5.211	1.607
7	trigonal bipyramid	5	$D_{3h}(131)$	-11.613	13.913	-1.348	1.085
8	square pyramid^a	5	$C_{4v}(14)$	-22.014	-70.442		
9	octahedron	6	$O_h(141)$	-31.466	19.711	-13.365	3.625
10	trigonal prism^b	6	$D_{3h}(33)$	-35.018	-35.202		
11	hexagon	6	$D_{6h}(6)$	23.669	6009.225	0	0.754
12	pentagonal bipyramid	7	$D_{5h}(151)$	-29.158	54.822	-1.375	1.143
13	capped trigonal antiprism^c	7	$C_{3v}(13\overline{3})$	-65.112	-195.592		
14	cube	8	$O_h(44)$	-110.757	-459.242		
15	square antiprism^d	8	$D_{4d}(4\overline{4})$	-57.363	-6.866		
16	hexagonal bipyramid	8	$D_{6h}(161)$	-8.074	5595.528	-2.8×10^{-6}	0.755
17	augmented trigonal prism^e	9	$D_{3h}(3\overline{3}\overline{3})$	-69.857	129.259	-3.401	1.656
18	capped square prism^f	9	$C_{4v}(144)$	-95.529	7771.152	-0.0024	0.773
19	capped square antiprism^g	9	$C_{4v}(14\overline{4})$	-68.025	106.362	-4.637	1.867
20	bicapped square antiprism^h	10	$D_{4d}(14\overline{4}1)$	-14.298	7318.885	-9.1×10^{-6}	0.755
21	icosahedron	12	$I_h(15\overline{5}1)$	204.873	145076.754	0	0.754
22	cuboctahedron	12	$O_h(4\overline{4}\overline{4})$	-5.296	97086.514	-2.6×10^{-9}	0.754
23	dodecahedron	20	$I_h(5555)$	-527.357	114166.566	-0.0019	0.772

J. A. Bowers, K. Rajagopal, Phys. Rev. D66, 065002 (2002).

However, the GL potential up to the sixth-order predicts strong first-order transition for BCC and, both beta and gamma are negative for FCC.

BCC: $\bar{\beta} = -31.466, \quad \bar{\gamma} = 19.711$

➡ Strong first-order phase transition at

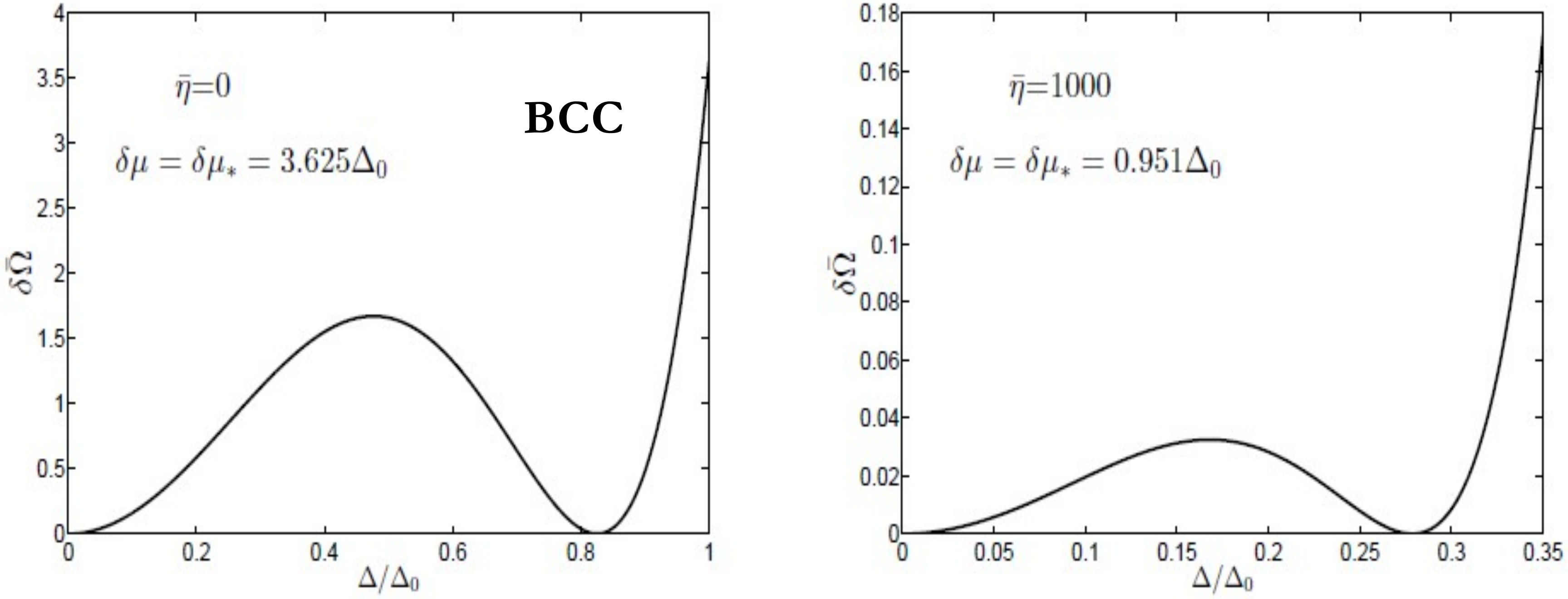
$$\delta\mu = \delta\mu_* = 3.625\Delta_0 \text{ where } \Delta = 0.83\Delta_0$$

FCC: $\bar{\beta} = -110.757, \quad \bar{\gamma} = -459.242$

➡ No prediction! But conjectured to be the preferred structure.

How important are the higher-order expansions?

$$\frac{\delta\Omega(\Delta)}{\mathcal{N}_{\mathrm{F}}} = P\alpha\Delta^2 + \frac{1}{2}\beta\Delta^4 + \frac{1}{3}\gamma\Delta^6 + \frac{1}{4}\eta\Delta^8$$



For eta→+infinity, the phase transition approaches second order and the upper critical field approaches 0.754.

Higher-order terms in the GL potential are rather important:

(1)Evaluate the higher-order expansions in GL approach;
(2)Or use a different approach without assuming a small condensate.

The order parameter forms a crystal structure

⇒ Fermions moving in a periodic (off-diagonal) pair potential

⇒ Solid-state physics, energy band structure

1D modulation: D. Nickel and M. Buballa, Phys. Rev. D79, 054009 (2009)

Consider a high-density effective Lagrangian for two-flavor pairing:

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left[i\partial_t - \varepsilon(\hat{\mathbf{p}}) + \hat{\mu} \right] \psi + \frac{g}{4} (\psi^\dagger \sigma_2 \psi^*)(\psi^{\text{T}} \sigma_2 \psi)$$

$\psi = (\psi_{\text{u}}, \psi_{\text{d}})^{\text{T}}$ → two-flavor quark field

$\hat{\mu} = \text{diag}(\mu_{\text{u}}, \mu_{\text{d}})$ → quark chemical potentials

$$\mu_{\text{u}} = \mu + \delta\mu$$

$$\mu_{\text{d}} = \mu - \delta\mu$$

g → contact coupling representing the attractive interaction

Order parameter of CSC: $\Delta(\mathbf{r}) = \langle \varphi(\tau, \mathbf{r}) \rangle$

$$\varphi(\tau, \mathbf{r}) = -g\psi_{\text{u}}(\tau, \mathbf{r})\psi_{\text{d}}(\tau, \mathbf{r})$$

At weak coupling, we may use the mean-field approach.

Mean-field Lagrangian: $\mathcal{L}_{\mathrm{MF}} = \Psi^{\dagger}(-\partial_{\tau} - \mathcal{H}_{\mathrm{MF}})\Psi - \frac{|\Delta(\mathbf{r})|^2}{g}$

$$\mathcal{H}_{\mathrm{MF}} = \begin{pmatrix} \varepsilon(\hat{\mathbf{p}}) - \mu - \delta\mu & \Delta(\mathbf{r}) \\ \Delta^{*}(\mathbf{r}) & -\varepsilon(\hat{\mathbf{p}}) + \mu - \delta\mu \end{pmatrix}$$

$$\Psi = (\psi_{\mathrm{u}}, \psi_{\mathrm{d}}^{*})^{\mathrm{T}} \longrightarrow$$ Nambu-Gor'kov spinor

To be specific, we consider the BCC and FCC crystals.

BCC $$\Delta(\mathbf{r}) = 2\Delta\left[\cos(2qx) + \cos(2qy) + \cos(2qz)\right]$$

FCC $$\Delta(\mathbf{r}) = 8\Delta\cos\left(\frac{2qx}{\sqrt{3}}\right)\cos\left(\frac{2qy}{\sqrt{3}}\right)\cos\left(\frac{2qz}{\sqrt{3}}\right)$$

The pair potential is periodic in coordinate space $\Delta(\mathbf{r}) = \Delta(\mathbf{r} + \mathbf{a}_i)$

Three linearly independent lattice vectors

$$\mathbf{a}_1 = a\mathbf{e}_x, \ \mathbf{a}_2 = a\mathbf{e}_y, \text{ and } \mathbf{a}_3 = a\mathbf{e}_z$$

$$a = \pi/q \text{ for BCC and } a = \sqrt{3}\pi/q \text{ for FCC}$$

Fourier expansion with reciprocal lattice vectors

$$\Delta(\mathbf{r}) = \sum_{\mathbf{G}} \Delta_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}} = \sum_{l,m,n=-\infty}^{\infty} \Delta_{lmn} e^{i\mathbf{G}_{lmn}\cdot\mathbf{r}}$$

$$\mathbf{G} = \mathbf{G}_{lmn} = \frac{2\pi}{a}\left(l\mathbf{e}_x + m\mathbf{e}_y + n\mathbf{e}_z\right), \quad l, m, n \in \mathbb{Z}.$$

Field theory with usual momentum representation

$$\Psi(\tau, \mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{\omega_n} \Psi(i\omega_n, \mathbf{p}) e^{-i\omega_\nu \tau + i\mathbf{p}\cdot\mathbf{r}}$$

$$\Psi^\dagger(\tau, \mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_{\omega_n} \Psi^\dagger(i\omega_n, \mathbf{p}) e^{i\omega_\nu \tau - i\mathbf{p}\cdot\mathbf{r}}$$

➡ Effective action

$$\mathcal{S}_{\mathrm{MF}} = \frac{V}{T} \sum_{\mathbf{G}} \frac{|\Delta_{\mathbf{G}}|^2}{g} - \frac{1}{T} \sum_{\omega_n, \omega_{n'}} \sum_{\mathbf{p}, \mathbf{p}'} \Psi^\dagger(i\omega_n, \mathbf{p}) \left(i\omega_n \delta_{\omega_n, \omega_{n'}} \delta_{\mathbf{p}, \mathbf{p}'} - \delta_{\omega_n, \omega_{n'}} \mathcal{H}_{\mathbf{p}, \mathbf{p}'} \right) \Psi(i\omega_{n'}, \mathbf{p}')$$

$$\mathcal{H}_{\mathbf{p}, \mathbf{p}'} = \begin{pmatrix} (\xi_{\mathbf{p}} - \delta\mu)\delta_{\mathbf{p}, \mathbf{p}'} & \sum_{\mathbf{G}} \Delta_{\mathbf{G}} \delta_{\mathbf{G}, \mathbf{p}-\mathbf{p}'} \\ \sum_{\mathbf{G}} \Delta^*_{\mathbf{G}} \delta_{\mathbf{G}, \mathbf{p}'-\mathbf{p}} & (-\xi_{\mathbf{p}} - \delta\mu)\delta_{\mathbf{p}, \mathbf{p}'} \end{pmatrix} \qquad \xi_{\mathbf{p}} = |\mathbf{p}| - \mu$$

➡ Infeasible for further analytical and numerical treatment

The usual momentum representation is not compatible with the periodic structure of the order parameter. Take a look at the eigenvalue equation which is known as BdG equation

$$\begin{pmatrix} \varepsilon(\hat{\mathbf{p}}) - \mu - \delta\mu & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -\varepsilon(\hat{\mathbf{p}}) + \mu - \delta\mu \end{pmatrix} \phi_\lambda(\mathbf{r}) = E_\lambda \phi_\lambda(\mathbf{r})$$

Eigenfunction takes the form of Bloch function (Bloch theorem)

$$\phi_\lambda(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} \phi_{\lambda\mathbf{k}}(\mathbf{r}) \qquad \phi_{\lambda\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{G}} \phi_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}}$$

$\mathbf{k} \in \mathrm{BZ}$

Matrix equation in the G-space $\sum_{\mathbf{G}'} \mathcal{H}_{\mathbf{G},\mathbf{G}'}(\mathbf{k}) \phi_{\mathbf{G}'} = E_\lambda(\mathbf{k}) \phi_{\mathbf{G}}$

$$\mathcal{H}_{\mathbf{G},\mathbf{G}'}(\mathbf{k}) = \begin{pmatrix} (\xi_{\mathbf{k}+\mathbf{G}} - \delta\mu)\delta_{\mathbf{G},\mathbf{G}'} & \Delta_{\mathbf{G}-\mathbf{G}'} \\ \Delta_{\mathbf{G}-\mathbf{G}'} & (-\xi_{\mathbf{k}+\mathbf{G}} - \delta\mu)\delta_{\mathbf{G},\mathbf{G}'} \end{pmatrix}$$

Band structure in a periodic pair potential

Field Theory: Bloch representation

$$\Psi(\tau, \mathbf{r}) \;=\; \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\in \mathrm{BZ}} e^{i\mathbf{k}\cdot\mathbf{r}} \sum_{\omega_n} \sum_{\mathbf{G}} \Psi_{\mathbf{G}}(i\omega_n, \mathbf{k}) e^{-i\omega_\nu \tau + i\mathbf{G}\cdot\mathbf{r}}$$

$$\Psi^\dagger(\tau, \mathbf{r}) \;=\; \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\in \mathrm{BZ}} e^{-i\mathbf{k}\cdot\mathbf{r}} \sum_{\omega_n} \sum_{\mathbf{G}} \Psi^\dagger_{\mathbf{G}}(i\omega_n, \mathbf{k}) e^{i\omega_\nu \tau - i\mathbf{G}\cdot\mathbf{r}}$$

Effective action

$$\mathcal{S}_{\mathrm{MF}} = \frac{V}{T} \sum_{\mathbf{G}} \frac{|\Delta_{\mathbf{G}}|^2}{g} - \frac{1}{T} \sum_{\omega_n, \omega_{n'}} \sum_{\mathbf{k}, \mathbf{k}'\in \mathrm{BZ}} \sum_{\mathbf{G}, \mathbf{G}'} \Psi^\dagger_{\mathbf{G}}(i\omega_n, \mathbf{k}) \left[i\omega_n \delta_{\omega_n, \omega_{n'}} \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\mathbf{G}, \mathbf{G}'} - \delta_{\omega_n, \omega_{n'}} \delta_{\mathbf{k}, \mathbf{k}'} \mathcal{H}_{\mathbf{G}, \mathbf{G}'}(\mathbf{k}) \right] \Psi_{\mathbf{G}'}(i\omega_{n'}, \mathbf{k}')$$

Grand potential

$$\Omega = \frac{1}{g} \sum_{\mathbf{G}} |\Delta_{\mathbf{G}}|^2 - \frac{T}{V} \sum_{\omega_n} \sum_{\mathbf{k}\in \mathrm{BZ}} \mathrm{Tr}\ln \frac{S^{-1}}{T}$$

$$(S^{-1})_{\mathbf{G}, \mathbf{G}'} = i\omega_n \delta_{\mathbf{G}, \mathbf{G}'} - \mathcal{H}_{\mathbf{G}, \mathbf{G}'}$$

Trace taken only in the Nambu-Gor'kov space and G-space

Numerical result for BCC structure

The Hamiltonian matrix $\mathcal{H}_{\mathbf{G},\mathbf{G}'}(\mathbf{k})$ has infinite dimensions. We have to make a truncation in order to perform a calculation.

Symmetrical truncation: $-D < l, m, n < D \; (D \in \mathbb{Z}^{+})$

➡ Matrix equation:

$$\mathbf{H}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = (E + \delta\mu)\begin{pmatrix} u \\ v \end{pmatrix}$$

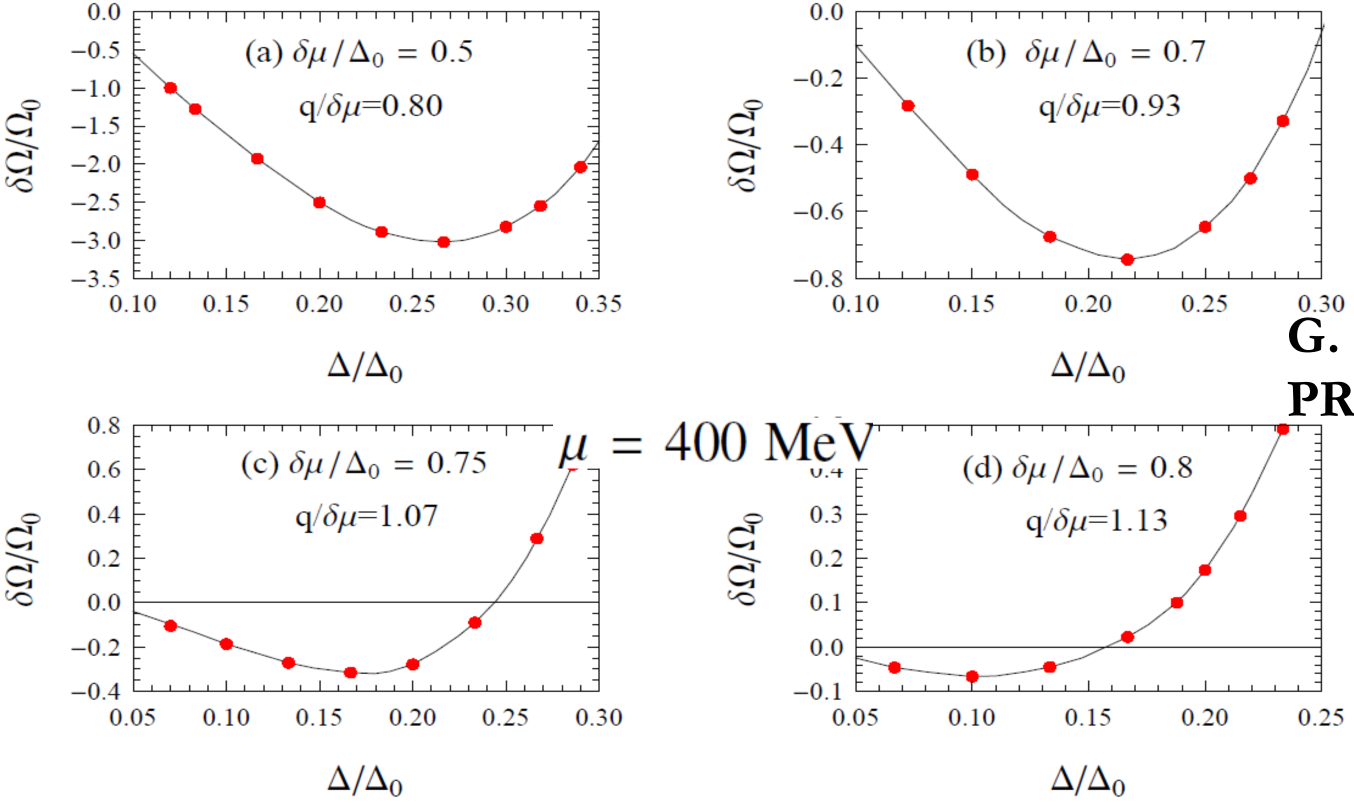
$$\mathbf{H}_{11}^{[l,m,n],[l',m',n']} = -\mathbf{H}_{22}^{[l,m,n],[l',m',n']} = \xi_{[l,m,n]}\delta_{l,l'}\delta_{m,m'}\delta_{n,n'},$$

$$\mathbf{H}_{12}^{[l,m,n],[l',m',n']} = \mathbf{H}_{21}^{[l,m,n],[l',m',n']} = \Delta_{[l-l',m-m',n-n']} = \Delta_{\mathbf{G}-\mathbf{G}'},$$

$\mathbf{H}_{ij}$ are $(2D+1)^3 \times (2D+1)^3$ matrices

For sufficiently large D, the contribution from the high-energy bands becomes vanishingly small. In practice, we choose a sufficiently large D and check the convergence by varying D.

From the numerical calculations we normally need D~30 for BCC and D~60 for FCC. The computing cost for BCC is affordable so far.



$\mu = 400$ MeV

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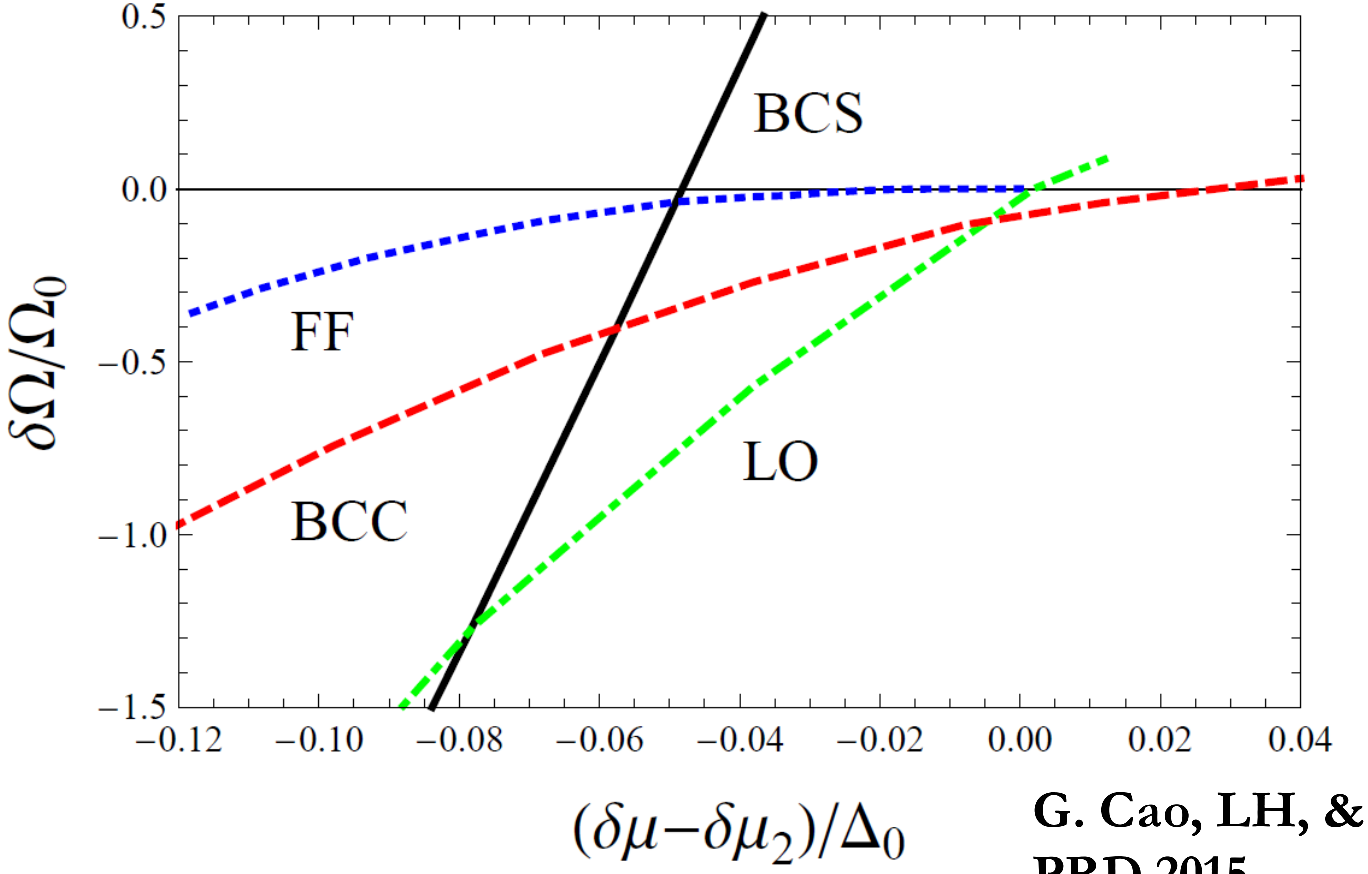
Energy comparison



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PRD 2015

(1)Upper critical field of BCC is only 4% higher than the FF value (0.754)

(2)BCC-N transition is of rather weak first order $\Delta \simeq 0.1\Delta_0$

Summary

- A nearly quantitative many-body theory for BCS-BEC crossover in cold Fermi gases: Gaussian pair fluctuation (GPF) theory

- BEC-BCS crossover may occur in dense QCD like theories

- A solid-state physics approach to crystalline color superconductivity: previous GL prediction is not reliable, needs further studies

- Further application of GPF: linear response, transport coefficients, …

Thank you for your attention!

Coupling constant renormalization: 3D vs 2D

Computing scattering amplitude with contact interaction (LS equation):

$$T_{2\mathrm{B}}^{-1}(E) = -U^{-1} - \mathcal{B}(E) \qquad \mathcal{B}(E) = \frac{1}{V}\sum_{\mathbf{p}} \frac{1}{E + i\epsilon - 2\varepsilon_{\mathbf{p}}}$$

UV divergence: cutoff regularization (dimensional reg. → epsilon expansion)

3D: $\mathcal{B}(E) = -\frac{m\Lambda}{2\pi^2} + \frac{m}{4\pi}\sqrt{-m(E+i\epsilon)}$ 2D: $\mathcal{B}(E) = -\frac{m}{4\pi}\ln\frac{\Lambda^2}{m} + \frac{m}{4\pi}\ln(-E - i\epsilon)$

Renormalization=Matching known scattering amplitude

3D: $$f(k) = -\frac{1}{a^{-1} + ik} \Longrightarrow \frac{1}{U(\Lambda)} = -\frac{m}{4\pi a} + \frac{m\Lambda}{2\pi^2} = -\frac{m}{4\pi a} + \frac{1}{V}\sum_{|\mathbf{p}|<\Lambda}\frac{1}{2\varepsilon_{\mathbf{p}}}$$

2D: $$f(k) = \frac{1}{\ln(E/\varepsilon_{\mathrm{B}}) - i\pi} \Longrightarrow \frac{1}{U(\Lambda)} = \frac{m}{4\pi}\ln\frac{\Lambda^2}{m\varepsilon_{\mathrm{B}}} = \frac{1}{V}\sum_{|\mathbf{p}|<\Lambda}\frac{1}{2\varepsilon_{\mathbf{p}} + \varepsilon_{\mathrm{B}}}$$

Gas parameters (interaction strength)

2D scattering length

3D: $\eta = \frac{1}{k_{\mathrm{F}}a}$ 2D: $\eta = \ln(k_{\mathrm{F}}a_{2\mathrm{D}})$ $\varepsilon_{\mathrm{B}} = \frac{4}{ma_{2\mathrm{D}}^2 e^{2\gamma}}$