# THE THEORY OF DYNAMICAL FRICTION

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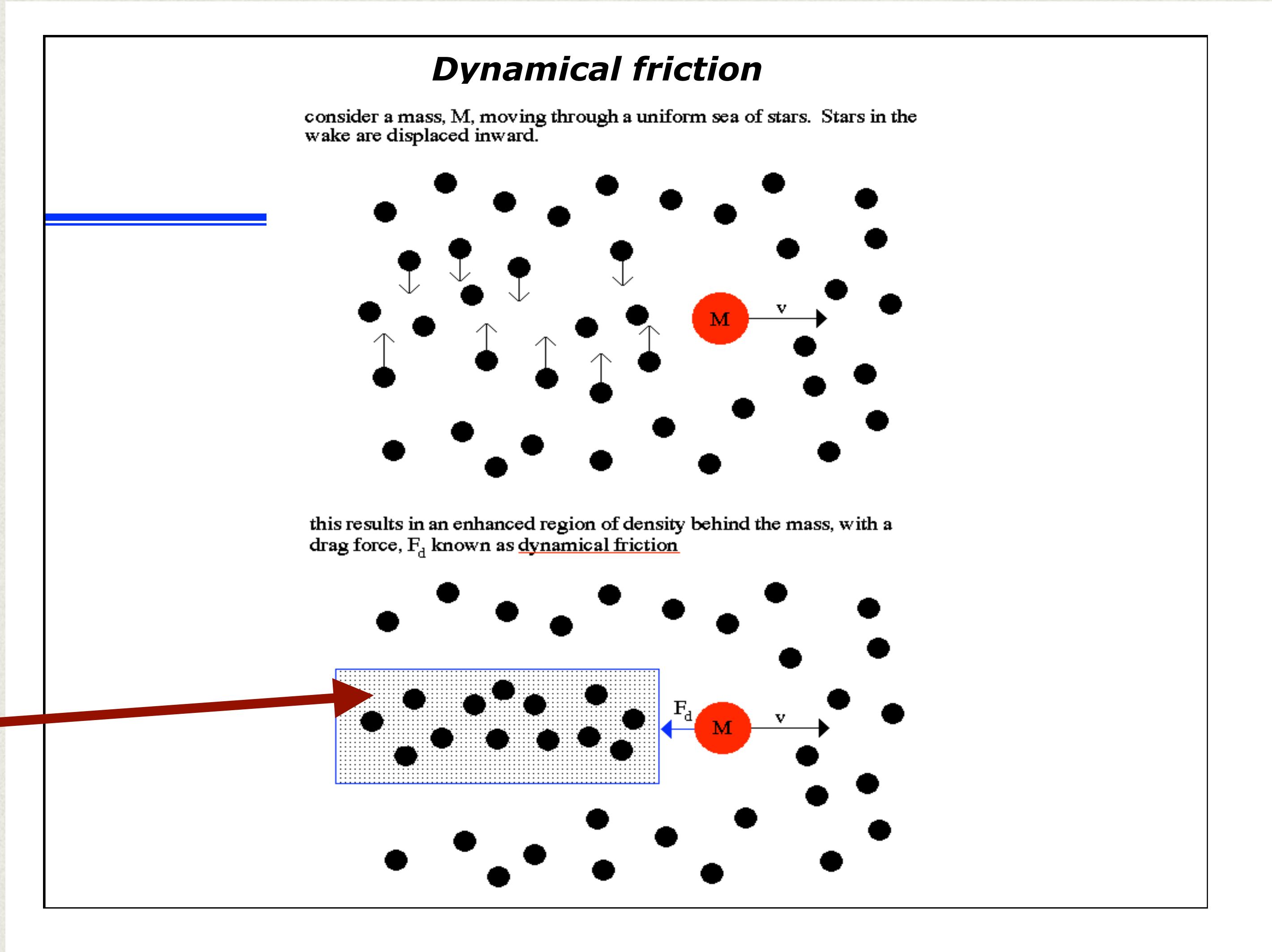
# OUTLINE

- Motivation: why is this important? What do we already know?
- Description of the formalism
- Calculation of the drag force:
  - ideal gas: recover the known cases, generic expression for arbitrary distribution and arbitrary "bullet" velocity
  - recovery of the ideal liquid results
  - viscous liquids
  - generic results of interacting gases in the relaxation time approximation, interpolation between the gas and the liquid
- Comments on the boundary conditions effects
- Outlook

# WHAT IS DYNAMICAL FRICTION?

The problem is not new:

A massive particles propagates through a medium with a constant velocity. **What is the gravitational drag force, induced by the wake?**




*First analysis: Chandrasekhar, 1942*

# DYNAMICAL FRICTION AND MODERN HEP

- Traditionally: used to calculate motion of bodies in the galactic dynamics — with classical NR results being perfectly adequate
- More modern questions: effects of self-interacting DM (component) on the galactic dynamics — more complicated, self-interactions effects should be taken into account
- Recently: calculate the the energy released by (primordial) black holes going through neutron stars and white dwarf — effects of matter in extreme conditions must be properly accounted for
- Exotic objects in the accretion discs of the black holes

# WHAT DO WE KNOW ABOUT THE DYNAMICAL FRICTION?

Ideal non-relativistic gas

$$F = \frac{4\pi G^2 m_b^2 m_p}{v^2} \int_0^v dv' v'^2 f(v') \ \log\Lambda$$

Photons gas, relativistic bullet

$$F = \frac{64\pi}{3} G^2 m_b^2 \rho v \gamma^2 \log\Lambda \ .$$

How do we interpolate between two these regimes?

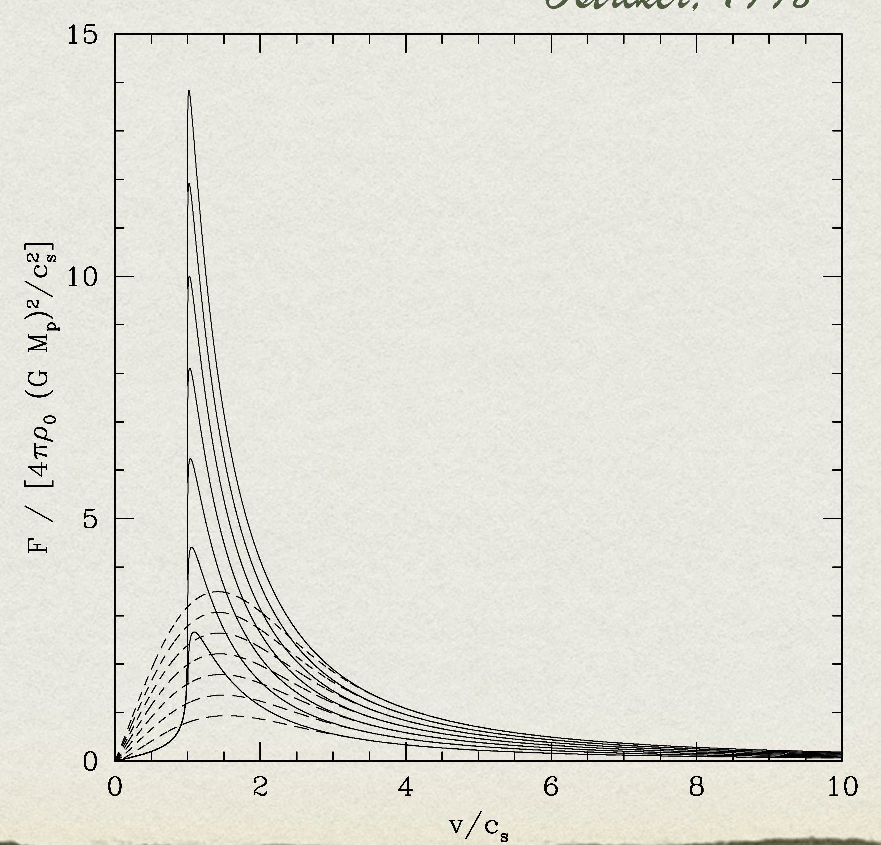
Ideal liquids:

Supersonic:

$$F = \frac{4\pi m^2 G^2 \gamma^2 (1+v^2)^2 (\rho_0 + P_0)}{v^2} \log\Lambda \ \theta(v - c_s)$$

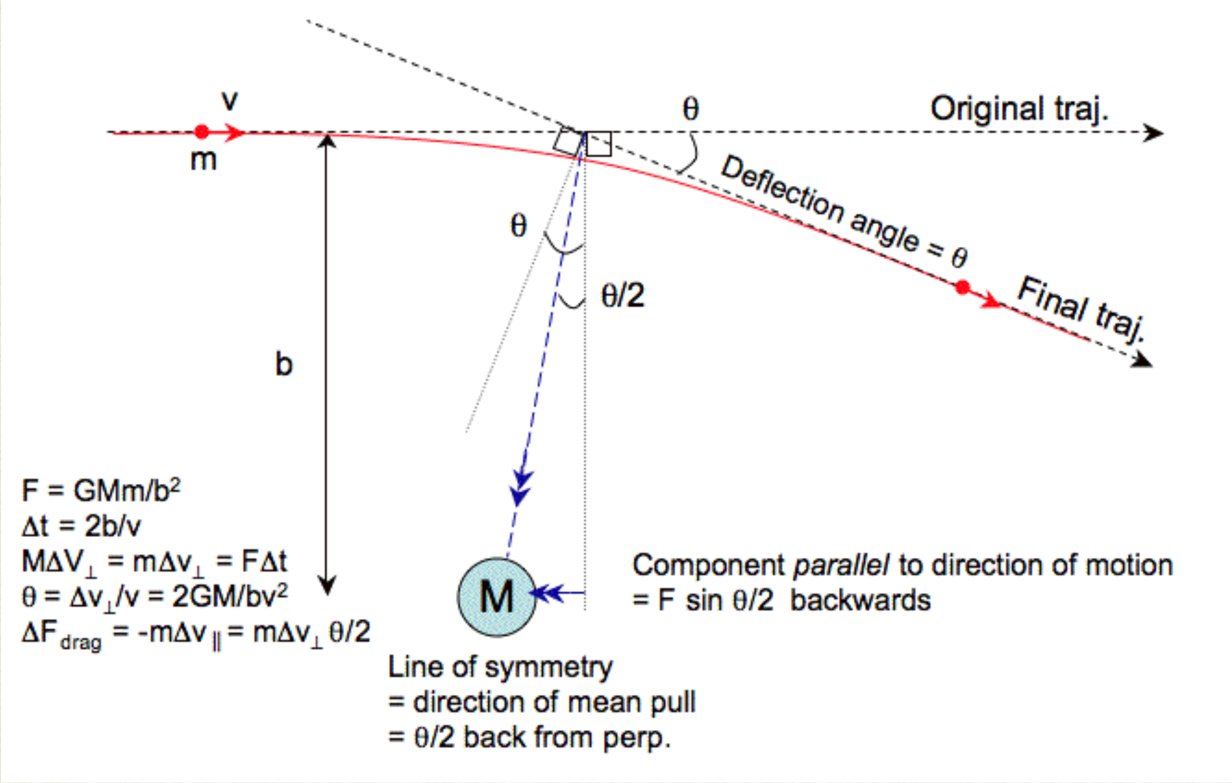
Vanishes subsonically, though certain boundary effects can change this conclusion

*Ostriker: 1998*



# CALCULATION OF DYNAMICAL FRICTION — HISTORICAL APPROACH

Chandrasekhar's approach: calculate the probability function of occurrence of force *F* acting on a star and an average time during which the force acts




In the case of the ideal fluid (Rephaeli & Salpeter, Ostriker) one calculates the drag due to the asymmetry in the flow associated with the front of the shock wave.

# A NEW APPROACH

(Thermal) field theory approach + Gravity as an EFT

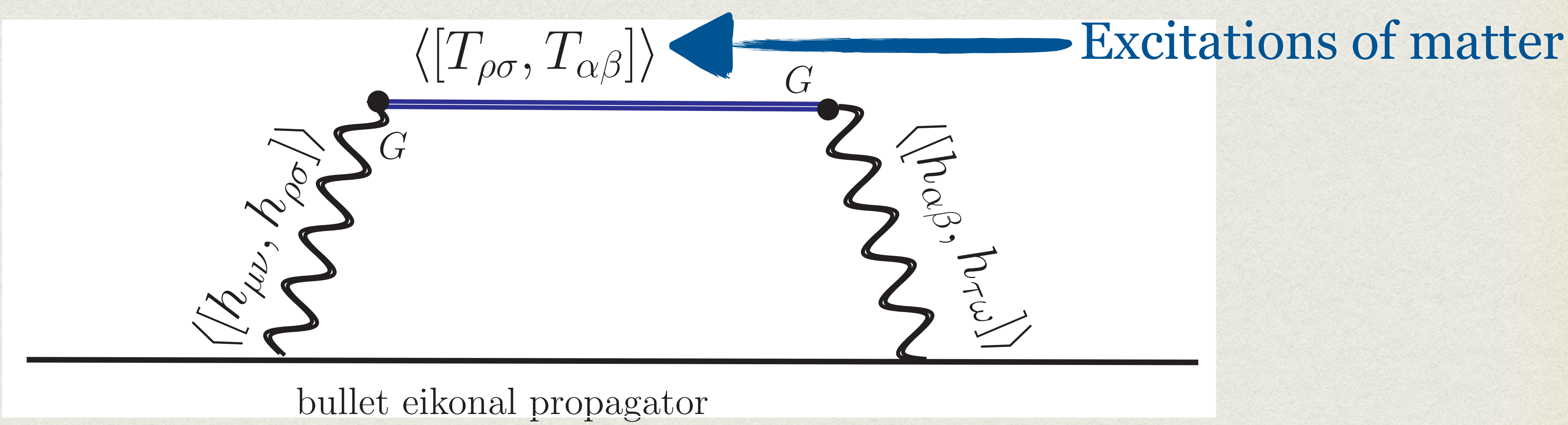
Definition of force induced by the wake:

$$F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}t} = -m\gamma\Gamma^\mu{}_{\alpha\beta}(h^{wake})v^\alpha v^\beta$$

The graviton field of the wake (Fourier space):

$$h^{wake}_{\mu\nu}(\omega,k) = G^{dressed}_{\mu\nu,\alpha\beta}(\omega,k)T^{\alpha\beta}_{bullet}(\omega,k)$$

Full object to calculate



# FORMAL EXPRESSION FOR THE FORCE

Let us first assume that we know the excitation of matter.

The energy-momentum tensor of the bullet:

$$T^{\mu\nu}_{bullet} = \frac{m_{bullet}}{\gamma} u^\mu u^\nu \delta(z - vt)\delta^2(x_\perp) \quad \Longrightarrow \quad \tilde{T}^{\mu\nu}_{bullet} = \frac{2\pi m_{bullet}}{\gamma} u^\mu u^\nu \delta(\omega - k^z v)$$

From here we can use the standard machinery: calculate the dressed propagator, convolve with the energy momentum tensor of the bullet and further the Christoffel symbols.

$$F^z = -\frac{m^2}{2\gamma^2} \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \frac{\mathrm{d}k_z}{2\pi} k_z \mathrm{Im}\left[G^{dressed}\right] \qquad G^{dressed} \equiv \left[G^{dressed}_{\mu\nu,\alpha\beta}\right] u^\alpha u^\beta u^\mu u^\nu$$

Qualities of the perturber and the matter perfectly factorize

Structure one could predict from optical theorem

# RECIPE: HOW TO CALCULATE THE DRAG FORCE IN ANY REGIME

- Find an expression for the excitation of matter: particle retarded propagator in the case of the ideal gas, sound modes for the ideal fluid in ideal liquid…
- Use it to calculate the retarded propagator of the dressed graviton
- Contract with the four-velocity of the bullet
- Calculate the drag force as the Fourier transform of the imaginary part of the contracted dressed graviton

# SCALES OF THE PROBLEM

Relevant scales:

- $R_{IR}$ — the largest relevant distance, the size of the cloud
- $R_{UV}$ — the smallest distance, the size of the bullet (Schwarzschild radius)
- The mean free path — at this scale the liquid EFT starts breaking down, must be replaced by full kinetic theory

What is the k-dependence in various regimes?

General dimensional considerations: $F \propto m^2 G^2$

*Comes from general expression* → $m^2$

*Gravitational coupling² from the dressed propagator* → $G^2$

# STRUCTURE OF THE DRAG FORCE EXPRESSION

- Lowest frequencies — largest wavelengths: ideal liquid regime

$$T_{\mu\nu} = (\rho_0 + P_0) u_\mu u_\nu + P_0 g_{\mu\nu} \quad \Longrightarrow \quad F \propto (\rho_0 + P_0) G^2 m^2$$

*Already has dimensions of force. Namely the force is k-finite or ∝ log k. Since gravity is a long-range force, we would expect a divergence log k.*

- Introduce first order corrections: viscous liquid.

$$\delta T_{\mu\nu} = -2\eta \sigma_{\mu\nu} - \zeta \nabla_\alpha u^\alpha \triangle_{\mu\nu}$$

*Both have one derivative — viscosities are dim. 3*

*From dimensional analysis considerations:*

$$\delta F \propto m^2 G^2 \, \eta_s \, k$$

# STRUCTURE OF THE DRAG FORCE EXPRESSION

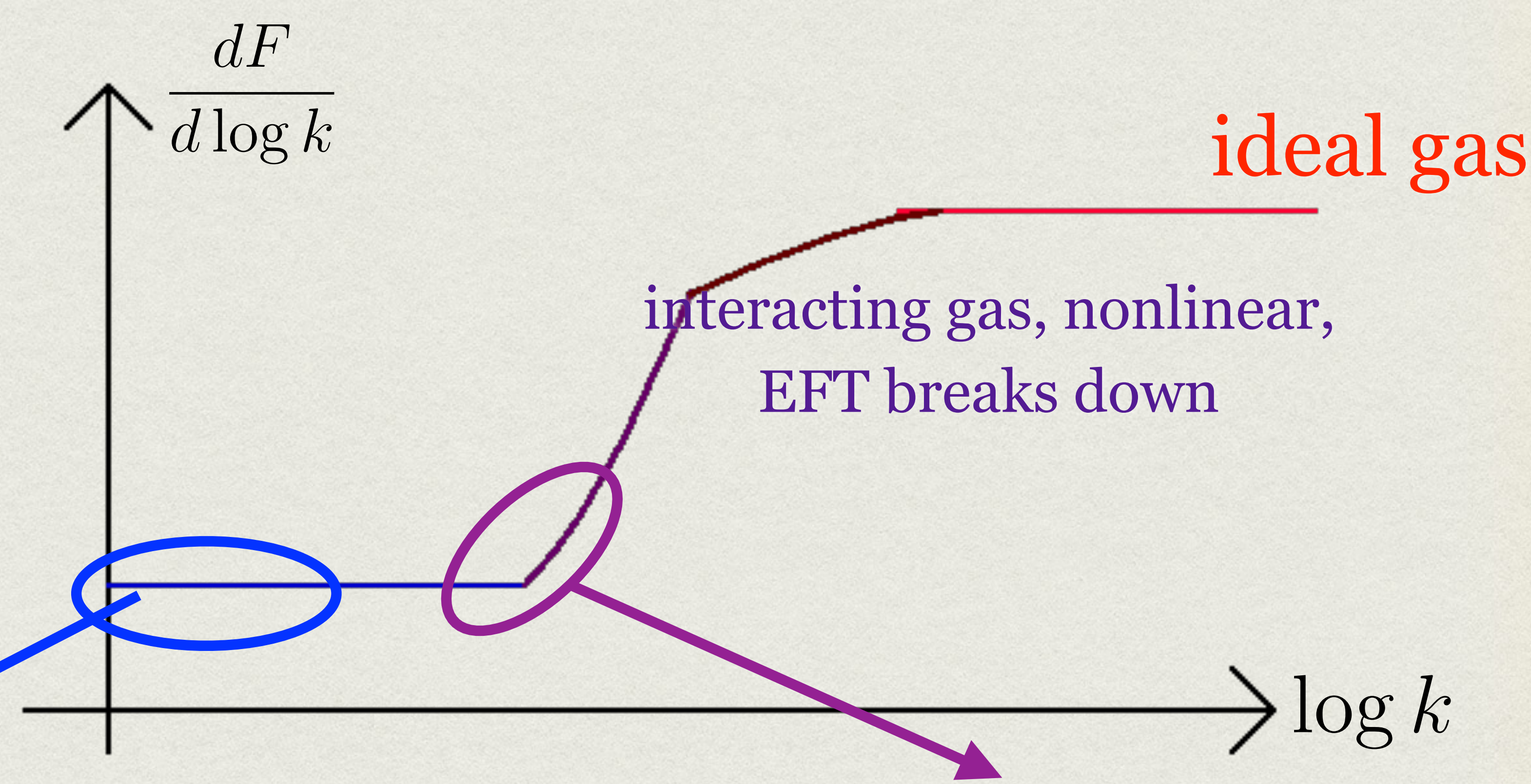
The viscous fluid EFT breaks down at the scale of MFP: $F \sim m^2 G^2 (\eta_s k + a_2 k^2 + a_3 k^3 + \ldots)$

The full theory should give a finite result

Another limit: very high frequencies, ideal gas. The only parameter again is the energy density, we expect:

$$F \propto m^2 G^2 \rho_0 \log k$$

Viscous fluid is an EFT, where we consider the first order corrections to the low-k approximation. We also know the high-k limit (ideal gas). Naive interpolation between them:



# THE KINETIC THEORY

To capture the effects of the interactions between the gas particles, use relaxation time approximation. $\tau$ is the time scale of the interactions

Boltzmann equation:

$$p^{\mu}\partial_{\mu}f - \Gamma^{\alpha}{}_{\beta\gamma}p^{\gamma}p^{\gamma}\nabla^{(p)}_{\alpha}f = \frac{p^{\alpha}u_{\alpha}}{\tau}(f - f_{eq})$$

*interaction kernel, with $\tau$ is the interactions time scale*

- Ideal liquid regime: (k$\tau$) << 1
- viscosities: ~$\tau$
- Interacting gas — the "transitional regime": (k$\tau$) ~ O(1)

limiting regimes quantified

# MATTER EXCITATIONS AND KINETIC THEORY

Machinery: how do we solve the kinetic theory (technical)

Basic assumptions: $f_{eq}$ is isotropic, depends only on $p^0$

Formal solution:

$$\delta f = \frac{1}{p^0}\frac{\delta f_{eq}/\tau + \Gamma^{\alpha}{}_{\beta\gamma} p^{\beta} p^{\gamma} \nabla^{(p)}_{\alpha} f_0}{-i\omega + i\vec{v}_p \cdot \vec{k} + 1/\tau}$$

*Energy momentum tensor*

$$\delta T^{\mu\nu} = \int \frac{d^3 p}{(2\pi)^3}\frac{p^{\mu} p^{\nu}}{p^0}\delta f$$

*Matter excitation (retarded correlator of energy momentum tensor):*

$$\left\langle [T^{\alpha\beta}, T^{\mu\nu}] \right\rangle = \frac{\delta T^{\alpha\beta}}{\delta h_{\mu\nu}} + \text{contact terms}$$

*analytic in $k$, $\omega$*

# WARM UP: IDEAL GAS

The formal solution is much simpler than in the generic case:

$$\delta f = \frac{1}{p^0}\frac{\Gamma^i{}_{\beta\gamma}p^\beta p^\gamma \nabla_i^{(p)} f_0}{-i\omega + i\vec{v}_p\cdot\vec{k}}$$

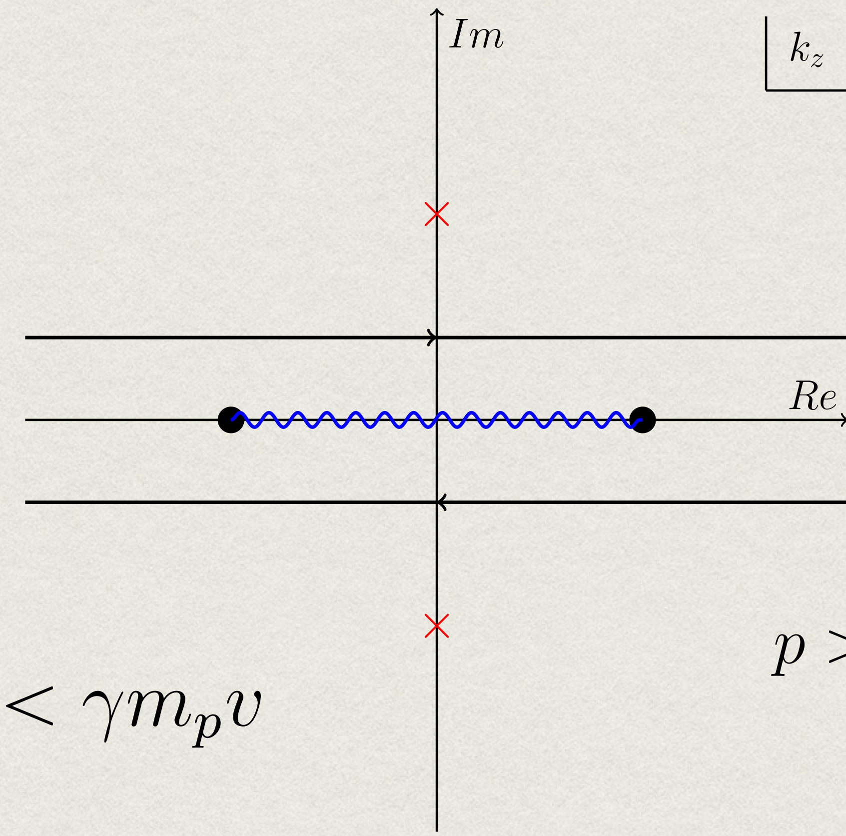
$$\Im \int k_z G^{dressed} dk_z = \Im \int k_z dk_z \int dp \frac{f'(p^0) p \gamma^4}{240\pi^2 (k_\perp^2 + k_z^2)^4 (p^0)^2 (k_\perp^2 - k_z^2(-1+v^2))^2} \times \left( R(k_z, k_\perp; p, p^0; v) + S(k_z, k_\perp; p, p^0; v) \log \frac{k_z p^0 v + \sqrt{k_z^2 + k_\perp^2}\, p}{k_z p^0 v - \sqrt{k_z^2 + k_\perp^2}\, p} \right)$$

*two completely analytic functions of $k_z$*

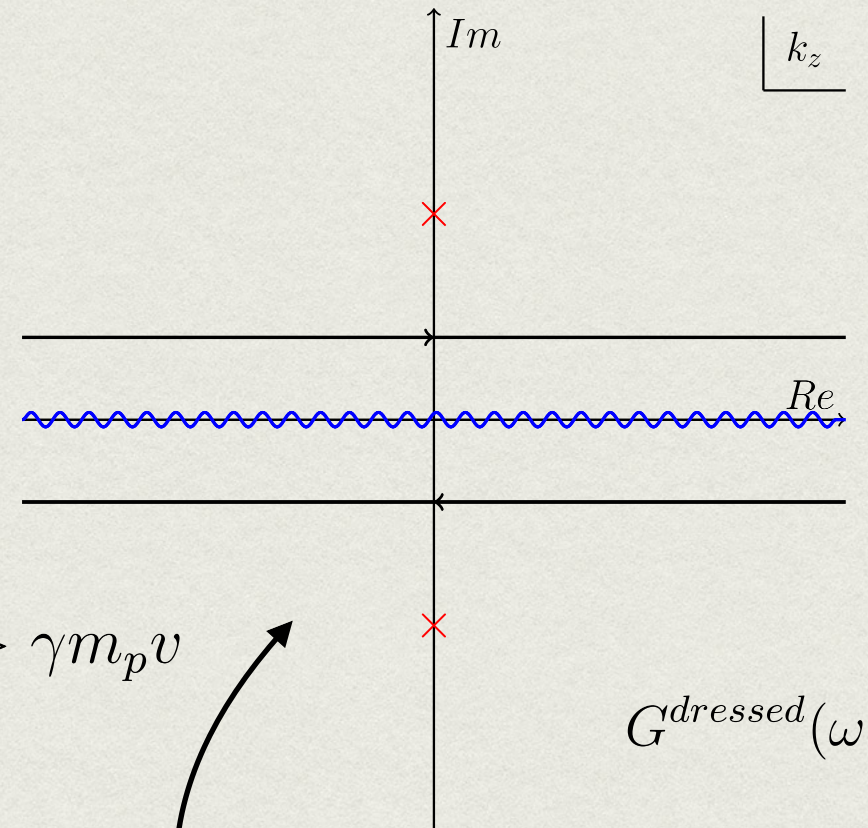
*contributes to the branch cut along the real axis*

For sufficiently big $p > \gamma m_p v$ the log is always negative and the entire imaginary part comes from the jump over the branch cut

# IDEAL GAS: ANALYTIC STRUCTURE OF THE DRESSED PROPAGATOR




$p < \gamma m_p v$




$p > \gamma m_p v$

without any dissipation:

$$G^{dressed}(\omega + i\epsilon) = G^{*dressed}(\omega - i\epsilon)$$

$$-\frac{k_\perp p}{\sqrt{p^2(v^2-1)+v^2 m_g^2}} < k_z < -\frac{k_\perp p}{\sqrt{p^2(v^2-1)+v^2 m_g^2}}$$

***Compare to the known results***

photons gas:

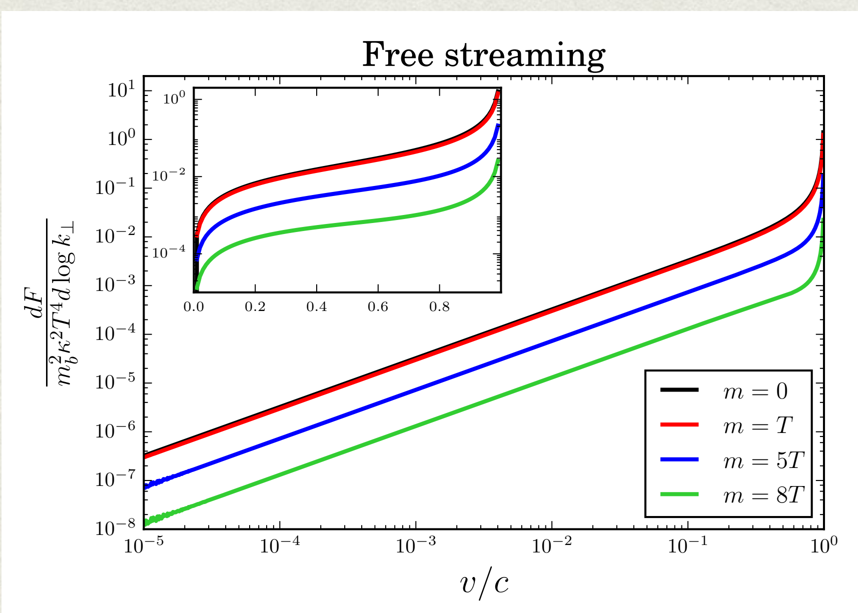
$$F = \frac{64\pi}{3} G^2 m_b^2 \rho v \gamma^2 \log\Lambda \ .$$

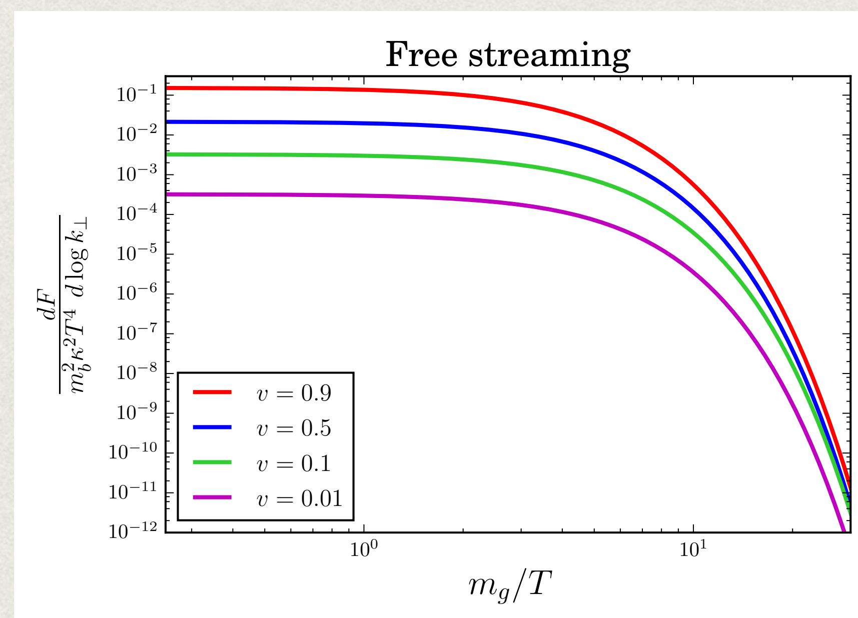
Chandrasekhar:

$$F = \frac{4\pi G^2 m_b^2 m_p}{v^2} \int_0^v dv' v'^2 f(v') \ \log\Lambda$$

# IDEAL GAS: GENERIC SOLUTION

Because of the analytic structure of the $k_z$ integral, the generic solution breaks down into two integrals over p, one of them running from $p=0$ to $p = \gamma m_p v$, and another running from $p = \gamma m_p v$ to infinity. The integrands are continuous at the stitching point.




The known limits (Chandrasekhar+photons gas) are easily reproduced from the general expression

# INVISCID FLUID

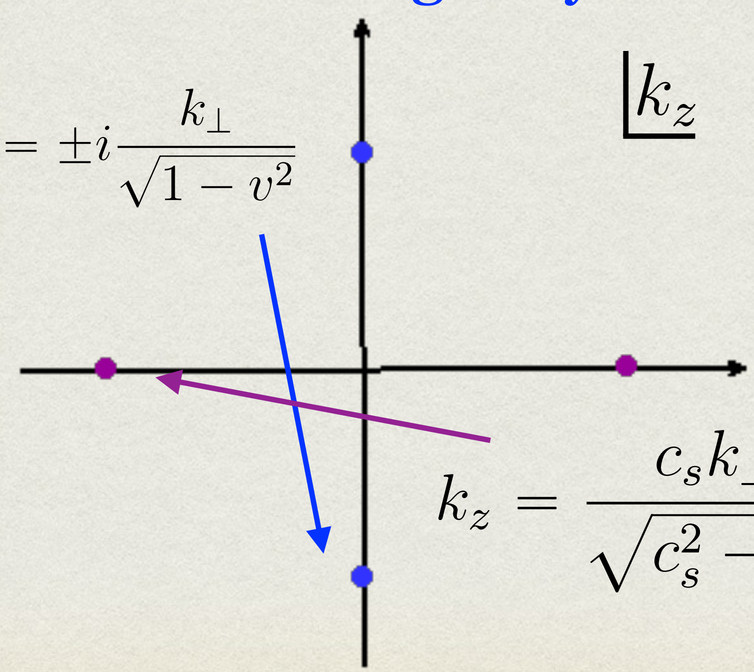
Here the energy momentum tensor and the propagators are well known. Dressed propagator:

$$G^{dressed}(\omega = vk^z + i\epsilon) = \left(\frac{\kappa\gamma^2}{(1-v^2)k_z^2 + \vec{k}_\perp^2}\right)^2 \frac{f^{\text{ideal}}_{\text{sound}}(\vec{k})}{(c_s^2 - v^2)k_z^2 + c_s^2\vec{k}_\perp^2 - i\epsilon v k_z}.$$

all the gravitational poles are on imaginary axis

sound poles: on the imaginary axis for subsonic velocities, real otherwise. The virtual phonons become real

$$k_z = \pm i\frac{k_\perp}{\sqrt{1-v^2}}$$



$$k_z$$

$$k_z = \frac{c_s k_\perp}{\sqrt{c_s^2 - v^2}}$$

As in the ideal gas case, without dissipation there is no contribution from the gravity poles

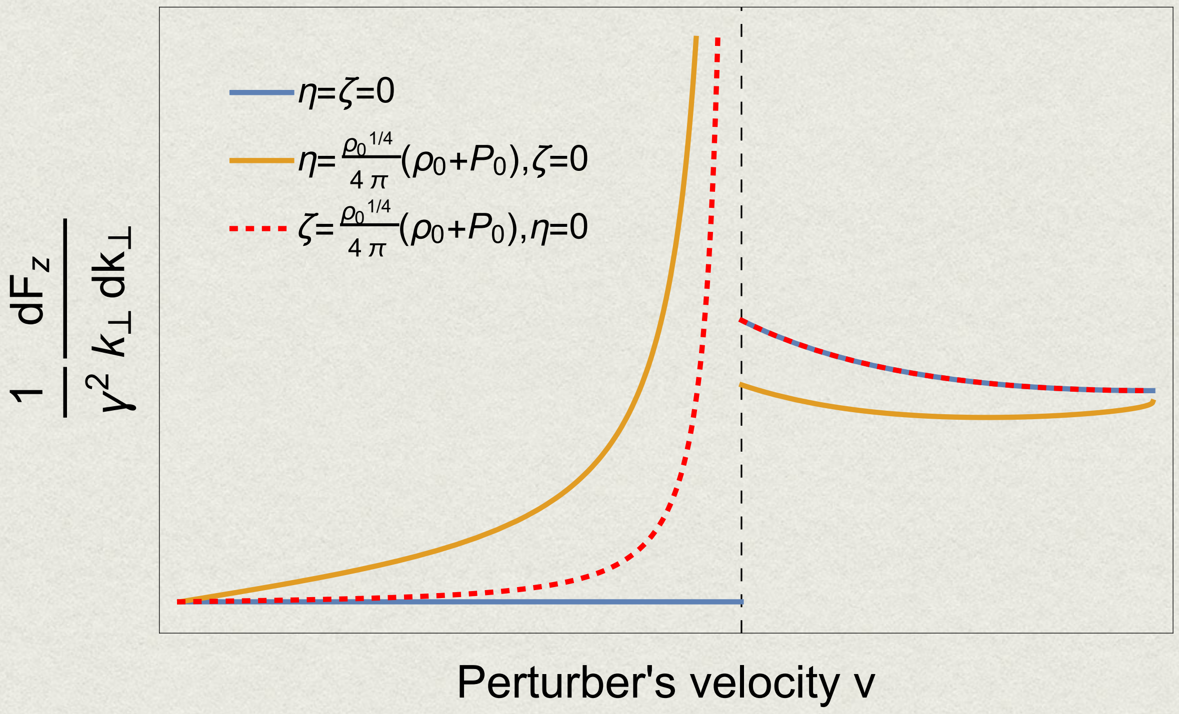
Subsonic bullet — no poles on the real axis, integral is perfectly real, confirming the result of Rephaeli-Slapeter

# MORE REALISTIC PICTURE: ADD THE VISCOSITY

In the system with dissipations the calculation is more involved and we cannot disregard the analytic structure outside the real axis.

$$G^{dressed}(\omega + i\epsilon) \neq G^{*dressed}(\omega - i\epsilon)$$

The viscosity contributions blow up at the speed of sound. The results are also linearly sensitive to the cutoff of the EFT



# SOLVE THE FULL KINETIC THEORY

Formal solution:

$$\delta f = \frac{1}{p^0} \frac{\delta f_{eq}/\tau + \Gamma^{\alpha}{}_{\beta\gamma} p^\beta p^\gamma \nabla_\alpha^{(p)} f_0}{-i\omega + i\vec{v}_p \cdot \vec{k} + 1/\tau}$$

$$\delta f_{eq} = f_0(p^0) \frac{p^0}{T} \left( \vec{v}_p \cdot \delta\vec{u} + \frac{\delta T}{T} \right)$$

*Here closely following Romatschke, 1512.02641*

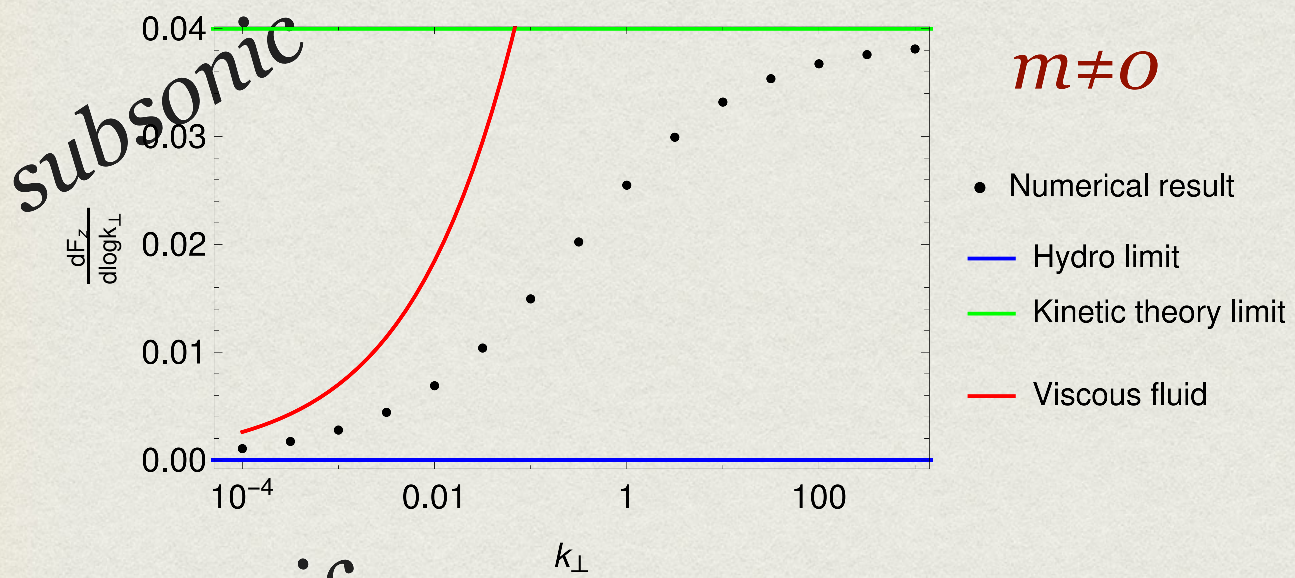
We have 4 unknown variables: $\delta$T, and three components of $\delta$u. Plugging this back into the expression for energy momentum tensor we have 4 different equations to solve:

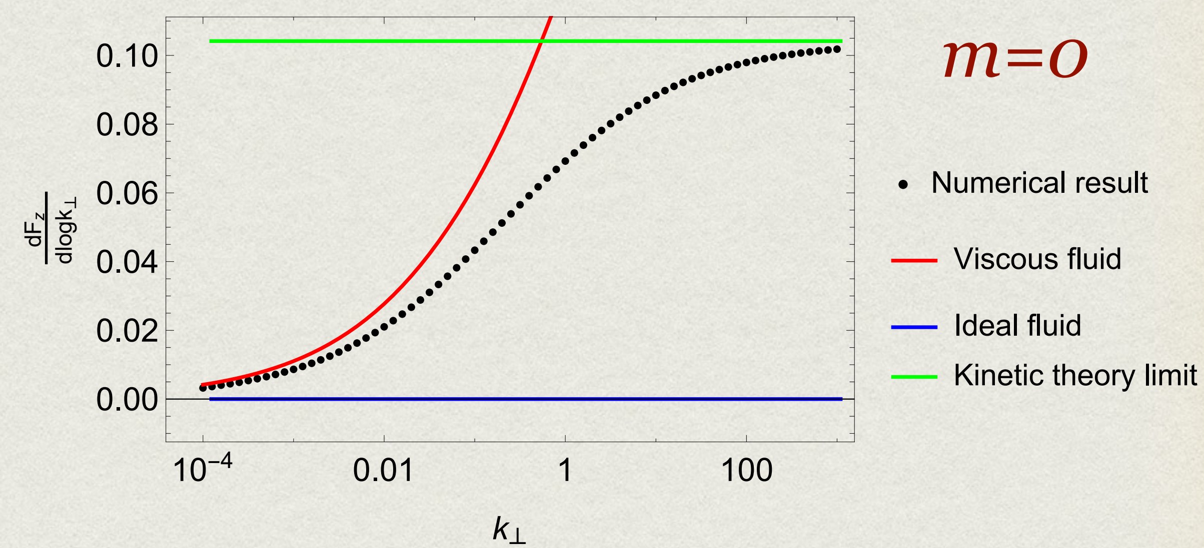
$$\delta\rho = \delta T^{00} \qquad \delta u^i = \frac{\delta T^{0i}}{\rho + P}$$

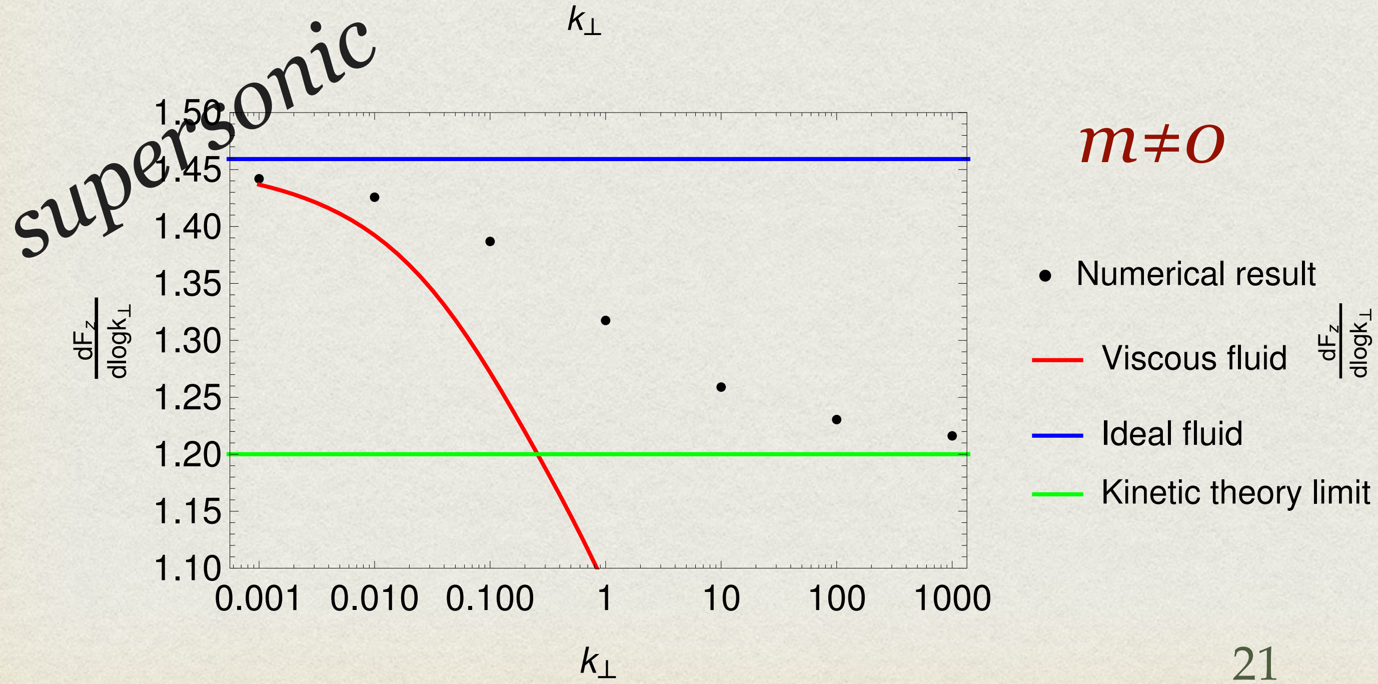
Use the unperturbed *f* to calculate the energy density

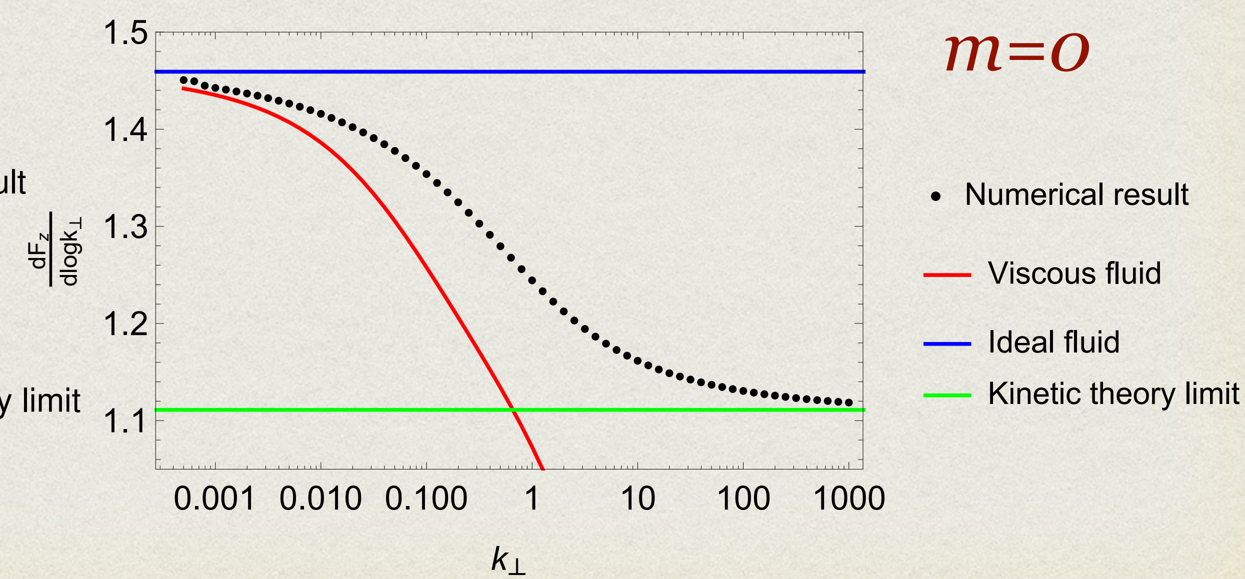
# RESULTS FROM THE FULL KINETIC THEORY

## Smooth interpolation:



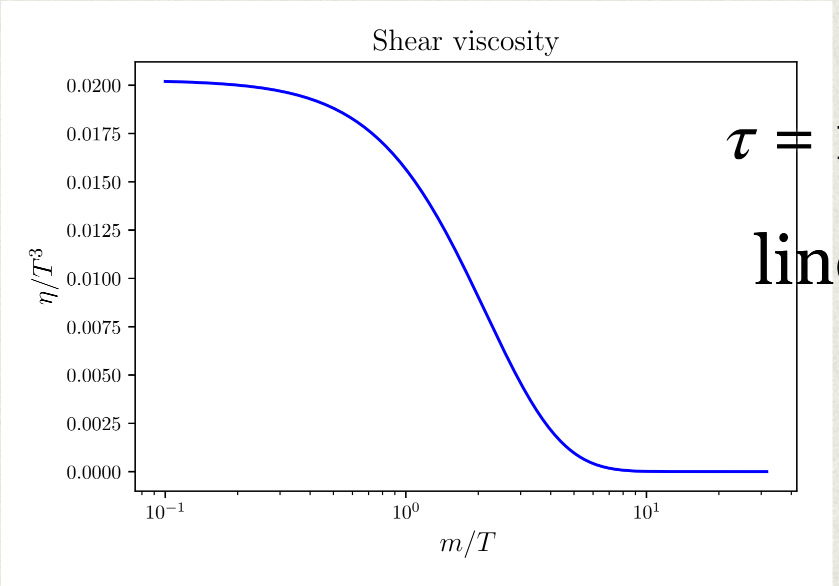



# MAPPING THE HYDRO EFT ONTO THE FULL THEORY

## The viscosities can be extracted from Kubo relations:

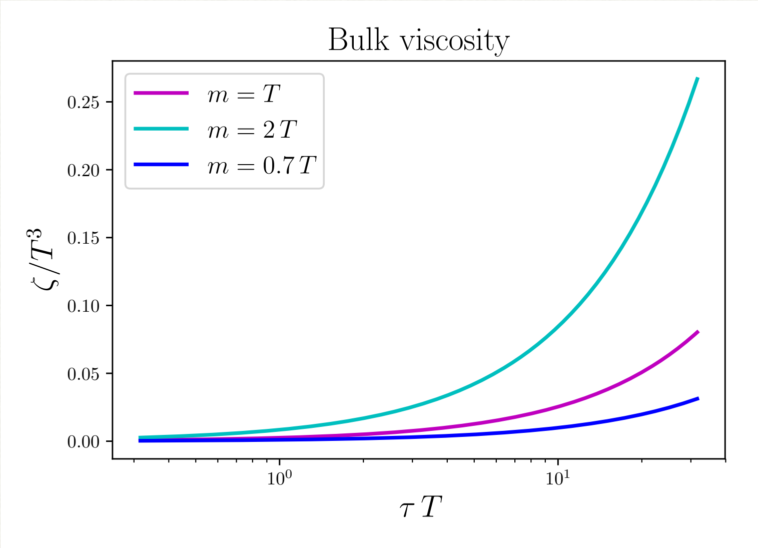
Viscosities for the Boltzmann distribution. In the conformal limit $\eta \to T\tau/5$.

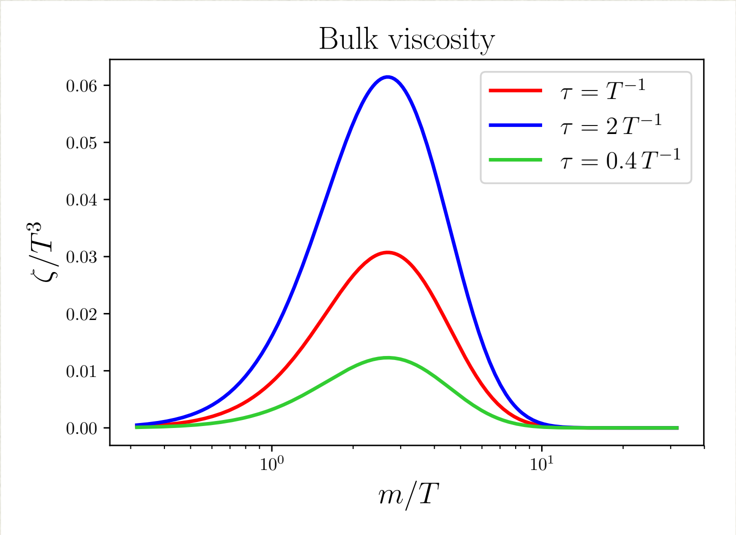



$\tau$ = 1/T, and

linear in $\tau$

Bulk viscosity mapped onto m and $\tau$ maximized at m~ T



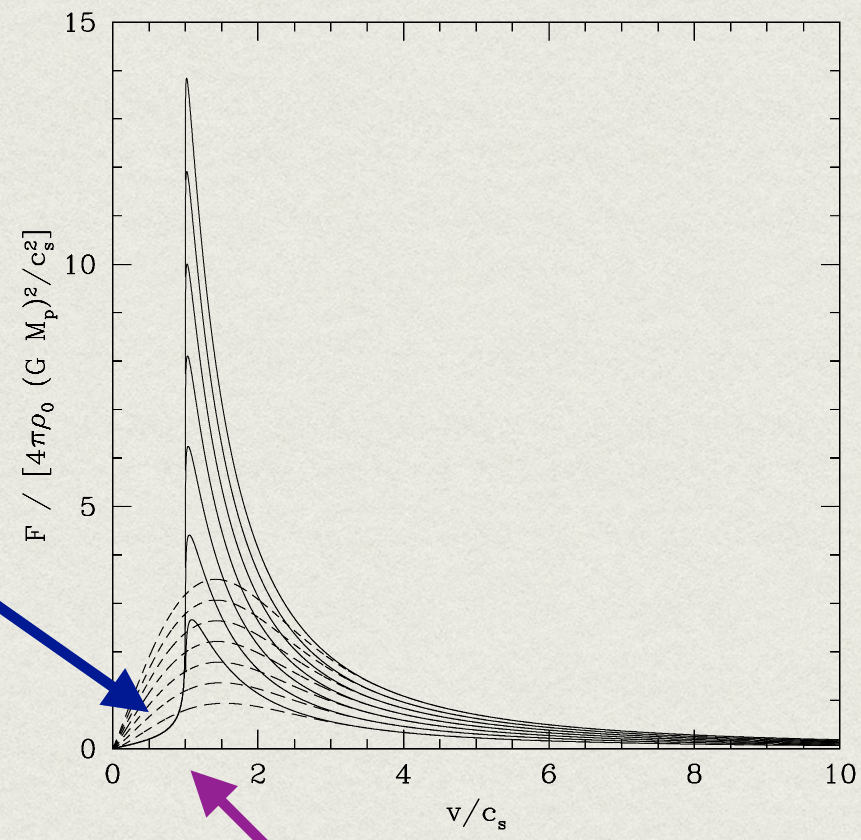



# COMMENTS ON "OSTRKIER'S CORRECTIONS"

In contrast with our results and the results of Rephaeli-Salpeter the force does not vanish identically in the subsonic regime

$$F = \frac{4\pi G^2 m_b^2 \rho_0}{v^2} \log\left(\frac{1+\mathcal{M}}{1-\mathcal{M}}\right)$$

Finite (but might be numerically important), boundary effect in time




speed of sound

Way to reproduce in our formalism:

$$T^{\mu\nu}_{bullet}(\vec{x},t) = \gamma\, m\, \delta(z - vt)\delta^2(\vec{x}_\perp) v^\mu v^\nu \times \theta(t)$$

$$T^{\mu\nu}_{bullet}(\vec{k},\omega) = \gamma\, m\, \frac{-i v^\mu v^\nu}{-\omega + k^z v - i\epsilon}$$

And proceed with a similar calculation

# CONCLUSIONS

- Proposed a completely generic method to calculate the classical dynamical friction in any regime
- Showed a completely generic analytic expression for the ideal gas in any regime of temperatures/ velocities
- Developed the expression for viscous fluids within the EFT that make important contributions in the subsonic regime
- Showed the smooth interpolations between the low and high-k regimes within the full kinetic theory
- These improved results can now be used in plethora of physical problems: interacting DM, propagating of the BHs through the compact objects, physics of the accretion discs etc.
- Extension of the existing results to include the boundary effects, post-Newtonian corrections is straightforward (though laborious) using our formalism