Families of spectral triples and foliations of spacetime

Koen van den Dungen

Scuola Internazionale Superiore di Studi Avanzati (SISSA)

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Spectral triples

- **Definition:** A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a $*$-algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$, and an unbounded self-adjoint operator $\mathcal{D}\colon \operatorname{dom}\mathcal{D} \to \mathcal{H}$ such that
  - The commutators $[\mathcal{D}, a]$ extend to bounded operators on $\mathcal{H}$ for all $a \in \mathcal{A}$;
  - The operators $a(1+\mathcal{D}^2)^{-1/2}$ are compact for all $a \in \mathcal{A}$.

- A complete Riemannian spin manifold (M, g) gives a spectral triple

$$\Big(C_c^\infty(M), L^2(S), \not{D} = \sum_j \gamma(e_j)\nabla^S_{e_j}\Big).$$

- An abstract spectral triple can be viewed as a 'noncommutative Riemannian manifold'.

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Lorentzian spectral triples

- Spectral triples can describe Riemannian manifolds but not Lorentzian manifolds (i.e., *spaces* but not *spacetimes*).
- Many possible definitions of 'Lorentzian spectral triples'; it is not clear which is the 'right one'.
- Idea: give explicit construction instead of abstract definition.
- Mimic decomposition of spacetime into spacelike hypersurfaces.
- The construction should also work in Riemannian signature.

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Outline

Spacetimes

- A *spacetime* is a time-oriented Lorentzian manifold.
- A *temporal function* on a spacetime (Z, g) is a smooth function $Z \to \mathbb{R}$ such that ∇T is timelike and past-directed everywhere.
- A spacetime is *stably causal* if it admits a temporal function [Bernal-Sanchez '05].
- A spacetime is *globally hyperbolic* if it admits a Cauchy hypersurface.

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Product spacetimes

- **Definition:** An oriented spacetime (Z, g) is called a product spacetime if it is of the form $(M \times \mathbb{R}, g_\bullet - N^2 dT^2)$, where
  - $M \times \{t\}$ is a smooth spacelike hypersurface;
  - the temporal function T is the canonical projection $M \times \mathbb{R} \to \mathbb{R}$;
  - the *lapse function* N is a smooth map $N \colon M \times \mathbb{R} \to (0, \infty)$;
  - $g_\bullet = \{g_t\}_{t \in \mathbb{R}}$ is a smooth family of Riemannian metrics on M.

- We often view N as a family $N_\bullet = \{N_t\}_{t \in \mathbb{R}}$ with $N_t(x) := N(x, t)$. Let $\nu := \sqrt{-g(\nabla T, \nabla T)}^{-1} \nabla T$ be the unit vector field orthogonal to M.

- **Example:** Every globally hyperbolic spacetime is a product spacetime, such that each $M \times \{t\}$ is a Cauchy hypersurface [Bernal-Sanchez '05].

- Every product spacetime is stably causal, but not conversely.

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A product spin spacetime

- Let $(Z, g) = (M \times \mathbb{R}, g_\bullet - N^2 dT^2)$ be a product spacetime of *even* dimension $n + 1 = 2m$ with a given spin structure.
- We write $M_t := M \times \{t\}$. Consider the spinor bundles

$$S := \mathrm{Spin}_g^+(Z) \times_{\mathrm{Spin}_{1,n}^+} \Delta_{1,n}, \qquad \Delta_{1,n} := \mathbb{C}^{2^m} = \Delta_n^+ \oplus \Delta_n^-,$$

$$S_t^\pm := \mathrm{Spin}_{g_t}^+(M_t) \times_{\mathrm{Spin}_n} \Delta_n^\pm, \qquad \Delta_n^\pm := \mathbb{C}^{2^{m-1}}.$$

  We have $S_t := S|_{M_t} = S_t^+ \oplus S_t^-$, where $S_t^+ \simeq S_t^-$.
- The Clifford multiplication with respect to (M_t, g_t) is given on S_t by

$$X \mapsto i\gamma(\nu)\gamma(X) = \begin{pmatrix} \gamma_{M_t}(X) & 0 \\ 0 & -\gamma_{M_t}(X) \end{pmatrix}.$$

- Let $\not{D}_{M_t} = \sum_j \gamma(e_j)\nabla_{e_j}^{S_t}$ be the canonical Dirac operator on (M_t, g_t, S_t^+). Then $-\not{D}_{M_t}$ is the canonical Dirac operator on (M_t, g_t, S_t^-).

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The Dirac operator

- The Weingarten map $W(X) = \nabla^Z_X \nu$ gives the relation between the Levi-Civita connections ∇^Z and ∇^{M_t} as

$$g(W(X), Y)\nu = \nabla^Z_X Y - \nabla^{M_t}_X Y,$$

for vector fields X, Y on M_t.

- The spin connections ∇^S and ∇^{S_t} are related via

$$\nabla^S_X - \nabla^{S_t}_X = -\frac{1}{2}\gamma(\nu)\gamma(W(X)).$$

- The Dirac operator restricted to M_t can then be written as

$$\not{D}_Z = -\gamma(\nu)\nabla^S_\nu - i\gamma(\nu)\begin{pmatrix} \not{D}_{M_t} & 0 \\ 0 & -\not{D}_{M_t} \end{pmatrix} - \frac{n}{2}H_t\gamma(\nu),$$

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The ‘spacelike’ spectral triples

- From the canonical indefinite hermitian structure $(\cdot|\cdot)$ on S we obtain a positive-definite hermitian structure by

$$(\phi|\psi)^{\text{pos}} := (\phi|\gamma(\nu)\psi).$$

- We obtain a Hilbert space $L^2(S_t^{\pm})$ as the completion of $\Gamma_c^\infty(S_t^{\pm})$ with respect to

$$\langle\phi|\psi\rangle^{\text{pos}} = \int_{M_t} (\phi|\psi)^{\text{pos}} \text{dvol}_{M_t}.$$

- **Assume** that the metric g_t on M_t is complete for each $t \in \mathbb{R}$.

- **Proposition:** $(C_c^\infty(M_t), L^2(S_t^{\pm}), \pm \not{D}_{M_t})$ are odd spectral triples (for each $t \in \mathbb{R}$).

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Parallel transport (1)

- For $x \in M$, parallel transport in S along the curves $t \mapsto (x,t)$ yields an isometry $\tau_t^s \colon S_{(x,t)} \to S_{(x,s)}$ w.r.t. the canonical *indefinite* hermitian structure on S.
- We then obtain an isometry $U_t \colon \Gamma_c^\infty(M_t, S_t) \to \Gamma_c^\infty(M_0, S_0)$ given by

$$(U_t\psi)(x) := \rho_t \tau_t^0 \psi(x),$$

  where we have defined the *volume function* $\rho_t := \left(|g_0|^{-1}|g_t|\right)^{\frac{1}{4}}$.
- For the positive-definite inner product we have

$$\langle U_t\phi|\gamma(\nu)U_t\psi\rangle^{\text{pos}} = \langle \phi|\tau_0^t\gamma(\nu)\tau_t^0\psi\rangle^{\text{pos}}.$$

  Therefore U_t is an isometry for $\langle\cdot|\cdot\rangle^{\text{pos}}$ if and only if $\nabla_\nu \nu = 0$.
- **Assume** that ν is geodesic, i.e. $\nabla_\nu \nu = 0$.

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Parallel transport (2)

- Consider the Hilbert space $\mathcal{H} := L^2(M_0, S_0^+)$. We identify S_0^+ with S_0^- via $\gamma(\nu)$, and then we can write

$$\gamma(\nu) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- We define $U \colon \Gamma_c^\infty(M \times \mathbb{R}, S) \to C_c^\infty(\mathbb{R}, \mathcal{H} \oplus \mathcal{H})$ by

$$(U\psi)(t) := N_t^{\frac{1}{2}} \cdot U_t(\psi|_{M_t}).$$

  Then U is an isometry for both $\langle\cdot|\cdot\rangle$ and $\langle\cdot|\cdot\rangle^{\text{pos}}$, and extends to a unitary isomorphism $U \colon L^2(M \times \mathbb{R}, S) \to L^2(\mathbb{R}, \mathcal{H} \oplus \mathcal{H})$.
- Under this isomorphism, the covariant derivative ∇_ν^S is related to the time derivative on $C_c^\infty(\mathbb{R}, \mathcal{H} \oplus \mathcal{H})$ by

$$U \nabla_\nu^S U^{-1} = N_t^{-\frac{1}{2}} \rho_t \partial_t \rho_t^{-1} N_t^{-\frac{1}{2}}.$$

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$$U\nabla_\nu^S U^{-1} = N_t^{-\frac{1}{2}} \rho_t \partial_t \rho_t^{-1} N_t^{-\frac{1}{2}}.$$

Parallel transport (2)

- Consider the Hilbert space $\mathcal{H} := L^2(M_0, S_0^+)$. We identify S_0^+ with S_0^- via $\gamma(\nu)$, and then we can write

$$\gamma(\nu) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- We define $U\colon \Gamma_c^\infty(M \times \mathbb{R}, S) \to C_c^\infty(\mathbb{R}, \mathcal{H} \oplus \mathcal{H})$ by

$$(U\psi)(t) := N_t^{\frac{1}{2}} \cdot U_t\big(\psi|_{M_t}\big).$$

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The Dirac operator rewritten

- On a hypersurface M_t, the Lorentzian Dirac operator decomposes as

$$\not{D}_Z = -\gamma(\nu)\nabla^S_\nu - i\gamma(\nu)\tilde{\not{D}}_{M_t} - \frac{n}{2}H\gamma(\nu),$$

where

$$\gamma(\nu) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\not{D}}_{M_t} = \begin{pmatrix} \not{D}_{M_t} & 0 \\ 0 & -\not{D}_{M_t} \end{pmatrix}.$$

- We have the expression $nH_t = N_t^{-1}\partial_t\big(\log|g_t|^{\frac{1}{2}}\big) = 2N_t^{-1}\rho_t^{-1}(\partial_t\rho_t)$.
- Under the isomorphism $U\colon L^2(M\times\mathbb{R},S) \to L^2(\mathbb{R},\mathcal{H}\oplus\mathcal{H})$, we obtain

$$U\not{D}_Z U^{-1} = \gamma(\nu)\left(-N_\bullet^{-\frac{1}{2}}\rho_\bullet\partial_t\rho_\bullet^{-1}N_\bullet^{-\frac{1}{2}} - i\tilde{\not{D}}_\bullet - N_\bullet^{-1}\rho_\bullet^{-1}(\partial_t\rho_\bullet)\right)$$

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The family

- Hence the Lorentzian Dirac operator $\not{D}_Z$ can be described (up to unitary isomorphism) by
  - a family of "spatial" Dirac operators $\{\not{D}_t\}_{t\in\mathbb{R}}$;
  - a (family of) lapse function(s) $N_\bullet = \{N_t\}_{t\in\mathbb{R}}$;
  - a family of volume functions $\rho_\bullet = \{\rho_t\}_{t\in\mathbb{R}}$.
- We can assemble these objects into spectral triples

$$\Big(C_c^\infty(M_0) \odot C_c^\infty(\mathbb{R}), L^2(\mathbb{R}, \mathcal{H} \oplus \mathcal{H}), \not{D}_\pm\Big)$$

for the Riemannian Dirac-type operators on $M \times \mathbb{R}$:

$$\not{D}_\pm := \gamma(\nu)\left(\pm i N_\bullet^{-\frac{1}{2}} \rho_\bullet \partial_t \rho_\bullet^{-1} N_\bullet^{-\frac{1}{2}} - i\tilde{\mathcal{D}}_\bullet \pm i N_\bullet^{-\frac{1}{2}} [\partial_t, \rho_\bullet] \rho_\bullet^{-1} N_\bullet^{-\frac{1}{2}}\right)$$

- We can recover the Lorentzian Dirac operator via a "reverse Wick rotation":

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1 Foliations of spacetime

2 Families of spectral triples

3 Conclusion

Families of spectral triples

- **Definition:** A weakly differentiable family of spectral triples $\{(\mathcal{A}, {}_{\pi_t}\mathcal{H}, \mathcal{D}_t)\}_{t\in\mathbb{R}}$ is a family of spectral triples $\{(\mathcal{A}, {}_{\pi_t}\mathcal{H}, \mathcal{D}_t)\}_{t\in\mathbb{R}}$ such that the following conditions are satisfied:
  - $W := \operatorname{Dom} \mathcal{D}_t$ is independent of t, and the graph norms of $\mathcal{D}_t$ are uniformly equivalent;
  - the map $\mathcal{D}_\bullet \colon \mathbb{R} \to \mathcal{L}(W, \mathcal{H})$ is weakly differentiable, and its weak derivative is uniformly bounded;
  - the family of representations $\{\pi_t\}_{t\in\mathbb{R}}$ of A on $\mathcal{H}$ is weakly differentiable, and for each $a \in \mathcal{A}$ the family $\{\pi_t(a) \colon W \to W\}$ is strongly continuous.

Lapse and volume operators

- **Definition:** Given a weakly differentiable family of spectral triples $\{(\mathcal{A}, \mathcal{H}, \mathcal{D}_t)\}_{t\in\mathbb{R}}$, we consider a family of lapse operators $\{N_t\}_{t\in\mathbb{R}}$ and a family of volume operators $\{\rho_t\}_{t\in\mathbb{R}}$ satisfying the following assumptions:
  1. the families of operators $\{N_t\}$ and $\{\rho_t\}$ are positive, invertible, and uniformly bounded;
  2. the operators $N_t^{\frac{1}{2}}$ and ρ_t and their inverses preserve the domain W, the families $\{N_t^{\frac{1}{2}} : W \to W\}$ and $\{\rho_t : W \to W\}$ are strongly differentiable and uniformly bounded, and the inverses $\{N_t^{-\frac{1}{2}} : W \to W\}$ and $\{\rho_t^{-1} : W \to W\}$ are strongly continuous and uniformly bounded;
  3. the strong derivatives $\{(\partial N^{\frac{1}{2}})_t\}$ and $\{(\partial\rho)_t\}$ and the commutators $\{[\mathcal{D}_t, N_t^{\frac{1}{2}}]\}$ and $\{[\mathcal{D}_t, \rho_t]\}$ on $\mathcal{H}$ are uniformly bounded;
  4. $[\rho_t, [\mathcal{D}_t, \rho_t]] = 0$, $[\rho_t, N_t^{\frac{1}{2}}] = 0$, $[N_t^{\frac{1}{2}}, \pi_t(a)] = 0$, and $[\rho_t, \pi_t(a)] = 0$ for all $a \in \mathcal{A}$.

The total spectral triple

- **Theorem:** Consider a weakly differentiable family of spectral triples $\{(\mathcal{A}, {}_{\pi_t}\mathcal{H}, \mathcal{D}_t)\}_{t\in\mathbb{R}}$ with a family of lapse operators $\{N_t\}_{t\in\mathbb{R}}$ and a family of volume operators $\{\rho_t\}_{t\in\mathbb{R}}$. Define the operators $\mathcal{D}_+$ and $\mathcal{D}_-$ on $L^2(\mathbb{R}, \mathcal{H})^{\oplus 2}$ by

$$\mathcal{D}_\pm := \mathcal{J}\left(\pm i N_\bullet^{-\frac{1}{2}} \rho_\bullet \partial_t \rho_\bullet^{-1} N_\bullet^{-\frac{1}{2}} - i\tilde{\mathcal{D}}_\bullet \pm i N_\bullet^{-\frac{1}{2}} [\partial_t, \rho_\bullet] \rho_\bullet^{-1} N_\bullet^{-\frac{1}{2}}\right),$$

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  Then the triples $(\mathcal{A} \odot C_c^\infty(\mathbb{R}), L^2(\mathbb{R}, \mathcal{H})^{\oplus 2}, \mathcal{D}_\pm)$ are spectral triples.
- **Remark:** For $N_t = \rho_t = 1$, the proof is (more or less) given in [vdD-Rennie '16] (which in turn is based on [Kaad-Lesch '13, §8]).

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Comparison

- Other definitions of 'Lorentzian spectral triples' are based on Krein spaces instead of Hilbert spaces, see e.g. [Strohmaier '06, Paschke-Sitarz '06, Franco '14, vdD '16].

- **Theorem:** Consider a weakly differentiable family of spectral triples $\{(\mathcal{A},\mathcal{H},\mathcal{D}_t)\}_{t\in\mathbb{R}}$ with a family of lapse operators $\{N_t\}_{t\in\mathbb{R}}$ and a family of volume operators $\{\rho_t\}_{t\in\mathbb{R}}$. Then the operator $i\mathcal{D}$ is Krein-self-adjoint. Furthermore, $(\mathcal{A}\odot C_c^\infty(\mathbb{R}), L^2(\mathbb{R},\mathcal{H})^{\oplus 2}, i\mathcal{D}, \mathcal{J})$ is a Krein spectral triple (as defined in [vdD '16]).

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Conclusion

- The Lorentzian Dirac operator $\not{D}$ on a product spacetime can be 'decomposed' into the spatial Dirac operators $\not{D}_\bullet$, the lapse function $N_\bullet$, and the volume function $\rho_\bullet$.
- Given an abstract family of spectral triples over A with 'lapse operator' $N_\bullet$ and 'volume operator' $\rho_\bullet$, we can construct a larger spectral triple over $C_0(\mathbb{R}, A)$.
- As a noncommutative analogue of Lorentzian spacetimes, we can define Lorentzian product spectral triples via the 'reverse Wick rotation'.

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