Unreachable points in cosmological evolution in polynomial $f(R)$ gravity

References:

- JCAP 1602 (2016) no.02, 030
- JCAP 1410 (2014) no.10, 009

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Plan....

- $f(R)$ equations...
- Cosmological properties near the solutions of $f''(R) = 0$.
- Conclusion.

The $f(R)$ equations...

- In general instead of only the Ricci scalar, R, the Gravitational action fas a general function $f(R)$
- Using FLRW metric $ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right]$, the cosmological equations become,

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\begin{aligned}
3H^2 &= \frac{\kappa}{F(R)}\rho_{\rm eff}\ , \\
3H^2 + 2\dot{H} &= \frac{-\kappa}{F(R)}P_{\rm eff}\ ,
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- The curvature induced $\rho_{\rm curv}$ and $P_{\rm curv}$ are:

$$\rho_{\rm curv} \equiv \frac{RF - f}{2\kappa} - \frac{3H\dot{R}F'}{\kappa}\,, \quad P_{\rm curv} \equiv \frac{\dot{R}^2F'' + 2H\dot{R}F' + \ddot{R}F'}{\kappa} - \frac{RF - f}{2\kappa}\,.$$

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- The talk will address the cosmological phase when

$$F'(R = R_c) = 0\,.$$

  These particular phase shows interesting behavior in $f(R)$ theories.

Polynomial $f(R)$ theories and their properties

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- On the other hand suppose $f(R)$ is such an odd order polynomial so that the even order polynomial $F(R)$ has no real roots and the condition $F(R) > 0$ holds for all R. But then $F'(R)$ is an odd order polynomial and has at least one real root and consequently one cannot have $F'(R) > 0$ for all R.

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- We will see the that the value of R_c, for which $F'(R = R_c) = 0$, separates one region of cosmological existence from another different region.

The variables near $R = R_c$

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$$f(R) = f(R_c) + F(R_c)\,\delta R \, ,$$

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- Using this assumptions the field equations, up to first order of the fluctuations, near $R = R_c$ are.....

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- If at $t = 0$ the system has $R \neq R_c$ while $\omega = -1$ (and $\rho = \mathrm{constant}$), or $\rho = 0$ that initial state can reach the (R_c, H_c, ρ_c) point and get stuck, as it again hits a de Sitter like phase where $\dot{H} = 0$.

Cosmological behavior near $R = R_c$

- On the other hand if at $t = 0$ the system has $R \neq R_c$ while $\omega \neq -1$ and $\rho \neq 0$ then one can show that the system will require an infinite time to reach the (R_c, H_c, ρ_c) point. The reason is as follows.

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- Consequently, any arbitrary cosmological system will not be able to cross the (R_c, H_c, ρ_c) point in finite time.

Conclusion

- There are some configurations in cosmological space which are unreachable from other possible initial cosmological set up, when cosmological evolution is guided by $f(R)$ theories of gravity. Regions where $f''(R) > 0$ will be unreachable if initially one starts in an universe where $f''(R) < 0$. The Ricci scalar R_c for which $f''(R_c) = 0$ separates the reachable and unreachable points.

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- Many theories of $f(R)$ gravity does not at all possess this kind of a configuration as $f(R) = R + \alpha R^2$ or $f(R) = \frac{1}{\alpha}\left[1 - \exp(\alpha R)\right]$. On the other hand most polynomial $f(R)$ theories whose order is cubic or more must have these kind of interesting configurations.

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- The possible roots of the equation $f''(R) = 0$ partitions the probable regions of cosmological existence in cubic or higher order polynomial theories of gravity.