Towards causal relations in noncommutative spacetime

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Warsaw University of Technology & Copernicus Center (Cracow)

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Elements of causality theory

Spacetime — the arena of classical and quantum physics

A *spacetime* $\mathcal{M}$ is a smooth manifold, which is

- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map g

$$\mathcal{M} \ni p \mapsto g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$$

  such that
  - $\forall p \in \mathcal{M} \quad g_p$ is a symmetric bilinear map of signature $(- + + +)$,
  - $\forall X, Y \in \mathscr{X}(\mathcal{M})$ the map $p \to g_p(X_p, Y_p)$ is smooth.
- time-oriented (will explain in a second)

Points of $\mathcal{M}$ are suggestively called events.

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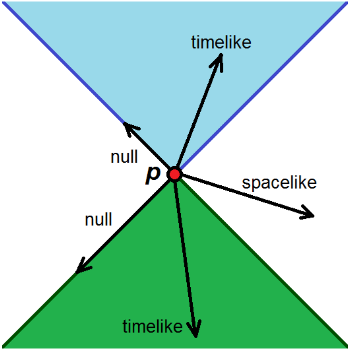
Classification of tangent vectors

Vector $v \in T_p\mathcal{M}$ is called

- *timelike* if $g_p(v, v) < 0$,
- *null* if $g_p(v, v) = 0$,
- *spacelike* if $g_p(v, v) > 0$,
- *causal* if $g_p(v, v) \leqslant 0$ and $v \neq 0$,



The set of timelike vectors tangent at p has two components. A *time-orientation* is a *continuous* choice of one of the components, containing (by definition) *future-directed timelike vectors*.

Above terms extend naturally to *vector fields* and to *piecewise C^1 curves*.

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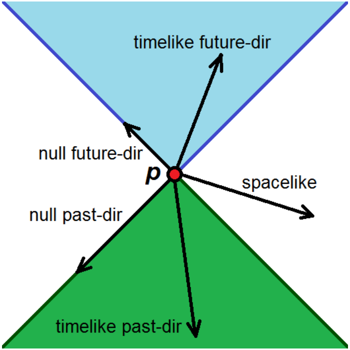

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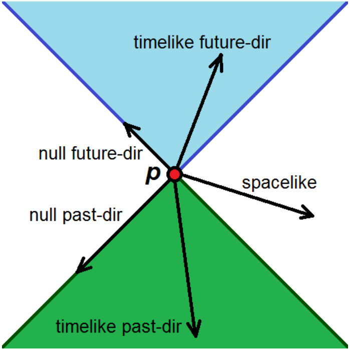

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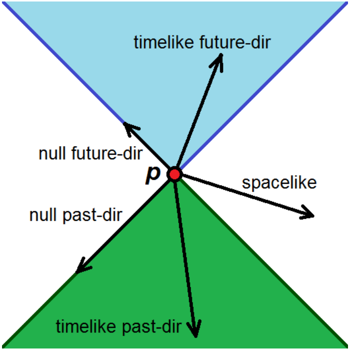

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Causal precedence relation $\preceq$

We say that $p \in \mathcal{M}$ **causally precedes** $q \in \mathcal{M}$, denoted $p \preceq q$, if there exists a (piecewise C^1) future-directed causal curve $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$, $\gamma(1) = q$ or if $p = q$.

Relation $\preceq$ is a *preorder* (reflexive + transitive).

Causal future and past

The **causal future** and the **causal past** of an event $p \in \mathcal{M}$ are the sets

$$J^+(p) := \{q \in \mathcal{M} : \ p \preceq q\} \qquad \text{and} \qquad J^-(p) := \{q \in \mathcal{M} : \ q \preceq p\}.$$

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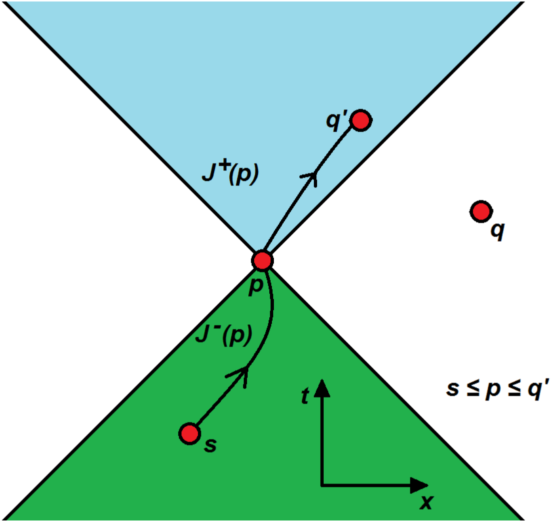
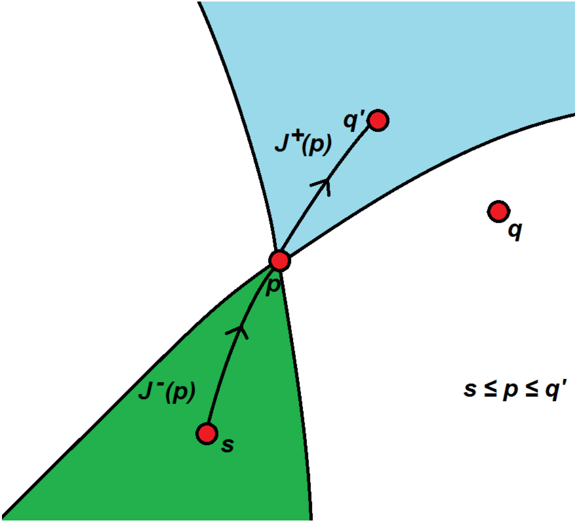
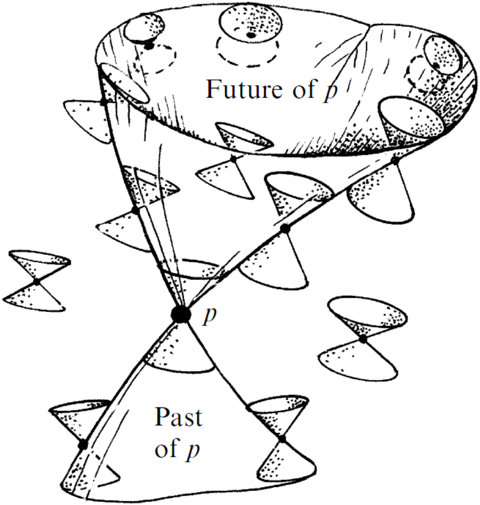


Causal structure of a spacetime in SR (Minkowski spacetime)



Causal structure of a spacetime in GR

Elements of causality theory



Causal structure of a spacetime in GR (from Penrose's "Road to Reality")

Motivations and the goal

- Causality (in the context of Lorentzian geometry) = certain binary relation between **events**.
- Pointlike events are **not** directly observable!
  - Measuring apparatus' imperfection
  - Wave-packets have nonzero spread
- Goal: extend the causality relation to **states** on a suitable (pre-)C^*-algebra $\mathcal{A}$
  $\rightsquigarrow$ Besnard, Franco, Eckstein
- In this talk: $\mathcal{A} = C_0(\mathcal{M})$, and so
  - Mixed states are of the form $f \mapsto \int_{\mathcal{M}} f d\mu$ for some *Borel probability measure* μ.
  - Pure states are the Dirac measures δ_p for $p \in \mathcal{M}$.

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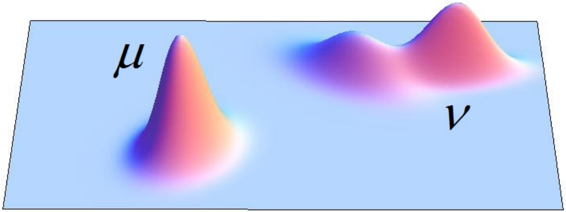
The **causal precedence** relation $\preceq$ on $\mathcal{M}$

$$p \preceq q \iff \exists \text{ future-directed causal curve } \gamma \text{ from } p \text{ to } q, \text{ or } p = q$$

- Goal: Define $\mu \preceq \nu$ for $\mu, \nu \in \mathscr{P}(\mathcal{M})$.



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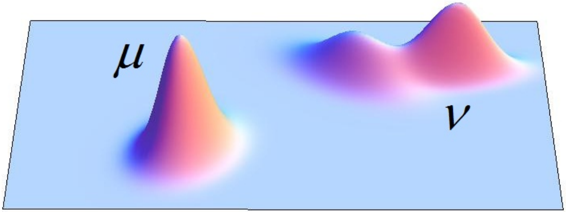

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Causality for probability measures

Causal functions

Function $f : \mathcal{M} \to \mathbb{R}$ is called **causal**, if it is nondecreasing along every future-directed causal curve.

What does it mean that $\mu \preceq \nu$ for $\mathcal{M}$ globally hyperbolic?

For any $\mu, \nu \in \mathscr{P}(\mathcal{M})$:

$$\mu \preceq \nu \quad \overset{\text{def}}{\Longleftrightarrow} \quad \int_{\mathcal{M}} f d\mu \leqslant \int_{\mathcal{M}} f d\nu \ \text{ for all causal functions } f.$$

Theorem [Minguzzi, Besnard, Franco]

For any $p, q \in \mathcal{M}$ we have that $p \preceq q$ iff $\delta_p \preceq \delta_q$ in the above sense.

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Better definition [M. Eckstein, TM '17]

Let $\mathcal{M}$ be a spacetime. Then for any $\mu, \nu \in \mathscr{P}(\mathcal{M})$

$\mu \preceq \nu \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists\, \omega \in \mathscr{P}(\mathcal{M}^2)$ such that:

- $\forall_{A \subseteq \mathcal{M}} \quad \omega(A \times \mathcal{M}) = \mu(A), \;\; \omega(\mathcal{M} \times A) = \nu(A),$
- $\omega(J^+) = 1,$

where $J^+ := \{(p,q) \in \mathcal{M}^2 \mid p \preceq q\}$.

- ω can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p$, $\nu = \delta_q$, the only coupling is $\omega = \delta_{(p,q)}$ and so $\delta_p \preceq \delta_q$ iff $p \preceq q$.

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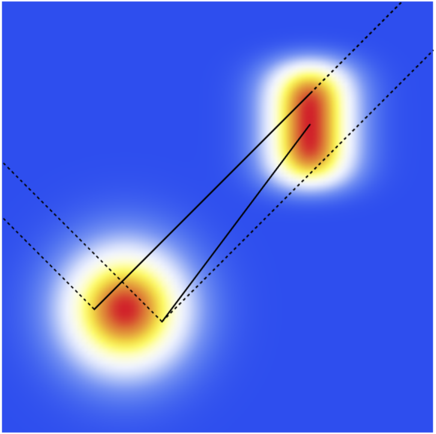
Each infinitesimal part of the probability measure should travel along a future-directed causal curve.



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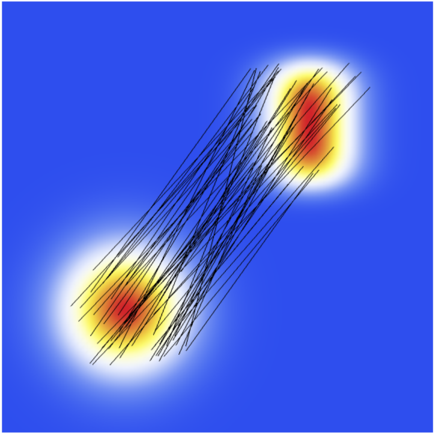
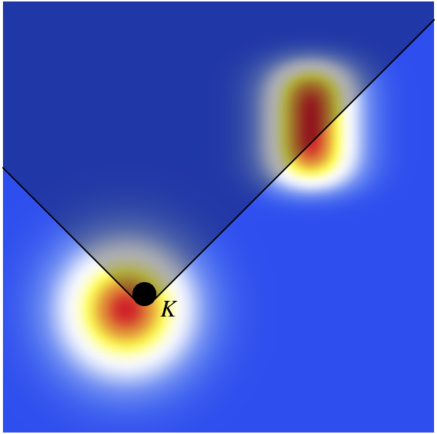


Causality for probability measures

For $\mathcal{M}$ causal and such that J^+ is closed topologically:

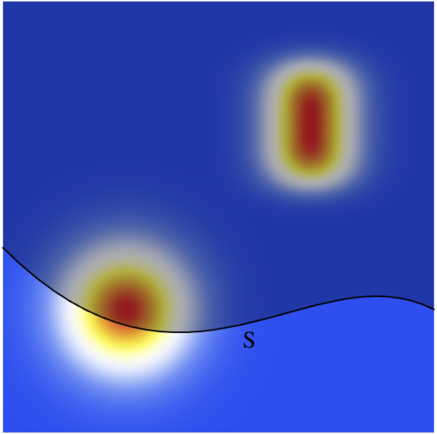
$$\mu \preceq \nu \iff \forall \text{compact } \mathcal{K} \subseteq \operatorname{supp} \mu \quad \mu(\mathcal{K}) \leqslant \nu(J^+(\mathcal{K}))$$



Causality for probability measures

For $\mathcal{M}$ globally hyperbolic (i.e. admitting Cauchy hypersurfaces):

$\mu \preceq \nu \quad \iff \quad \forall$ Cauchy hypersurface $\mathcal{S} \quad \mu(J^+(\mathcal{S})) \leqslant \nu(J^+(\mathcal{S}))$



Causal time-evolution of measures ($\mathcal{M}$:= Minkowski)

Causal time-evolution of a pointlike particle

A curve $\gamma : I \to \mathcal{M}$, $\gamma(t) = (t, x(t))$ is a worldline of a physical particle if

$$\forall s,t \in I \quad s \leqslant t \;\Rightarrow\; \gamma(s) \preceq \gamma(t).$$

Causal time-evolution of a probability measure

A map $\mu : I \to \mathscr{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\operatorname{supp} \mu_t \subseteq \{t\} \times \mathbb{R}^3$ for all $t \in I$ a **causal evolution of a measure** if

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Causal time-evolution of measures ($\mathcal{M}$:= Minkowski)

The Schrödinger equation

$$i\hbar\partial_t\psi = \hat{H}\psi$$

yields the $\mathscr{P}(\mathcal{M})$-valued map

$$d\mu_t = \delta_t \times |\psi(t,x)|^2\, d^3x.$$

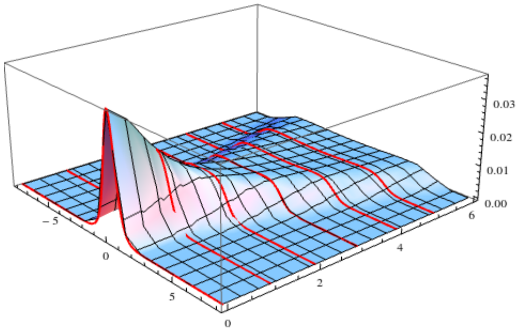
Which Hamiltonians lead to a causal evolution?

Causal time-evolution of a probability measure

A map $\mu : I \to \mathscr{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\operatorname{supp}\mu_t \subseteq \{t\} \times \mathbb{R}^3$ for all $t \in I$ a **causal evolution of a measure** if

$$\forall s,t \in I \quad s \leqslant t \;\Rightarrow\; \mu_s \preceq \mu_t.$$

Causal time-evolution of measures ($\mathcal{M} :=$ Minkowski)



The Schrödinger equation

$$i\hbar\partial_t\psi = \hat{H}\psi$$

yields the $\mathscr{P}(\mathcal{M})$-valued map

$$d\mu_t = \delta_t \times |\psi(t,x)|^2\, d^3x.$$

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Theorem [M. Eckstein, TM '17]

Suppose $\rho(t, x)$ satisfies the *continuity equation*

$$\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0$$

with a velocity field such that $\|\mathbf{v}(t, x)\| \leqslant c$. Then μ_t defined via

$$d\mu_t = \delta_t \times \rho(t, x)\, d^3x$$

evolves causally.

- Dirac's Hamiltonian respects causality.
- Same for Białynicki-Birula's "Dirac-like" Hamiltonian of the wave function of a photon.
- However, the $\sqrt{\hat{p}^2 + m^2}$ Hamiltonian violates causality!

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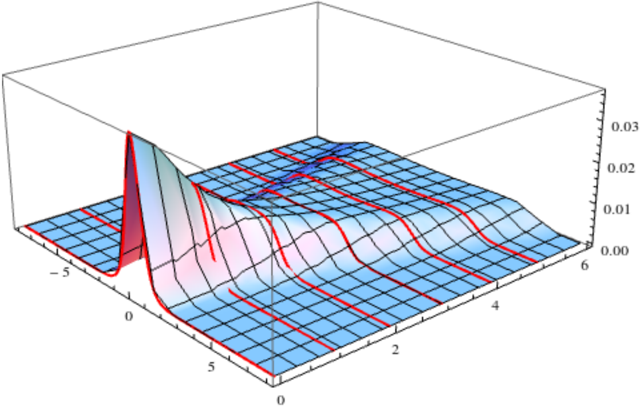
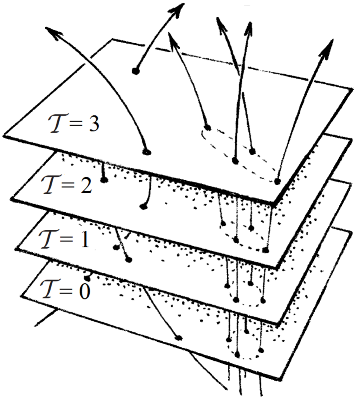


What about other observers?

Theorem [TM '17]

Consider a map $t \mapsto \mu_t \in \mathscr{P}(\mathcal{M})$ satisfying $\mathrm{supp}\ \mu_t \subseteq \{t\} \times \mathbb{R}^3$ for every $t \in I$. TFAE:

- The map $t \mapsto \mu_t$ is *causal*, i.e.
  $$\forall s, t \in I \quad s \leqslant t \ \Rightarrow \ \mu_s \preceq \mu_t.$$
- There exists a **probability measure on the space of worldlines**, from which one can recover μ_t for all $t \in I$.



Adapted from Penrose's "Road to Reality"

Conclusions

- Any spacetime $\mathcal{M}$ possesses an inherent **causal structure**, encoded by the binary relation $\preceq$.
- This relation can be **naturally** extended onto $\mathscr{P}(\mathcal{M})$, encapsulating the requirement that probability can only flow subluminally.
- **Claim**: Probability distributions obtained from QM wave packets should respect this causal relation.
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Thank you for your attention!

- M. Eckstein and T. Miller, *Causality for nonlocal phenomena*, Annales Henri Poincaré **18**(9) 2017, 3049–3096,
- T. Miller, *Polish spaces of causal curves*, Journal of Geometry and Physics **116** 2017, 295–315,
- M. Eckstein and T. Miller, *Causal evolution of wave packets*, Physical Review A **95** 2017, 032106,
- T. Miller, *On the causality and K-causality between measures*, Universe 2017 3(1):27,
- R. Penrose, *The Road to Reality: A complete Guide to the Laws of the Universe*, Jonathan Cape, London 2004