Time-like splitting functions at NNLO

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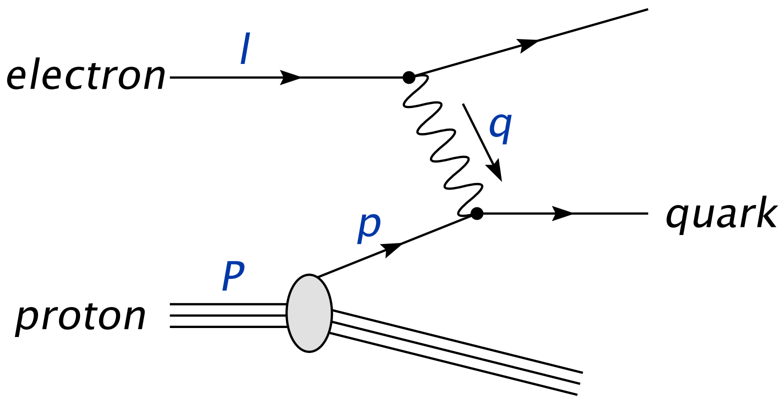
Based on work done in collaboration with:

- *Towards three-loop QCD corrections to the time-like splitting functions*
  O. Gituliar and S. M. arXiv:1505.02901
- *Progress on double-logarithmic large-x and small-x resummations for (semi-)inclusive hard processes*
  A. Vogt, C.H. Kom N.A. Lo Presti, G. Soar, A.A. Almasy, S. M. J.A.M. Vermaseren and K. Yeats arXiv:1212.2932
- *On the Next-to-Next-to-Leading Order Evolution of Flavour-Singlet Fragmentation Functions*
  A.A. Almasy, S. M. and A. Vogt arXiv:1107.2263
- *Higher-order threshold resummation for semi-inclusive* e^+e^- *annihilation*
  S. M. and A. Vogt arXiv:0908.2746
- *On third-order timelike splitting functions and top-mediated Higgs decay into hadrons*
  S. M. and A. Vogt arXiv:0709.3899
- *Next-to-Next-to-Leading Order Evolution of Non-Singlet Fragmentation Functions*
  A. Mitov, S. M. and A. Vogt hep-ph/0604053

Setting the stage

Deep-inelastic scattering



- Kinematic variables
  - momentum transfer $Q^2 = -q^2$ (space-like)
  - Bjorken variable $x = Q^2/(2p \cdot q)$
- Parton distributions PDF
  - scale evolution governed by splitting functions P_{ij}

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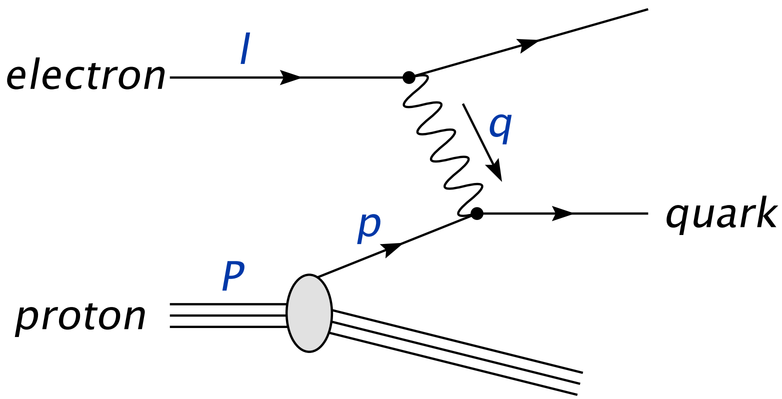

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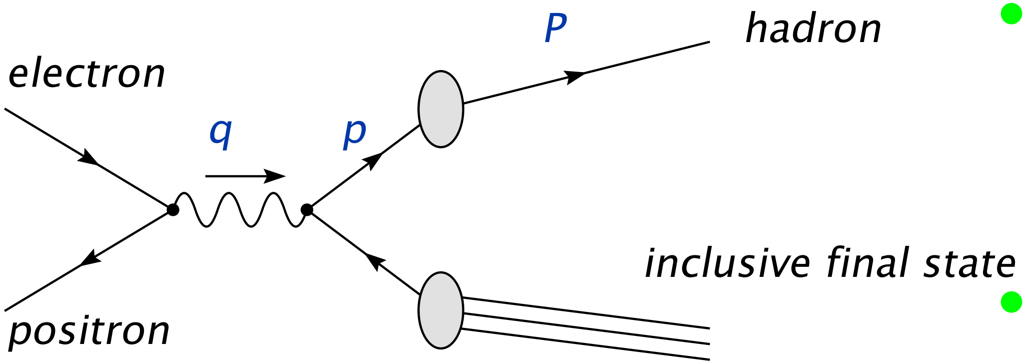
e^+e^- annihilation



- Kinematic variables
  - momentum transfer $Q^2 = +q^2$ (time-like)
  - scaling variable $x = (2p \cdot q)/Q^2$
- Fragmentation functions D
  - scale evolution governed by (time-like) splitting functions P_{ij}

Relating space- and time-like kinematics

Crewther relation

- From conformal and chiral invariance of leading singularity of short distance OPE simple relation between Crewther '72
  - amplitude $\pi^0 \to \gamma\gamma$
  - polarized Bjorken sum rule $\int_0^1 dx\, g_1^{ep-en}(x, Q^2)$
  - Adler function D_V (derivative of correlator $Q^2 \frac{\partial}{\partial Q^2} \Pi_V$)

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- Higher order radiative QCD corrections exhibit relations between
  - polarized Gross-Llewellyn Smith sum-rule C_{GLS} at $\mathcal{O}(\alpha_s^4)$ (first Mellin moment of $F_3^{\bar{\nu}p+\nu p}$) Baikov, Chetyrkin, Kühn '10
  - Adler function D_V at $\mathcal{O}(\alpha_s^4)$ Baikov, Chetyrkin, Kühn '10
- $C_{\rm GLS}$ and D_V related by running coupling (β-function) through $\mathcal{O}(\alpha_s^4)$ Broadhurst, Kataev '96; Maxwell, Broadhurst, Kataev '06; Baikov, Chetyrkin, Kühn '10; Baikov, Chetyrkin, Kühn, Rittinger '12

$$C_{\rm GLS}(\alpha_s)\, D_V(\alpha_s) \quad = \quad d_R \left(1 + \frac{\beta(\alpha_s)}{\alpha_s} K(\alpha_s)\right)$$

Drell-Yan-Levy relation

- Analytic continuation in energy $-q^2 \to +q^2$ (exploit analyticity properties)
  Curci, Furmanski, Petronzio '80; Floratos, Kounnas, Lacaze '81; Stratmann, Vogelsang '96; Blümlein, Ravindran, van Neerven '00; ...
- Relation between DIS structure function $F_1^{\rm s-like}$ and fragmentation function $F_T^{\rm t-like}$

$$F_T^{\rm t-like}(x) \;=\; -x F_1^{\rm s-like}\left(\frac{1}{x}\right)$$

- Leading order splitting function $P_{\rm qq}^{(0)}$
  - respects "naive" Drell-Yan-Levy relation (with $\delta(1-x) \to \delta(1-x)$)

$$P_{qq}^{(0)}(x) \;=\; 2C_F\left(\frac{2}{1-x} - 1 - x\right) + 3C_F\delta(1-x)$$

- Beyond leading order naive version of Drell-Yan-Levy relation not valid

Gribov-Lipatov reciprocity

- Leading order diagonal splitting functions identical for space- and time-like kinematics $P^{\rm t-like}(x) = P^{\rm s-like}(x) = -xP^{\rm s-like}\left(\frac{1}{x}\right)$
- reciprocity relation implied (realized in $N=4$ SYM theory)

Mapping DIS to e^+e^- annihilation

- Use physical cross sections in dimensional regularization ($D = 4 - 2\epsilon$)

Real and virtual contributions

- Partonic forward Compton amplitude $\mathcal{T}_n$ combines
  - virtual corrections $\mathcal{F}_n$ (QCD form factor, $\mathcal{F}_n \propto \delta(1-x)$)
  - real-emission contributions $\mathcal{R}_n(x)$
    (depend on harmonic polylogarithms in x)
- D-dimensional +-distributions in $\mathcal{R}_n$ for soft/collinear region

$$[\,(1-x)^{-1-k\epsilon}\,]_+ \;=\; -\frac{1}{\epsilon k}\,\delta(1-x) + \sum_{i=0} \frac{(-k\epsilon)^i}{i\,!} \left(\frac{\ln^i(1-x)}{1-x}\right)_+$$

Kinoshita – Lee-Nauenberg theorem

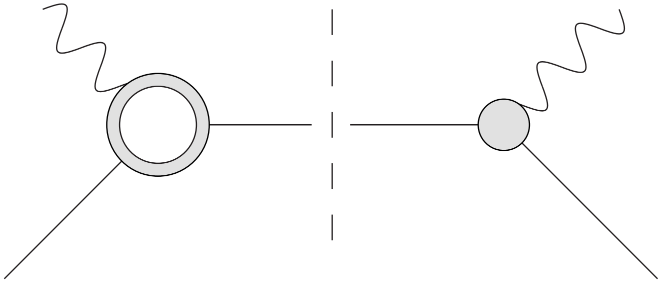
- Laurent-series for $\mathcal{T}_n$ in ϵ at n$^{\rm th}$-order
  - soft and collinear singularities in $\mathcal{F}_n$ and $\mathcal{R}_n$ behave as $1/\epsilon^{2n}$
  - mass-factorization predicts $1/\epsilon^n$
- Infrared safety ($\to$ KLN Kinoshita '62; Lee, Nauenberg '64)
  - constructive approach to $\mathcal{F}_n$ and $\mathcal{R}_n$

Mass factorization

- Universal structure of bare quantity after cancellation of poles between $\mathcal{F}_n$ and $\mathcal{R}_n$
  - structure function F_1 (space-like)
  - transverse fragmentation function F_T (time-like)
- Laurent-series at n$^{\mathrm{th}}$-order behaves as $\dfrac{1}{\epsilon^n}$

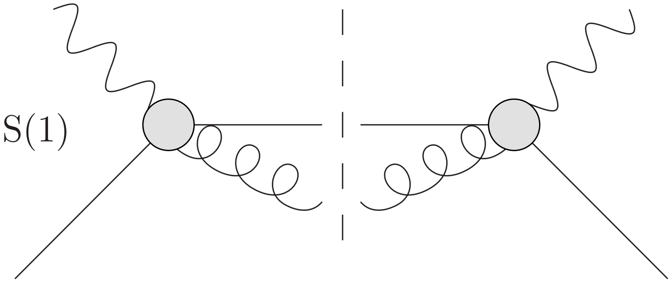
$$
\begin{aligned}
F^{(1)} &= -\frac{1}{\epsilon} P^{(0)} + c^{(1)} + \epsilon\, a^{(1)} + \epsilon^2\, b^{(1)} + \epsilon^3\, d^{(1)} + \dots \\
F^{(2)} &= \frac{1}{2\epsilon^2} P^{(0)}(P^{(0)} + \beta_0) - \frac{1}{2\epsilon}\left[P^{(1)} + 2P^{(0)}c^{(1)}\right] + c^{(2)} - P^{(0)}a^{(1)} \\
&\quad +\epsilon\left[a^{(2)} - P^{(0)}b^{(1)}\right] + \dots \\
F^{(3)} &= -\frac{1}{6\epsilon^3} P^{(0)}(P^{(0)} + \beta_0)(P^{(0)} + 2\beta_0) \\
&\quad + \frac{1}{6\epsilon^2}\left[P^{(1)}(3P^{(0)} + 2\beta_0) + P^{(0)}(3P^{(0)}c^{(1)} + 3\beta_0 c^{(1)} + 2\beta_1)\right] \\
&\quad - \frac{1}{6\epsilon}\left[2P^{(2)} + 3P^{(1)}c^{(1)} + P^{(0)}(6c^{(2)} - 3P^{(0)}a^{(1)} - 3\beta_0 a^{(1)})\right] + \dots
\end{aligned}
$$

Anatomy of DIS result (1 loop)



$\mathcal{F}^{(1)}$

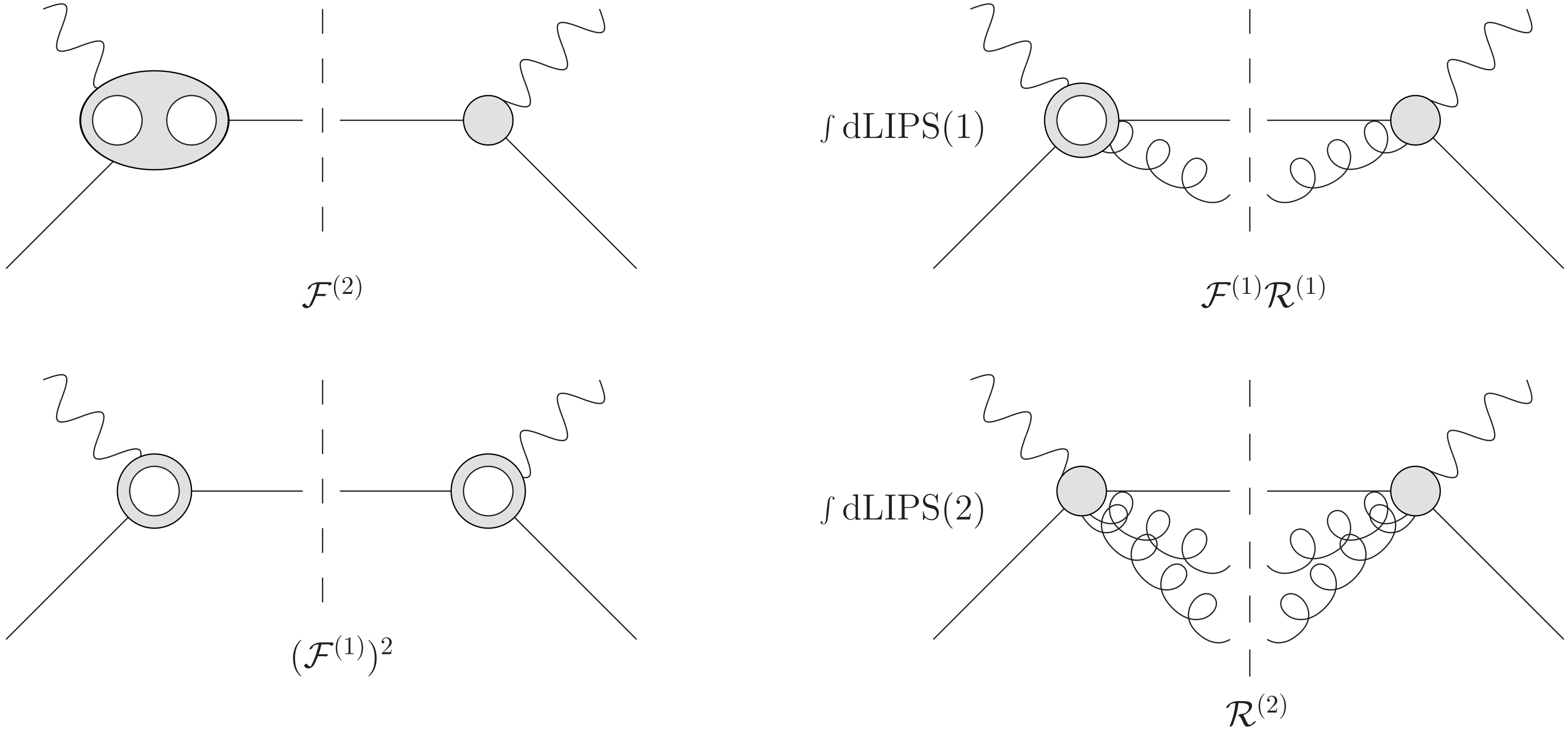
$\int$ dLIPS(1)



$\mathcal{R}^{(1)}$

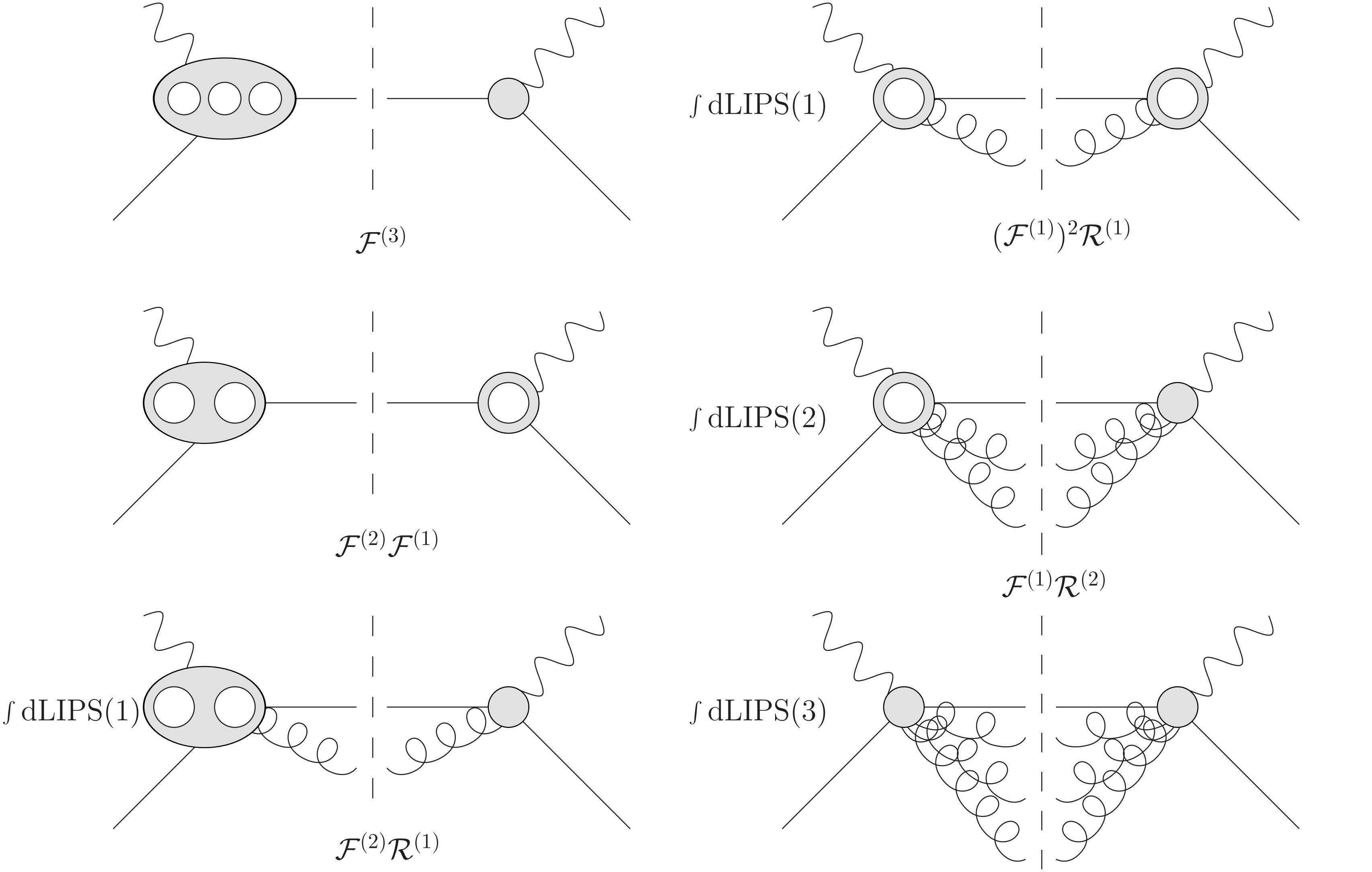
$$\mathcal{T}_1^{\mathrm{b}} \quad = \quad 2\,\mathsf{Re}\,\mathcal{F}_1\,\delta(1-x) + \mathcal{R}_1$$

Anatomy of DIS result (2 loops)



$$\mathcal{T}_2^{\mathrm{b}} \quad = \quad (2\,\mathsf{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\delta(1-x) + 2\,\mathsf{Re}\,\mathcal{F}_1\mathcal{R}_1 + \mathcal{R}_2$$

Anatomy of DIS result (3 loops)



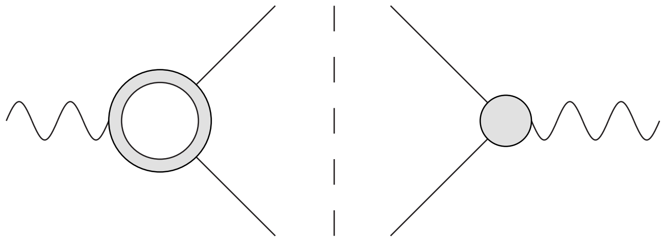
$\mathcal{R}^{(3)}$

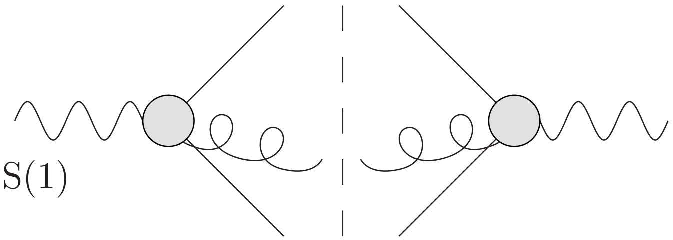
$$\mathcal{T}_3^{\rm b} \quad = \quad (2\,\textsf{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\,\delta(1-x) + (2\,\textsf{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{R}_1 + 2\,\textsf{Re}\,\mathcal{F}_1\mathcal{R}_2 + \mathcal{R}_3$$

Analytic continuation

- $\mathcal{R}_n$ is Laurent series in ϵ with coefficients being harmonic polylogarithms $H_{m_1,\dots,m_k}(x)$ or polynomials in $x, (1-x), (1+x)$
- Analytic continuation from space-like to time-like kinematics requires
  Curci, Furmanski, Petronzio '80; Stratmann, Vogelsang '96; . . .
  - mapping $-q^2 \to +q^2$ and $x \to \frac{1}{x}$
- Some subtleties
  - harmonic polylogarithms with basic functions of lowest weight
    $H_0(x) = \ln x\,, \quad H_1(x) = -\ln(1-x)\,, \quad H_{-1}(x) = \ln(1+x)$
    Goncharov '98; Borwein, Bradley,Broadhurst, Lisonek '99; Remiddi, Vermaseren '99
  - all branch cuts from analytic continuation $\frac{1}{x} - \mathrm{i}\delta$ to $x > 0$ uniquely defined through $H_1(1/x - \mathrm{i}\delta) = H_1(x) + H_0(x) - \mathrm{i}\pi$
  - phase space of detected parton in e^+e^--annihilation in D-dimensions (take phase space factor $x^{1-2\epsilon}$)
- Constructive approach to $\mathcal{R}_n^{\mathrm{t-like}}$ (given we know $\mathcal{R}_n^{\mathrm{s-like}}$)

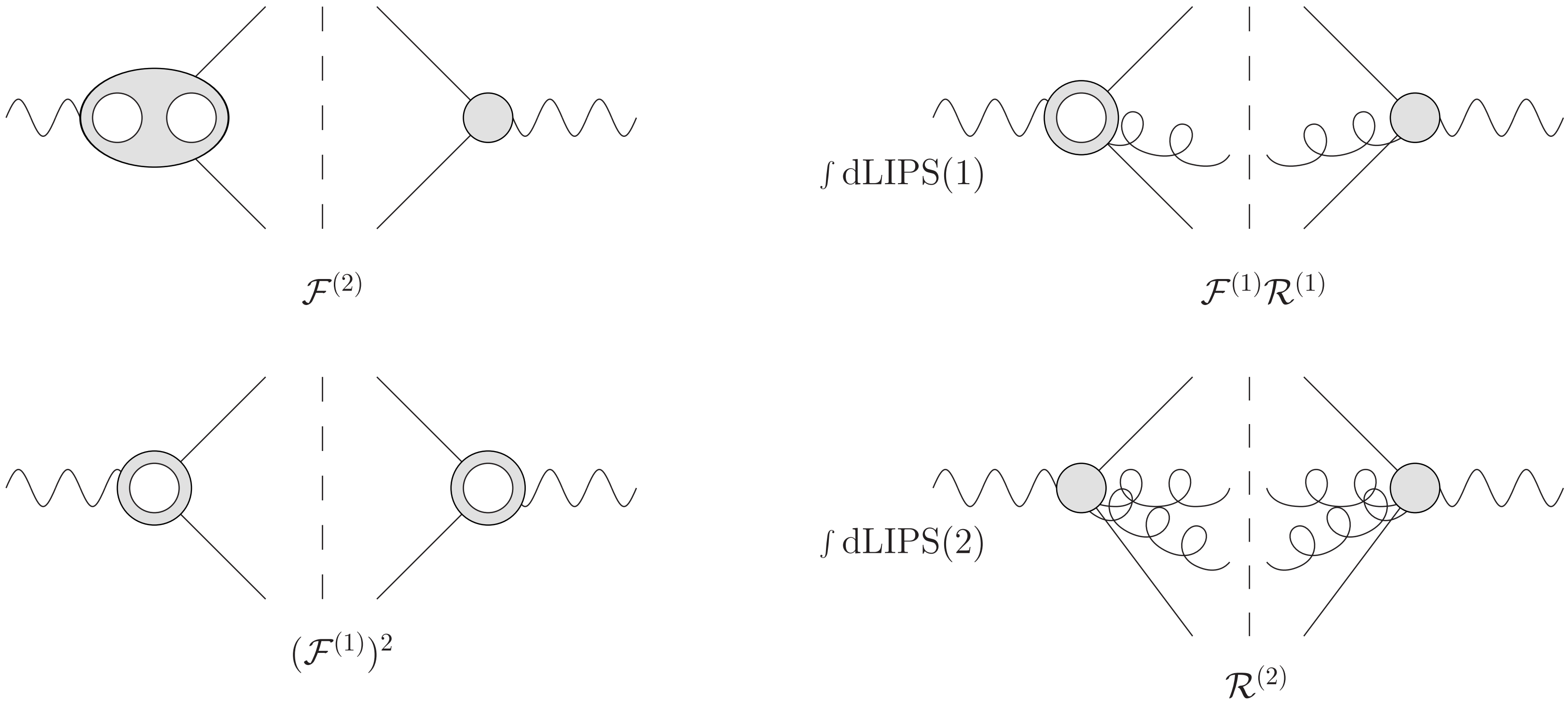
Assembly of e^+e^- (1 loop)





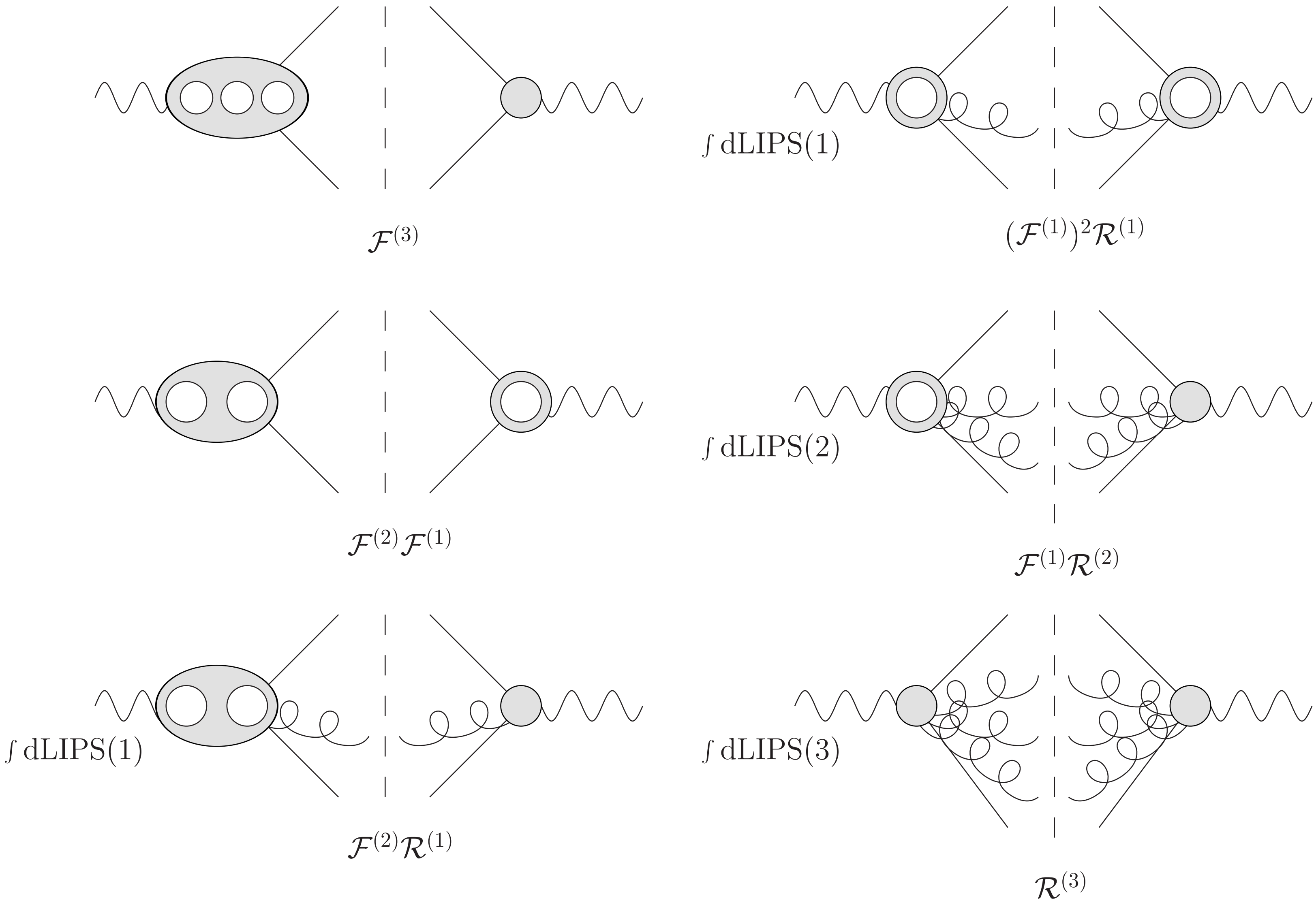
$$\mathcal{T}_1^{\mathrm{b}} \;=\; 2\,\mathsf{Re}\,\mathcal{F}_1\,\delta(1-x) + \mathcal{R}_1$$

Assembly of e^+e^- (2 loop)



$$\mathcal{T}_2^{\mathrm{b}} \quad = \quad (2\,\mathsf{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\delta(1-x) + 2\,\mathsf{Re}\,\mathcal{F}_1\mathcal{R}_1 + \mathcal{R}_2$$

Assembly of e^+e^- (3 loop)



$$\mathcal{T}_3^{\rm b} \quad = \quad (2\,\mathrm{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\,\delta(1-x) + (2\,\mathrm{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{R}_1 + 2\,\mathrm{Re}\,\mathcal{F}_1\mathcal{R}_2 + \mathcal{R}_3$$

Time-like results

- Read off results in time-like kinematics from mass factorization of bare transverse fragmentation function F_T

$$
\begin{aligned}
F^{(1)} &= -\frac{1}{\epsilon}\,P^{(0)} \;+\; c^{(1)} \;+\; \epsilon\,a^{(1)} \;+\; \epsilon^2\,b^{(1)} \;+\; \epsilon^3\,d^{(1)} \;+\; \dots \\
F^{(2)} &= \frac{1}{2\epsilon^2}\,P^{(0)}(P^{(0)}+\beta_0) - \frac{1}{2\epsilon}\left[P^{(1)} + 2P^{(0)}c^{(1)}\right] + c^{(2)} - P^{(0)}a^{(1)} \\
&\quad +\epsilon\left[a^{(2)} - P^{(0)}b^{(1)}\right] + \dots \\
F^{(3)} &= -\frac{1}{6\epsilon^3}\,P^{(0)}(P^{(0)}+\beta_0)(P^{(0)}+2\beta_0) \\
&\quad +\frac{1}{6\epsilon^2}\left[P^{(1)}(3P^{(0)}+2\beta_0) + P^{(0)}(3P^{(0)}c^{(1)} + 3\beta_0 c^{(1)} + 2\beta_1)\right] \\
&\quad -\frac{1}{6\epsilon}\left[2P^{(2)} + 3P^{(1)}c^{(1)} + P^{(0)}(6c^{(2)} - 3P^{(0)}a^{(1)} - 3\beta_0 a^{(1)})\right] + \dots
\end{aligned}
$$

- NNLO time-like splitting function $P^{(2)}$

Diagonal splitting functions (I)

- Results for mapping space-like to time-like processes ($F_1^{\mathrm{s-like}} \leftrightarrow F_T^{\mathrm{t-like}}$), ($F_3^{\mathrm{s-like}} \leftrightarrow F_A^{\mathrm{t-like}}$) and ($F_\phi^{\mathrm{s-like}} \leftrightarrow F_\phi^{\mathrm{t-like}}$)
  - read off three-loop splitting function $P_{\mathrm{ns}}^{(2)T}$, $P_{\mathrm{ps}}^{(2)T}$ and $P_{gg}^{(2)T}$
- Checks
  - one- and two-loops (even order ϵ with loop technology) Mitov, S.M. '06
  - three loops through order $1/\epsilon$ (soft/collinear limit)
- Sum rules

$$\int_0^1 dx P_{\mathrm{ns}}^{(2)T}(x) = 0 \qquad \text{and} \qquad \int_0^1 dx \left(P_{qg}^{(2)T}(x) + P_{gg}^{(2)T}(x)\right) = 0$$

  - problems in sum rule check:
    coefficient $\frac{1}{\epsilon} C_{F/A}^3 \, P_{qq/gg}^{(0)} \, \zeta_2 \, \ln^2 x$ incorrect in $P_{\mathrm{ns}/gg}^{(2)T}(x)$
- Alternative approach based on universal splitting function (kinematics independent) Dokshitzer, Marchesini, Salam '05
  - anomalous dimension (Mellin transform)
    $\gamma^{T/S}(N) = f(N \mp \gamma^{T/S}(N)/2 - \beta(\alpha_s)/2)$
  - reciprocity respecting function $f(N)$ (exact in $N = 4$ SYM theory)

Diagonal splitting functions (II)

- Result for $P^{(2)T}_{\mathrm{ns}/gg}(x)$
  - analytic continuation with correct sum rule
    Mitov, S.M., Vogt '06; S.M. Vogt '07
  - agreement with approach based on universal splitting functions
    Dokshitzer, Marchesini, Salam '05
- Additional check
  - compute three-loop coefficient functions for one-particle inclusive Higgs decay
  - second-moment combination enters the Higgs (ϕ) decay rate

$$(C^{T}_{\phi,q} + C^{T}_{\phi,g})(N=2) \;=\; 1 + \alpha_s\, c^{(1)}_{\phi} + \alpha_s^2\, c^{(2)}_{\phi} + \alpha_s^3\, c^{(3)}_{\phi} + \ldots \;,$$

- agreement
  - NNLO with Chetyrkin, Kniehl, Steinhauser '97; Schreck, Steinhauser '07
  - N^{3}LO with Baikov, Chetyrkin '06 (up to ζ_2-terms)

Off-diagonal splitting functions

- Alternative factorization in terms of physical kernels

$$\frac{d}{d\ln Q^2} F^T \;=\; K^T \otimes F^T \;=\; \underbrace{\left\{\left(\beta \frac{dC^T}{d\alpha_s} + C^T \otimes P^T\right) \otimes (C^T)^{-1}\right\}}_{\text{scheme invariant}} \otimes F^T$$

- Use photon γ and Higgs ϕ probes

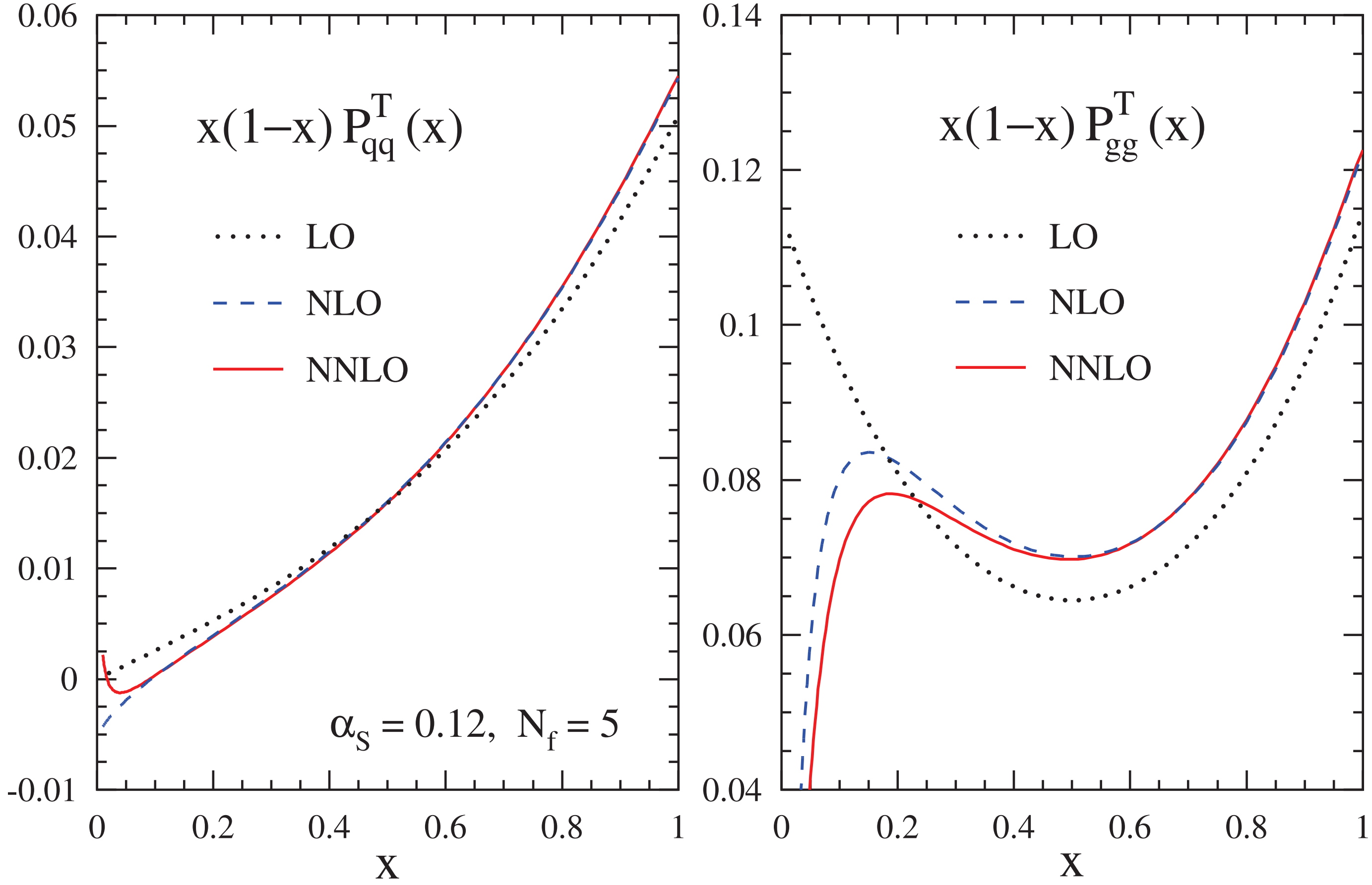
$$F^T = \begin{pmatrix} F_T \\ F_\phi^T \end{pmatrix}, \quad K^T = \sum_{n=0} \alpha_s^{n+1} \begin{pmatrix} K_{\mathrm{TT}}^{(n)} & K_{\mathrm{T}\phi}^{(n)} \\ K_{\phi\mathrm{T}}^{(n)} & K_{\phi\phi}^{(n)T} \end{pmatrix}, \quad C^T = \sum_{n=0} \alpha_s^{n} \begin{pmatrix} C_{T,\mathrm{q}}^{(n)} & C_{\phi,\mathrm{q}}^{(n)T} \\ C_{T,\mathrm{g}}^{(n)} & C_{\phi,\mathrm{g}}^{(n)T} \end{pmatrix}$$

- Analytic continuation *AC* of physical kernel elements provides off-diagonal splitting functions *AC* $\left[K^{(n)S}(x)\right] \;=\; K^{(n)T}(x)$ Almasy, S.M., Vogt '11
- Problem
  - physical kernel insufficient to fix unambigously $P_{\mathrm{qg}}^{(2)T}$
  - residual uncertainty proportional to color factor $\beta_0(C_A - C_F)$

$$\delta P_{\mathrm{qg}}^{(2)T} \;=\; \pm 2\,\zeta_2\,\beta_0(C_A - C_F)\,(11 + 24\,\ln x)\,P_{\mathrm{qg}}^{(0)T}(x)$$

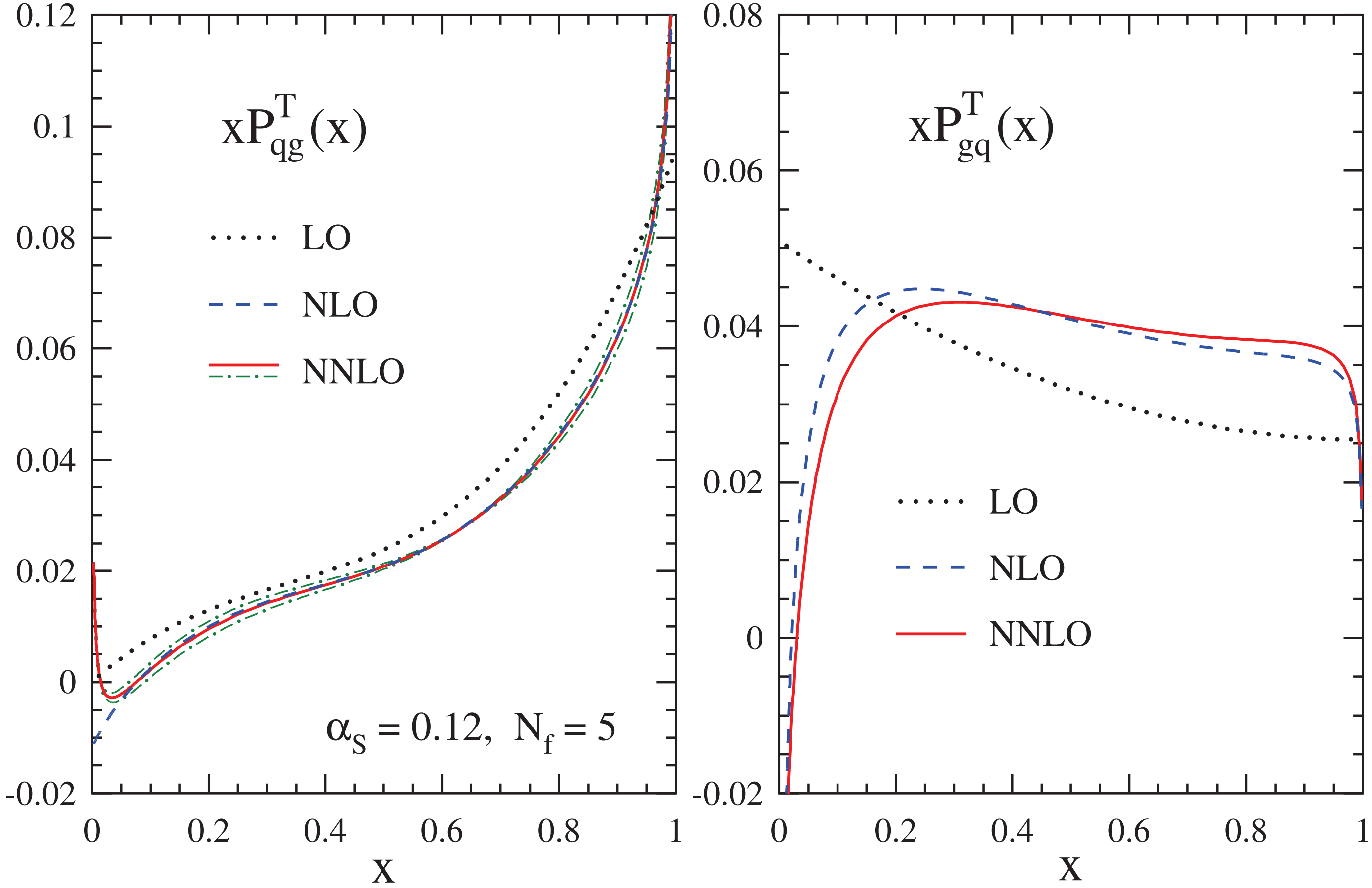
  - to be resolved by direct computation Gituliar, S.M. in progress

Diagonal time-like splitting functions



- Perturbative expansion of P^T_{qq} and P^T_{gg} (multiplied by $x\,(1-x)$)
  S.M., Vogt '07

Off-diagonal time-like splitting functions



- Perturbative expansion of P^T_{qg} and P^T_{gq} (multiplied by x)
  Almasy, S.M., Vogt '11
  - $P^{(2)T}_{\rm qg}$ contains residual uncertainty $\delta P^{(2)T}_{\rm qg}$

The small x-limit: $x \to 0$

- Structure of (diagonal) splitting functions at small x
  - double-logarithmic contributions with very large coefficients
  - huge enhancement already at $x \gtrsim 10^{-3}$

$$xP_{\rm qq}^{(2)T}(x) =$$

$$-\frac{32}{9}\, C_A C_F n_f \,(2 \ln^3 x + \ln^2 x) \;+\; \frac{8}{27}\,(155 + 72\,\zeta_2)\, C_A C_F n_f \;\ln x \;+\; \mathcal{O}(1)$$

$$xP_{\rm gg}^{(2)T}(x) =$$

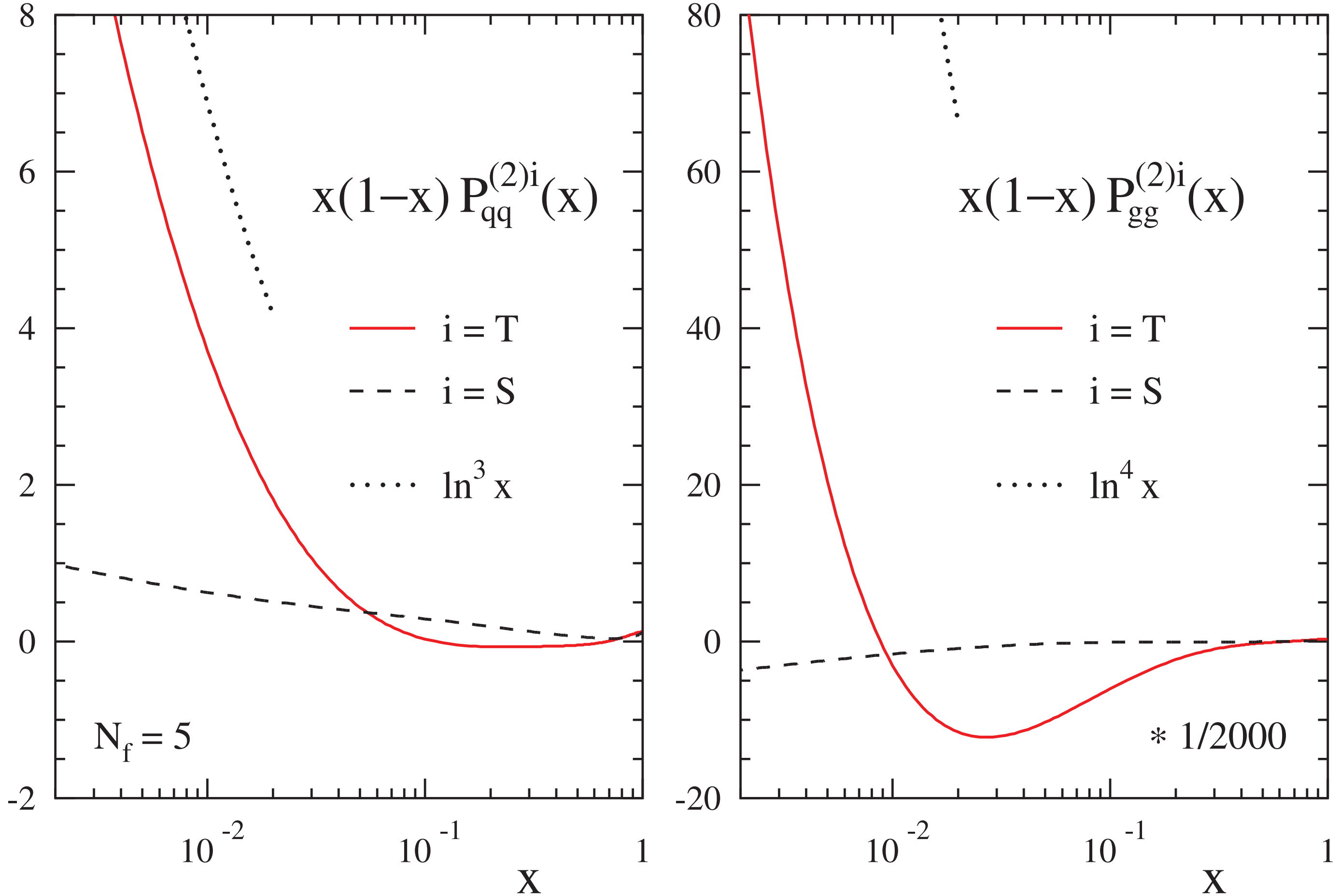
$$\frac{64}{3}\, C_A^3 \ln^4 x \;+\; \frac{32}{9}\,(33\, C_A t + 6\, C_A^2 n_f - 10\, C_A C_F n_f) \ln^3 x$$

$$+ \frac{8}{9}\left[(389 - 144\,\zeta_2)\, C_A^3 \;+\; 136\, C_A^2 n_f \;-\; 232\, C_A C_F n_f \;+\; 4\, n_f^2 (C_A - 2\, C_F)\right] \ln^2 x$$

$$+ \frac{8}{27}\Big[(4076 - 990\,\zeta_2 - 972\,\zeta_3)\, C_A^3 \;+\; (739 - 36\,\zeta_2)\, C_A^2 n_f$$

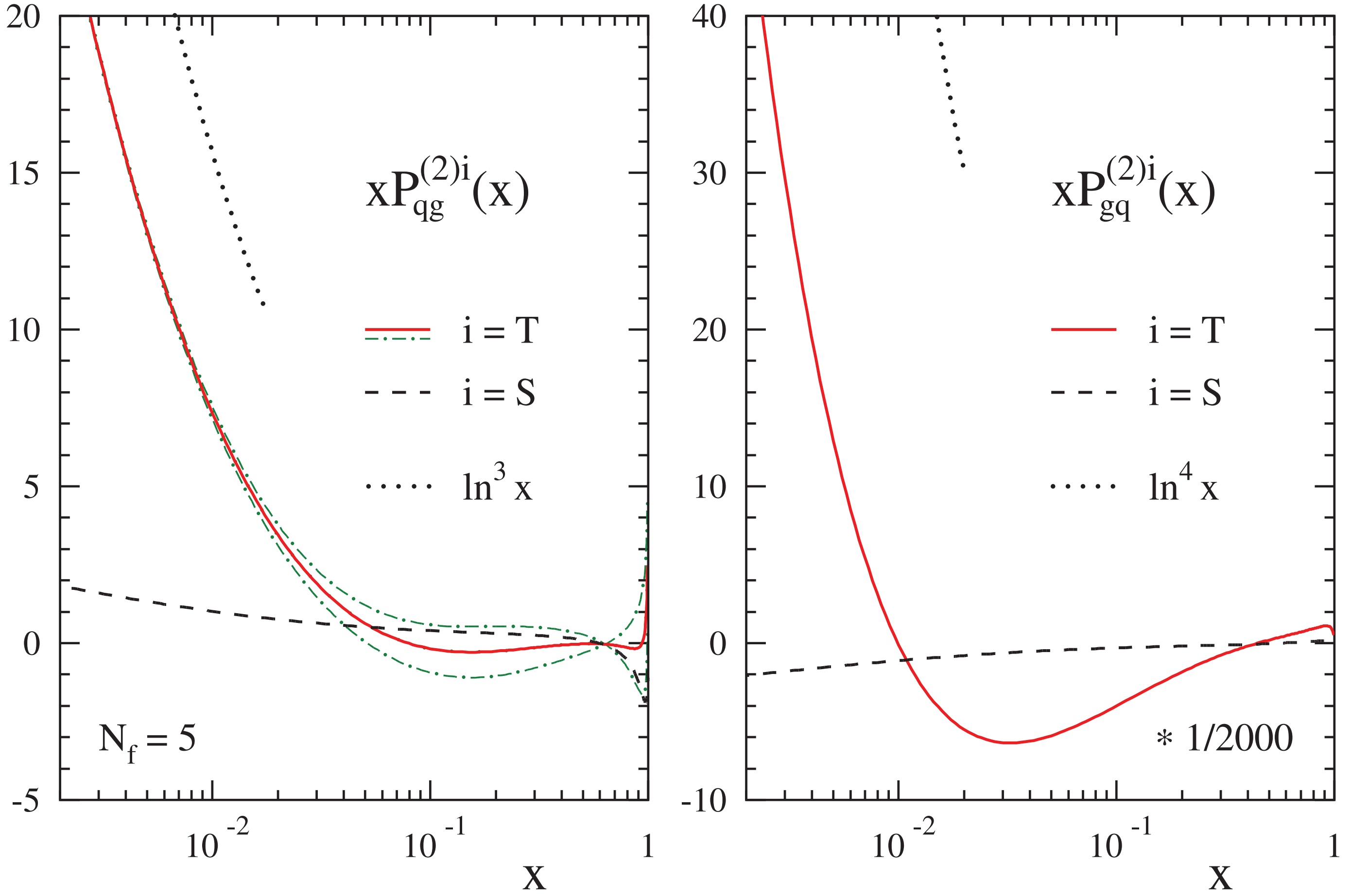
$$-\;(1819 - 144\,\zeta_2)\, C_A C_F n_f \;+\; 108\, C_F^2 n_f \;+\; 46\, n_f^2 (C_A - 2\, C_F)\Big] \ln x \;+\; \mathcal{O}(1)$$

$P_{qq}^{(2)T}$ and $P_{gg}^{(2)T}$ for $x \to 0$



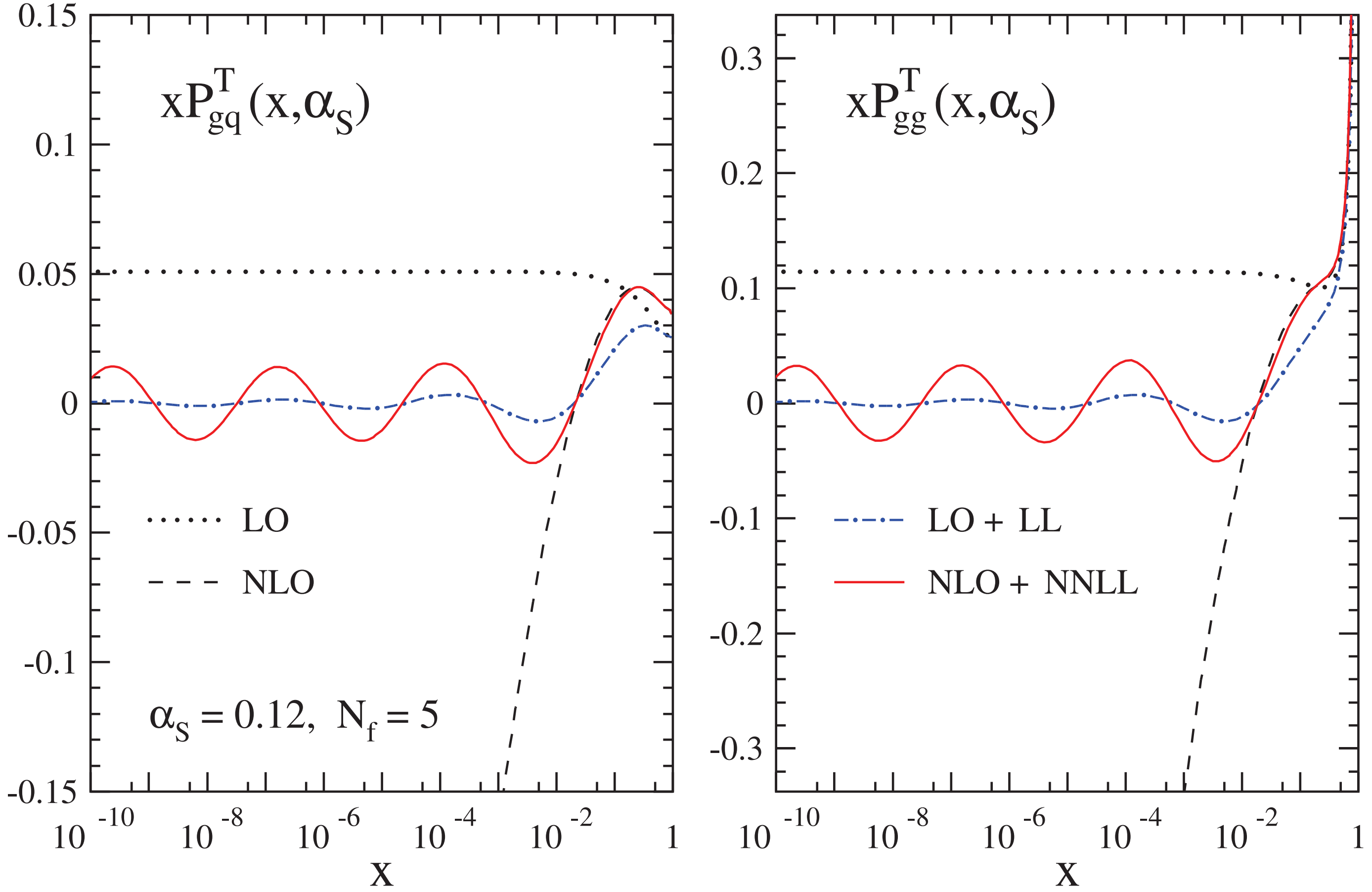
- Splitting function $P^{(2)}_{\rm qq}$ (left) and $P^{(2)}_{\rm gg}$ (right) S.M., Vogt '07
  - five flavours, multiplied by $x\,(1-x)$, divided by $2000 \simeq (4\pi)^3$
  - comparison with space-like splitting functions

$P_{gq}^{(2)T}$ and $P_{gq}^{(2)T}$ for $x \to 0$



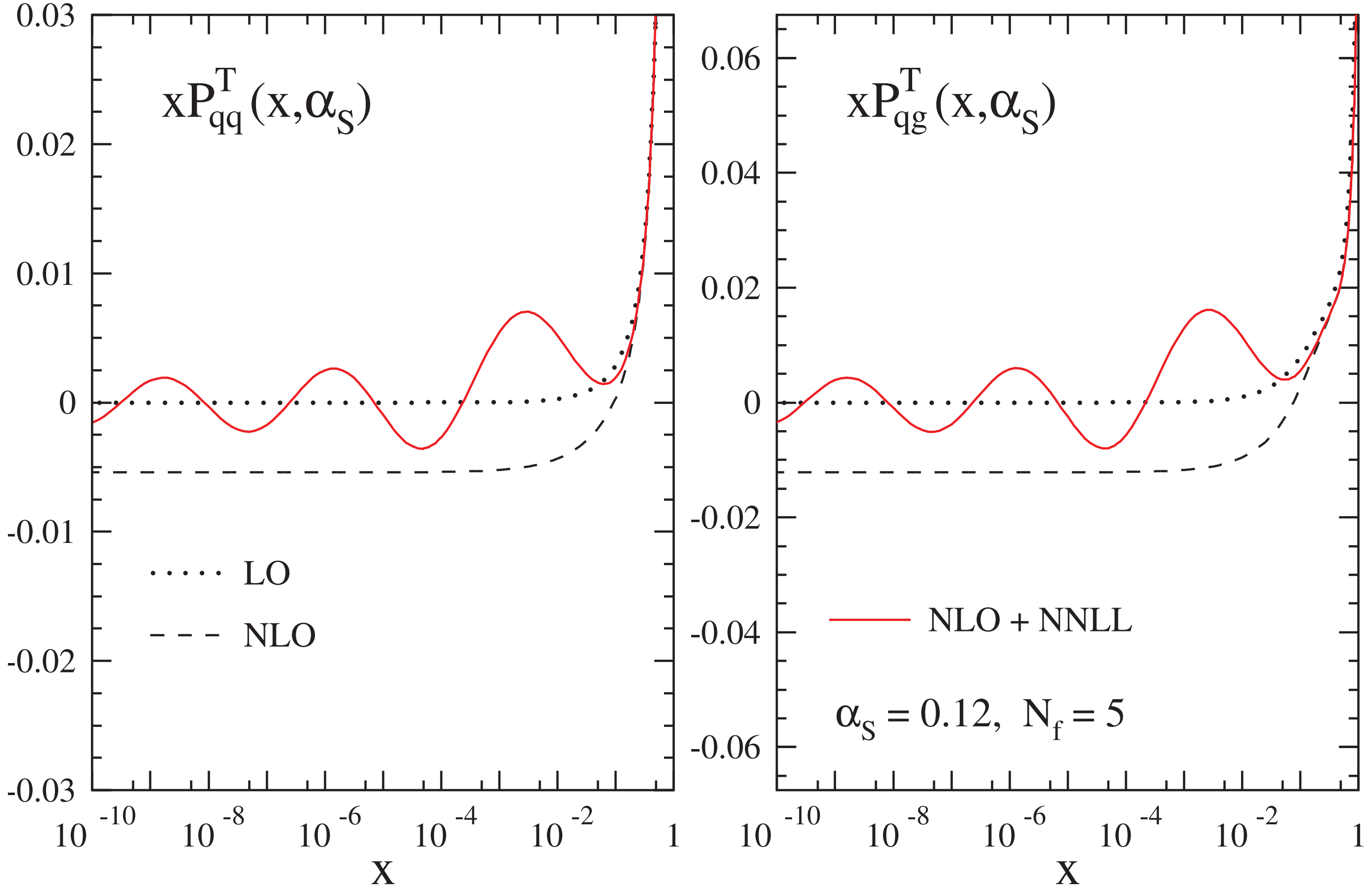
- Splitting function $P_{\rm qg}^{(2)}$ (left) and $P_{\rm gq}^{(2)}$ (right) Almasy, S.M., Vogt '11
  - five flavours, multiplied by x, divided by $2000 \simeq (4\pi)^3$
  - comparison with space-like splitting functions

Small-x resummation (I)



- Splitting function $P^{(2)}_{\rm gq}$ (left) and $P^{(2)}_{\rm gg}$ (right) Kom, Vogt, Yeates '12
  - all-x (minimal $N = 1$ finite) 'LO + resummed' and 'NLO + resummed' approximations
  - comparison with LO and NLO results (valid only at large $x \gtrsim 10^{-2}$)

Small-x resummation (II)



- Splitting function $P^{(2)}_{\mathrm{qq}}$ (left) and $P^{(2)}_{\mathrm{qg}}$ (right) Kom, Vogt, Yeates '12
  - all-x (minimal $N = 1$ finite) 'NLO $+$ resummed' approximations
  - LO contributions does not include $1/x$ terms; resummation starts at NLL level
  - comparison with LO and NLO results (valid only at large $x \gtrsim 10^{-2}$)

Summary

Mapping DIS to e^+e^- annihilation

- Theory results transferred to time-like evolution
  - successful recycling of DIS
  - use factorization, infrared safety (KLN) of observables and scheme invariance of physical kernels
- Theory predictions for time-like singlet splitting functions at NNLO
  - complete: $P_{qq}^{(2)T}$, $P_{gg}^{(2)T}$, $P_{gq}^{(2)T}$
  - almost complete: $P_{qg}^{(2)T}$

Outlook

- Complete derivation of $P_{qg}^{(2)T}$
- Improvement of small-x behaviour through resummation