Physical models from noncommutative causality

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ICNFP 2017 – Kolymbari, Crete – 24 Augustus 2017

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- An operator D densely defined on $\mathcal{H}$ such that
  - $a(1+\langle D\rangle^2)^{-\frac{1}{2}}$ is compact $\forall a \in \mathcal{A}$, with $\langle D\rangle^2 = \frac{1}{2}(DD^*+D^*D)$
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  - $[D,a]$ is bounded $\forall a \in \widetilde{\mathcal{A}}$
- A bounded operator $\mathcal{J}$ with $\mathcal{J}^2 = 1$, $\mathcal{J}^* = \mathcal{J}$, $[\mathcal{J}, a] = 0$,
  - $D^* = -\mathcal{J} D \mathcal{J}$
  - $\mathcal{J} = -N[D,\mathcal{T}]$ for $N \in \widetilde{\mathcal{A}}$, $N > 0$ and for some (possibly unbounded) self-adjoint operator $\mathcal{T}$ such that $\left(1+\mathcal{T}^2\right)^{-\frac{1}{2}} \in \widetilde{\mathcal{A}}$

The operator $\mathcal{J}$ is called the fundamental symmetry. Its role is to turn the positive definite inner product of the Hilbert space $\langle \cdot, \cdot \rangle$ into an indefinite inner product $(\cdot, \cdot) = \langle \cdot, \mathcal{J} \cdot \rangle$ (Krein space).

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The traditional condition on D to be selfadjoint is recovered here by requesting D to be (skew)-selfadjoint within the Krein space $(D\cdot,\cdot) = (\cdot,-D\cdot)$ which is equivalent to request $D^* = -\mathcal{J}D\mathcal{J}$ on $\mathcal{H}$.

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For a general $\mathcal{J}$ (and even for a general one-form), the signature can correspond to a pseudo-Riemannian one.

A globally hyperbolic spacetime can always be expressed with a split metric $g = -N^2 d^2\mathcal{T} + g_{\mathcal{T}}$ where $g_{\mathcal{T}}$ is a set of Riemannian metrics, $\mathcal{T}$ is a smooth global time (temporal) function and N is the *lapse* function.

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Proposition. *If $(\mathcal{M}, g)$ is a globally hyperbolic manifold (complete under spacelike reflexion), then*

$$\mathcal{J} = -N[D, \mathcal{T}] \; \left(= iNc(d\mathcal{T}) = i\gamma^0\right)$$

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Proposition. *Let us assume that a triple* $(\mathcal{A}, \mathcal{H}, D)$ *with fundamental symmetry* $\mathcal{J}$ *corresponds to a pseudo-Riemannian spin manifold. If* $\mathcal{J} = -N[D, \mathcal{T}]$ *for some smooth functions* N *and* $\mathcal{T}$*, then the geometry is Lorentzian and the metric admits a global splitting.*

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Proposition. *Let us assume that a triple $(\mathcal{A}, \mathcal{H}, D)$ with fundamental symmetry $\mathcal{J}$ corresponds to a pseudo-Riemannian spin manifold. If $\mathcal{J} = -N[D, \mathcal{T}]$ for some smooth functions N and $\mathcal{T}$, then the geometry is Lorentzian and the metric admits a global splitting.*

Conclusion (assuming a more general reconstruction theorem for pseudo-Riemannian manifolds):

Globally hyperbolic $\subseteq$ "$\mathcal{J} = -N[D, \mathcal{T}]$" $\subseteq$ "stably causal" Lorentzian manifolds

2. An algebraic formulation of causality for noncommutative geometry

On a Lorentzian manifold with metric g, tangent vectors can be classified in different groups:

- v is spacelike if $g(v,v) > 0$
- v is causal (timelike or null) if $g(v,v) \leq 0$

Two points are causally related, with $p \preceq q$, if and only if there is a future directed causal curve from p to q, i.e. a curve whose tangent vector is causal everywhere.

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The notion is purely geometric and needs to be algebraized using the elements of a Lorentzian spectral triple.

In particular, we want a relation between the pure states of the algebra, since there are in one-to-one correspondance with the points of the manifold.

We define $\mathcal{C} \subset C_b^\infty(\mathcal{M}, \mathbb{R})$ as the convex cone of *causal functions* which are the real-valued functions non-decreasing along every future directed causal curve, i.e.

$$\forall p, q \in \mathcal{M}, \quad p \preceq q \quad \Longrightarrow \quad \forall f \in \mathcal{C},\ f(p) \leq f(q)$$

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Using the correspondence between points and pure states, a corresponding partial order structure can be constructed on the space of pure states $P(\widetilde{\mathcal{A}})$ by the relation:

$$\forall \chi, \xi \in P(\widetilde{\mathcal{A}}), \quad \chi \preceq \xi \quad \Longleftrightarrow \quad \forall f \in \mathcal{C},\ \chi(f) \leq \xi(f)$$

Theorem. *(N.F., M. Eckstein 2013])*

Let $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, D, \mathcal{J})$ be a commutative Lorentzian spectral triple constructed from a globally hyperbolic spacetime, then $f \in \widetilde{\mathcal{A}}$ is a causal function if and only if

$$\forall \phi \in \mathcal{H}, \quad \langle \phi, \mathcal{J}[D, f]\phi \rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the positive definite inner product on $\mathcal{H}$.

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For noncommutative spacetime, we define the following set:

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and the causal relation between (pure) states is:

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Theorem. *If the Lorentzian spectral triple is commutative and constructed from a globally hyperbolic spacetime $\mathcal{M}$, then the relation $\preceq$ corresponds to the usual causal relation on $\mathcal{M}$.*

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The almost commutative spacetime $\mathcal{M} \times M_2(\mathbb{C})$

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- $(\mathcal{A}_{\mathcal{M}}, H_{\mathcal{M}}, D_{\mathcal{M}}) = (C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), -i\gamma^\mu\nabla^S_\mu)$ is the Lorentzian spectral triple built on an even spin Lorentzian manifold $\mathcal{M}$

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- $(\mathcal{A}_F, H_F, D_F)$ is a noncommutative (Riemannian) spectral triple constructed in the following way:
  - $\mathcal{A}_F = M_2(\mathbb{C})$
  - $\mathcal{H}_F = \mathbb{C}^2$
  - $D_F = \mathrm{diag}(d_1, d_2), \quad d_1, d_2 \in \mathbb{R}, \quad d_1 \neq d_2$

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The fundamental symmetry of the product is given by $\mathcal{J} = i\gamma^0 \otimes 1$.

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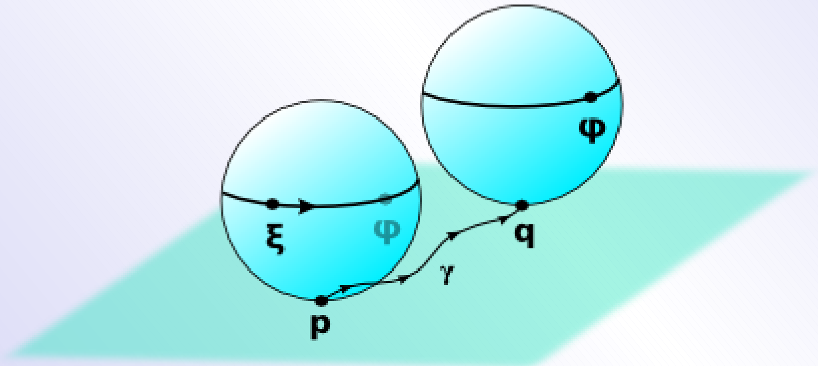
- $p \preceq q$ on $\mathcal{M}$ (so there exists a causal curve γ from p to q)
- $|\xi_1| = |\varphi_1|$ and $|\xi_2| = |\varphi_2|$ (ξ_i, φ_i are the components of ξ, φ), which means that ξ and φ are on the same parallel of latitude



Lorentzian spectral triple
Causality in NCG
$\mathcal{M} \times M_2$ model
Two-sheeted spacetime
Moyal spacetime

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If $\omega_{p,\xi}$ and $\omega_{q,\varphi}$ are on the same parallel of latitude, they can be defined by two angles θ_ξ and θ_φ by setting $\xi_1 = \varphi_1 \in \mathbb{R}^*$ and $\xi_2 = |\varphi_2|\, e^{i\theta_\xi}$, $\varphi_2 = |\varphi_2|\, e^{i\theta_\varphi}$.

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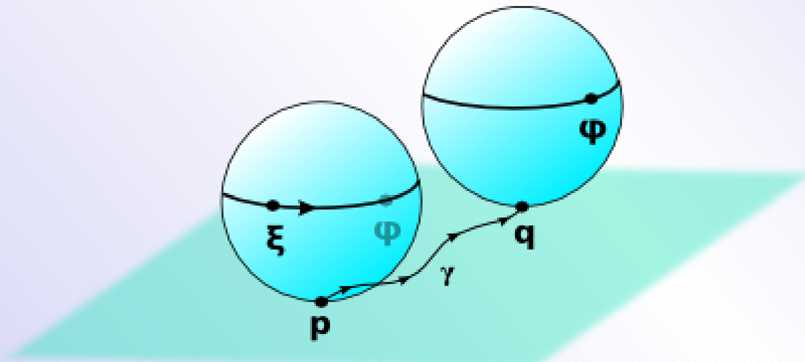
Theorem. *(N.F., M Eckstein 2014) Two pure states $\omega_{p,\xi}$ and $\omega_{q,\varphi}$ are causally related, with $\omega_{p,\xi} \preceq \omega_{q,\varphi}$, if and only if $p \preceq q$, they have the same latitude and*

$$l(\gamma) \geq \frac{|\theta_\varphi - \theta_\xi|}{|d_1 - d_2|}$$

where $l(\gamma)$ represents the length of a causal curve $\gamma : p \to q$.



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- $\left|\theta_\varphi - \theta_\xi\right|$ represents some distance within the parallel between two vectors in the internal space.
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This condition can be seen as a constant upper bound to the ratio between a distance in the internal space and a proper time in the continuous space, and can be interpreted as a *constant speed of light* existing within the finite space.

4. Second model: *The "two-sheeted" spacetime*

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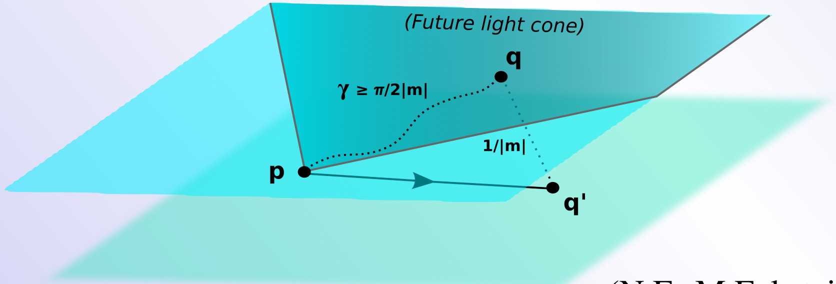
Theorem. *The causal structure is preserved on each sheet and two points p and q' on separated sheets are causally related with $p \preceq q'$ iff*

- *They are causally related if considered on the same sheet ($p \preceq q$)*
- *$l(\gamma) \geq \frac{\pi}{2|m|}$ where $l(\gamma)$ represents the length of a causal curve $\gamma : p \to q$ on $\mathcal{M}$*



(N.F., M Eckstein 2015)

Extension to mixed states

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$$\int_0^t ds\, |\psi| \sqrt{-g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))} \geq \left|\arcsin\sqrt{\varphi} - \arcsin\sqrt{\xi}\right|$$

The traditional proper time $l(\gamma)$ is replaced by a ”weighted proper time” depending on the scalar field value (Higgs field).

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We can identify our two-sheeted model with Zitterbewegung, assuming the oscillations occur between our two pure states.

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$$S_F = \int_{\mathcal{M}} \left[\psi_-^* D_{\mathcal{M}} \psi_- + \psi_+^* D_{\mathcal{M}} \psi_+ + m(\psi_-^* \psi_+ + \psi_+^* \psi_-) \right]$$

which is the action describing a single Dirac fermion of mass m propagating in a curved spacetime $\mathcal{M}$.

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Restoring the physical dimensions on the causality constraint, we get :

$$(p,-) \preceq (q,+) \iff p \preceq q \text{ and } l(\gamma) \geq \frac{\pi \hbar}{2\,|m|\,c^2} = \frac{T_{\text{ZB}}}{2}$$

So Zitterbewegung can be interpreted as a particule moving as the maximal speed within the internal space.
(M Eckstein, N.F., T. Miller 2017)

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- But there is an observable impact from a variation of the mass of the particle (order 10^{-20}s)

5. Third model: *Moyal spacetime*

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$$(f \star g)(x) := \frac{1}{(\pi\theta)^2} \int d^2y\, d^2z\; f(x+y)g(x+z)e^{-2i\, y^\mu\, \Theta^{-1}_{\mu\nu} z^\nu},$$

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- $\mathcal{J} = i\gamma^0$ is the fundamental symmetry

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Functions and states on Moyal can be easily described using as orthonormal basis the Wigner eigenfunctions of the two-dimensional harmonic oscillator $a = \sum_{mn} a_{mn} f_{mn}$ where

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They are the states that minimize the uncertainty equally distributed in position and momentum. The classical limit of the coherent states, when $\theta \to 0$, corresponds to the usual pure states on $\mathbb{R}^{1,1}$, hence to the points of the usual Minkowski space. (P. Martinetti, L. Tomassini 2013)

Theorem. *(N.F., J.-C. Wallet 2015)*

Let us suppose that two coherent states $\omega_\xi, \omega_\varphi$ *correspond to the complex scalars* $\kappa_1, \kappa_2 \in \mathbb{C}$. *Those coherent states are causally related, with* $\omega_\xi \preceq \omega_\varphi$, *if and only if* $\Delta\kappa = \kappa_2 - \kappa_1$ *is inside the convex cone of* $\mathbb{C}$ *defined by* $\lambda = \frac{1+i}{\sqrt{2}}$ *and* $\bar{\lambda} = \frac{1-i}{\sqrt{2}}$ *(i.e. the argument of* $\Delta\kappa$ *is within the interval* $[-\frac{\pi}{4}, \frac{\pi}{4}]$*).*

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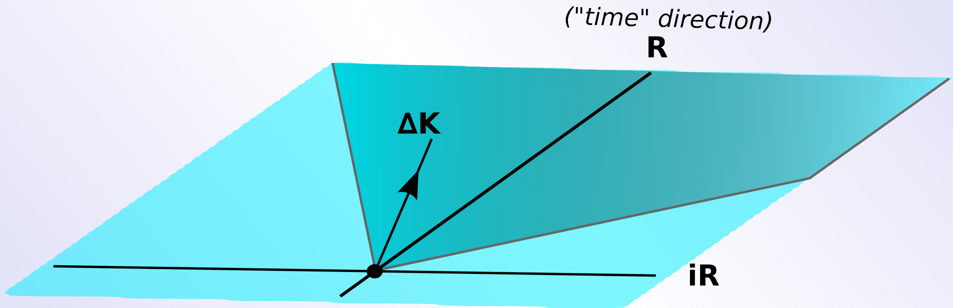

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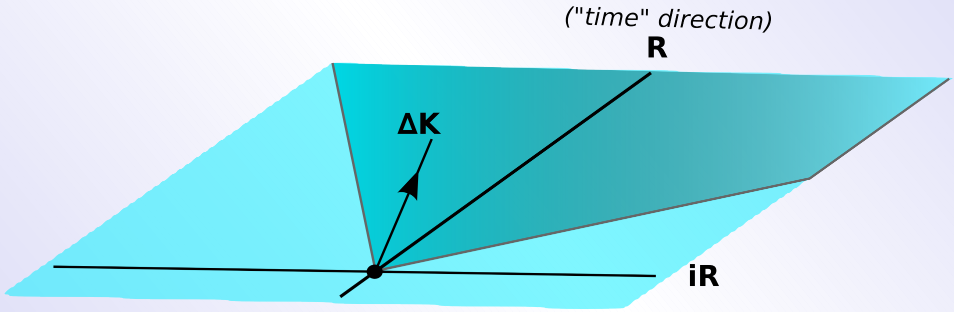

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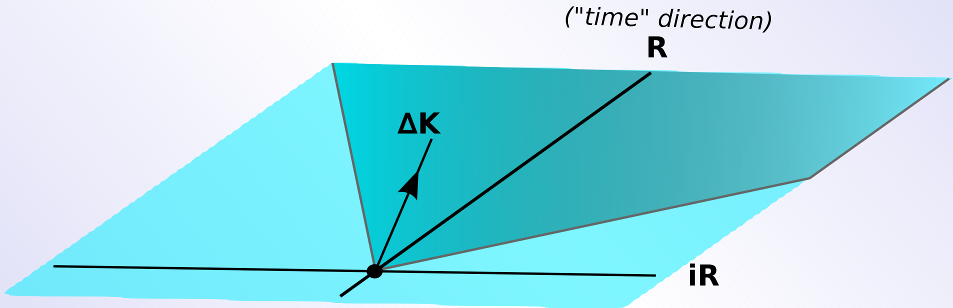

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In such a case, we can define a kind of ”time” as translations under positive real scalars κ.

Causal structure between generalized coherent states

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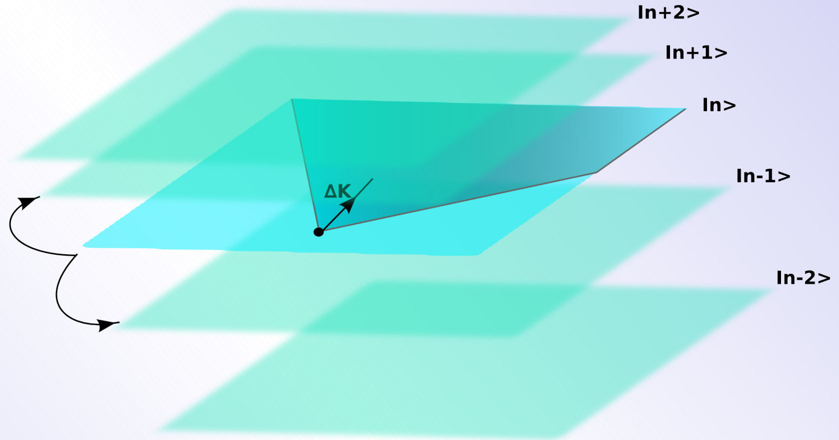
Proposition. *Let us write $\alpha_{\Delta\kappa}|n>$ the translation of the eigenstate $|n>$ under $\Delta\kappa$ (using $\mathbb{C} \cong \mathbb{R}^{1,1}$).*

If $\Delta\kappa \in \mathbb{R}$ such that

$$\Delta\kappa \geq \frac{\pi}{2}\sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}},$$

then $\quad |n> \preceq \alpha_{\Delta\kappa}|n+1> \quad$ *and also* $\quad |n+1> \preceq \alpha_{\Delta\kappa}|n>$



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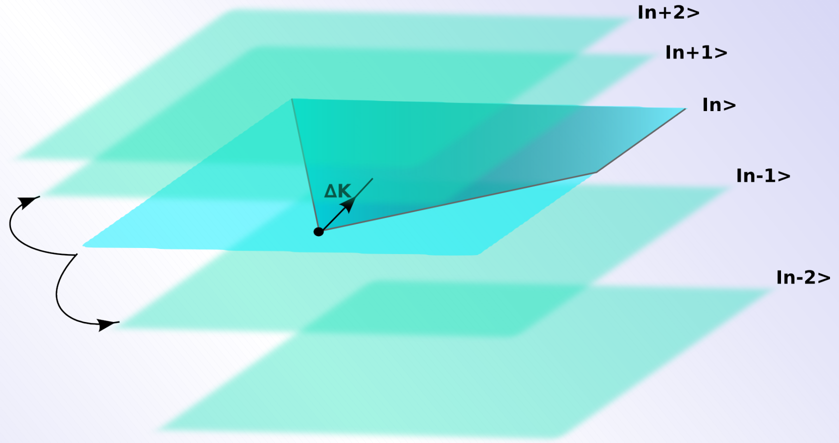
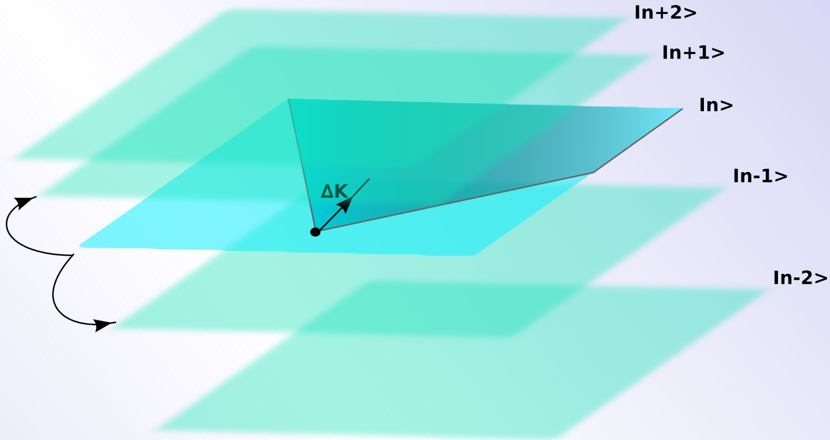


Those states are a part of the total set of pure states, but the structure is similar to the product of Minkowski and an ”infinite number of points”. Hence Moyal contains such a structure at least as a subset.



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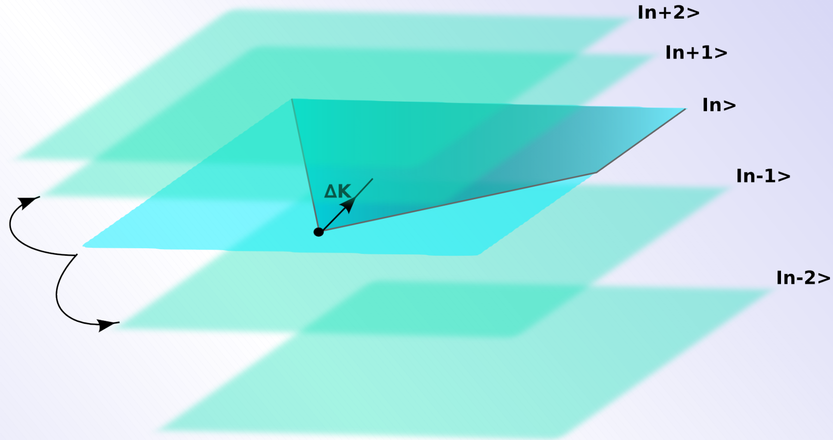

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This model represents waves packets under causal translations with a lower bound on time in order to change the energy level. However, the testability of this lower bound is problematic since, restoring physical dimensions, it is of the order of the Plank time.

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From the same limit of the causal condition, one can also recover the energy-momentum dispersion relation $E^2 = m^2 + p^2$ for a fermion traveling on the product of 4D-Minkowski and two-points.

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- Massless particles move on the continuous spacetime always at speed c (since the internal space is reduced to one point)