Introduction to noncommutative digital geometry

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based on arXiv:1701.06919 in collaboration with S. Majid

[supported by EU Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Individual Fellowship No 660061]

Plan of the talk:

Bonus:

1. Motivation - quantum space-times and noncommutative geometry

Main ingredients:

2. First order differential calculus over algebra & differential graded algebra

3. Quantum differentials, metrics and connections

Results:

4. Classification of noncommutative (inner) differential calculi in $n = 2, 3$ and 4 dim

NCG motivation from Quantum Gravity

Continuum differential geometry cannot be the geometry when both quantum and gravitational effects are present!

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One of the possibilities is to consider Noncommutative Geometry where the idea is to "algebralize" geometric notions and then generalize them to noncommutative algebras

Two faces of Noncommutative Geometry

C^* **algebras and spectral triples**
"Alain Connes NCG"

Quantum Groups* and quantum spacetimes
"Julius Wess NCG"
- deformations of (algebras of functions on) groups
- Hopf algebras and their modules

* The term "quantum group" first appeared in the theory of quantum integrable systems in the 80's, and later was formalized by V. Drinfeld, M. Jimbo, Y. Manin as a particular class of Hopf algebra.

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  - on quantum groups
  - on differential calculus on quantum planes and groups (quantum matrix groups can be seen as symmetries of nc planes).
  Together with Zumino - differential calculus on n-dim NC plane covariant under $Gl_q(n)$ quantum group : example of **NC differential geometry**.

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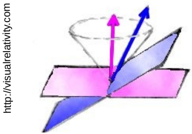
- Similar results were obtained by **S.L. Woronowicz**, but both different from Connes version.
  S. L. Woronowicz - compact matrix quantum groups(1987)

Noncommutative Geometry: origin of quantum space-times

Space–time



Minkowski spacetime (M, η) with the position of an event $p = (x^0, x^1, x^2, x^3)$

Minkowski space-time is the simplest example of a vacuum solution of the **Einsten equations**:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$\downarrow$$

$$g_{\mu\nu} = \eta_{\mu\nu}.$$

Noncommutative Geometry: origin of quantum space-times

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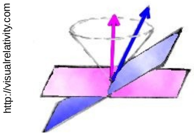

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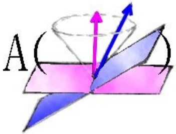
Coordinate algebra



$A = C^\infty(M) : [\, x^\mu, x^\nu \,] = 0$

coordinate functions

Quantum (noncommutative) spacetimes

1. Canonical (Heisenberg) spacetime A_θ:
$$[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu}$$
with deformation parameter h of length^2 (L_P) dim.

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"Moyal space"

S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172 (1995), [arXiv:hep-th/0303037].

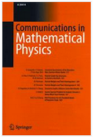
2. Lie-algebraic type spacetime:
$$[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu}_\rho \hat{x}^\rho$$
with deformation parameter h of mass (M_P) dim.

Special case: A_κ

$$[\hat{x}^0, \hat{x}^k] = \frac{i}{\kappa}\hat{x}^k \quad , \quad [\hat{x}^i, \hat{x}^k] = 0$$

- the so-called: κ-Minkowski spacetime.

S. Majid, H. Ruegg Phys.Lett. B334 (1994) [hep-th/9405107] ;
S. Zakrzewski J. Phys. A 127 (1994).



Communications in Mathematical Physics
June 2005, Volume 256, Issue 2, pp 305–374

Renormalisation of ϕ^4-Theory on Noncommutative $\mathbb{R}^4$ in the Matrix Base

Harald Grosse, Raimar Wulkenhaar

Article
First Online: 18 February 2005
DOI: 10.1007/s00220-004-1285-2

Cite this article as:
Grosse, H. & Wulkenhaar, R. Commun. Math. Phys. (2005) 256: 305. doi:10.1007/s00220-004-1285-2

Abstract

We prove that the real four-dimensional Euclidean noncommutative ϕ^4-model is renormalisable to all orders in perturbation theory. Compared with the commutative case, the bare action of relevant and marginal couplings contains necessarily an additional term: an harmonic oscillator potential for the free scalar field action. This entails a modified dispersion relation for the free theory, which becomes important at large distances (UV/IR-entanglement). The renormalisation proof relies on flow equations for the expansion coefficients of the effective action with respect to scalar fields written in the matrix base of the noncommutative $\mathbb{R}^4$. The renormalisation flow depends on the topology of ribbon graphs and on the asymptotic and local behaviour of the propagator governed by orthogonal Meixner polynomials.

Back to the main part of the talk

- Noncommutative geometry (NCG) - generalised notion of geometry.
- The noncommutative nature allows for obtaining quantum gravitational corrections to the classical solutions.
- 'Digital geometry'- proposing new kind of discretisation scheme.

Aim

- A version of quantum geometry - noncommutative Riemannian geometry.
- On a curved space one must use the methods of Riemannian geometry but in their quantum version.
- The formalism of noncommutative differential geometry does not require functions and differentials to commute, so is more general even when the algebra is classical.

Set up

- M - manifold and $C^\infty(M)$ - functions on a manifold $\rightarrow$ 'coordinate algebra' A.
- to define the differential geometry over A we need to find suitable space of **1-forms**.
- Noncommutative differential structure is expressed as **differential bimodule** (Ω^1, d) of 1-forms with d - obeying the Leibniz rule and $(dx)y \neq y(dx)$

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Module $E \ni e$ over algebra $A \ni a, b$:

$$\text{left } A\text{-module: } a.(b.e) = (ab).e$$
$$\text{right } A\text{-module: } (e.a).b = e.(ab)$$
$$A\text{-bimodule: } a.(e.b) = (a.e).b$$

Bimodule means we can associatively multiply such 1-forms by elements of A from the left and the right.

Differential calculus on algebra

- A is a (noncommutative) 'coordinate' algebra

Definition

A first order differential calculus (Ω^1, d) over A means:

1. Ω^1 is an A-bimodule
2. A linear map $\mathrm{d} : A \to \Omega^1$ such that

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b \quad , \forall a, b \in A$$

3. $\Omega^1 = span\{adb\}$
4. (optional) $kerd = k.1$ - connectedness condition

Differential graded algebra -DGA

- A is a (noncommutative) 'coordinate' algebra

Definition

DGA on an algebra A is:

1. A graded algebra $\Omega = \oplus_{n \geq 0} \Omega^n$, $\Omega^0 = A$
2. $d : \Omega^n \to \Omega^{n+1}$, s.t. $d^2 = 0$ and

$$d(\omega\rho) = (d\omega) \wedge \rho + (-1)^n \omega \wedge d\rho$$

$\forall \omega, \rho \in \Omega, \quad \omega \in \Omega^n.$

3. A, dA generate Ω
(optional surjectivity condition - if it holds we say it is an **exterior algebra** on A)

Differential calculus of the 'good' classical dimension

S. Majid, W.-Q. Tao, Pacific J. Math. 284 (2016), arXiv:1412.2284

Connected translation-invariant differential structures of the correct classical dimension on the enveloping algebra of a Lie algebra $\mathfrak{g}$ are in one - to - one correspondence with the **pre-Lie algebra** structures on $\mathfrak{g}$.

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Pre-Lie algebra structure
A map $\circ : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that:

$$(x^\mu \circ x^\nu) \circ x^\rho = (x^\mu \circ x^\rho) \circ x^\nu + x^\mu \circ (x^\nu \circ x^\rho - x^\rho \circ x^\nu), \quad \forall x^\mu, x^\nu, x^\rho \in \mathfrak{g}$$

A pre-Lie algebra with $\circ$ is a Lie algebra with the Lie bracket:

$$x^\mu \circ x^\nu - x^\nu \circ x^\mu = [x^\mu, x^\nu]$$

Differential calculus for coordinate algebra A

- For the coordinate algebra A a natural class of its calculi are obtained by the pre-Lie algebra structures.
- The differential calculus has generators $\mathrm{d}x^\mu$ where $\{x^\mu\}$ is a basis of $\mathfrak{g}$ and bimodule relations

$$[\mathrm{d}x^\mu, x^\nu] = \lambda \mathrm{d}(x^\mu \circ x^\nu).$$

(we take $\lambda = 1$)

The Jacobi identity

$$[[\mathrm{d}x^\mu, x^\nu], x^\rho] = [[\mathrm{d}x^\mu, x^\rho], x^\nu] + [\mathrm{d}x^\mu, [x^\nu, x^\rho]]$$

translates immediately to
$(x^\mu \circ x^\nu) \circ x^\rho = (x^\mu \circ x^\rho) \circ x^\nu + x^\mu \circ (x^\nu \circ x^\rho - x^\rho \circ x^\nu)$.

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The Leibniz rule

$$\mathrm{d}[x^\mu, x^\nu] = [\mathrm{d}x^\mu, x^\nu] + [x^\mu, \mathrm{d}x^\nu]$$

corresponds to $[x^\mu, x^\nu] = x^\mu \circ x^\nu - x^\nu \circ x^\mu$.

'Purely' noncommutative differential calculus

- We are interested in the noncommutative diff. calculus of certain type - inner

Definition

A differential algebra (A, Ω^1, d) is said to be **inner** if there exists $\theta \in \Omega^1$ such that:

$$[\theta, a] = da, \quad \forall a \in A$$

'Purely' noncommutative differential calculus

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If the pre-Lie algebra is unital with identity element e then the calculus is inner with $\theta = \mathrm{d}e$.

Quantum Riemannian Geometry

Ingredients of noncommutative Riemannian geometry as Quantum Geometry include:

- quantum differentials
- quantum metrics
- quantum-Levi Civita connections
- quantum curvature

Quantum metrics

Working with algebraic differential forms by **metric** we mean an element

$$g \in \Omega^1 \otimes_A \Omega^1$$

which is:

- 'quantum symmetric': $\wedge(g) = 0$,
- invertible
- central in the 'coordinate algebra' $A \ni x^\mu$:

$$[g, x^\mu] = 0$$

The general form of the quantum metric:

$$g = g_{\mu\nu} dx^\mu \otimes_A dx^\nu$$

Quantum connections

- Left connection on Ω^1 is $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ such that:

$$\nabla(a\omega) = a\nabla\omega + da \otimes \omega$$

for $a \in A$ and $\omega \in \Omega^1$.

Quantum connections

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- Bimodule connection is a left connection s.t. in addition for some (invertible) bimodule map:
$\sigma \quad : \quad \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ satisfies:

$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes da)$$

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- Metric compatible connection:

$$\nabla(g) = 0$$

+ TF = QLC

Torsion and curvature

Torsion of a connection on Ω^1 is

$$T_\nabla \omega = \wedge \nabla \omega - d\omega \quad : \quad \Omega^1 \to \Omega^2$$

Curvature:

$$R_\nabla \omega = (d \otimes id - \wedge(id \otimes \nabla))\nabla \omega \quad : \quad \Omega^1 \to \Omega^2 \otimes_A \Omega^1$$

Classification of noncommutative differential geometries

S. Majid, A. P., arXiv:1701.06919

We are interested in the classification of noncommutative differential geometries which are:

"digital"

- considered over the finite field $\mathbb{F}_2$ (instead that of $\mathbb{C}$) - proposing a new kind of 'discretisation scheme'.
- inner ($[\theta, x^\mu] = dx^\mu$).
- with constant coefficients (for the pre-Lie algebra structure, for quantum metrics and quantum connections).

'Digital geometry' set up

$A = \mathbb{F}_2[x^0, ...x^{n-1}]$ - affine / 'flat space' .

The pre-Lie algebra structure of the form:

$$x^\mu \circ x^\nu = V^{\mu\nu}_\rho x^\rho, \qquad V^{\mu\nu}_\rho \in \mathbb{F}_2$$

The corresponding **noncommutative differential calculus** is then given by:

$$[dx^\mu, x^\nu] = V^{\mu\nu}_\rho dx^\rho, \qquad V^{\mu\nu}_\rho \in \mathbb{F}_2$$

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$A = \mathbb{F}_2[x^0, ...x^{n-1}]$ - affine / 'flat space' .

The pre-Lie algebra structure of the form:

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The corresponding **noncommutative differential calculus** is then given by:

$$[dx^\mu, x^\nu] = V_\rho^{\mu\nu} dx^\rho, \qquad V_\rho^{\mu\nu} \in \mathbb{F}_2$$

And we look also for the (invertible and symmetric) quantum

metrics:

$$g = g_{\mu\nu} dx^\mu \otimes_A dx^\nu, \qquad g_{\mu\nu} \in \mathbb{F}_2$$

such that $[g, x^\mu] = 0$.

Remark for $n = 1$

$A = \mathbb{F}_2[x^0]$

- There is only **one** possibility which gives the finite difference calculus

$$\boxed{[\mathrm{d}x^0, x^0] = \lambda \mathrm{d}x^0}$$

for deformation parameter λ (which we take to be 1).

- The only candidate for a quantum metric is

$$g = \mathrm{d}x^0 \otimes \mathrm{d}x^0$$

which is central over $\mathbb{F}_2$.

Classification for $n = 2$

2 dim. inner nc geometries on $\mathbb{F}_2[x^0, x^1]$.

- possibilities for pre-Lie algebra $\circ$ with x^0, x^1 as basis: $x^0 \circ x^0 = x^0, \quad x^0 \circ x^1 = x^1 = x^1 \circ x^0$ and $x^1 \circ x^1 = \alpha x^0 + \beta x^1$ for constants α, β.
- Over $\mathbb{F}_2$ this means four possibilities

$$x^1 \circ x^1 = 0, \quad x^1 \circ x^1 = x^0, \quad x^1 \circ x^1 = x^1, \quad x^1 \circ x^1 = x^0 + x^1$$

with the first two isomorphic by $x^1 \mapsto x^1 + x^0$.

Classification for $n = 2$

2 dim. inner nc geometries on $\mathbb{F}_2[x^0, x^1]$.

- possibilities for pre-Lie algebra $\circ$ with x^0, x^1 as basis: $x^0 \circ x^0 = x^0, \quad x^0 \circ x^1 = x^1 = x^1 \circ x^0$ and $x^1 \circ x^1 = \alpha x^0 + \beta x^1$ for constants α, β.
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There are **three** inequivalent noncommutative calculi.

Next step is to find the metrics in each case and the metric compatible quantum Levi-Civita connections.

A) - First family in $n = 2$

Relations :

$$\left[\mathrm{d}x^0, x^0\right] = \mathrm{d}x^0, \left[\mathrm{d}x^0, x^1\right] = \mathrm{d}x^1 = \left[\mathrm{d}x^1, x^0\right], \left[\mathrm{d}x^1, x^1\right] = 0.$$

Quantum metrics and nonzero quantum Levi-Civita connections (metric compatible and torsion-free):

$$g_{A.I} = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0$$

5 possible QLCs

$$\nabla \mathrm{d}x^0 = \mathrm{d}x^1 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = 0$$

$$\nabla \mathrm{d}x^0 = \mathrm{d}x^1 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^1 \otimes \mathrm{d}x^1$$

$$\nabla \mathrm{d}x^0 = \mathrm{d}x^1 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^1 \otimes \mathrm{d}x^1$$

$$\nabla \mathrm{d}x^0 = 0, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^0 \otimes \mathrm{d}x^0$$

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$$g_{A.I} = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0$$
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→ 3QLCs

$$g_{A.II} = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1$$
$$\nabla \mathrm{d}x^0 = \mathrm{d}x^1 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = 0$$
$$\nabla \mathrm{d}x^0 = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^1 \otimes \mathrm{d}x^1$$
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B) - Second family in $n = 2$

Relations :

$$\left[\mathrm{d}x^0, x^0\right] = \mathrm{d}x^0, \left[\mathrm{d}x^0, x\right] = \mathrm{d}x = \left[\mathrm{d}x, x^0\right], [\mathrm{d}x, x] = \mathrm{d}x.$$

Quantum metric and quantum Levi-Civita connection:

$$g_B = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0$$

$$\nabla \mathrm{d}x^0 = 0, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^0 \otimes \mathrm{d}x^0$$

C - Third family in $n = 2$

Relations : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0$, $[\mathrm{d}x^0, x^1] = \mathrm{d}x^1 = [\mathrm{d}x^1, x^0]$, $[\mathrm{d}x^1, x^1] = \mathrm{d}x^0 + \mathrm{d}x^1$.

Quantum metrics and nonzero quantum Levi-Civita connections:

$$\longrightarrow \quad g_{C.I} = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1$$

$$\nabla \mathrm{d}x^0 = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^1 \otimes \mathrm{d}x^1$$

$$\longrightarrow \quad g_{C.II} = \mathrm{d}e \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0$$

$$\nabla \mathrm{d}x^0 = 0, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^0 \otimes \mathrm{d}x^0$$

$$\longrightarrow \quad g_{C.III} = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1$$

$$\nabla \mathrm{d}x^0 = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0$$

Curvature

For $n = 2$ all quantum Levi-Civita connections* are flat.
* bimodule quantum Levi-Civita (torsion free metric compatible) connections with constant coefficients and invertible σ.

Check:
For $g_{A.I} = \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0$ we have five connections.
The first one is given by:

$$\nabla \mathrm{d}x^0 = \mathrm{d}x^1 \otimes \mathrm{d}x^1, \quad \nabla \mathrm{d}x^1 = 0$$

The curvature is then:
$R_\nabla \mathrm{d}x^0 = (\mathrm{d} \otimes \mathrm{id} - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla))\nabla \mathrm{d}x^0 =$
$-(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)\left(\mathrm{d}x^1 \otimes \mathrm{d}x^1\right) = 0,$
$R_\nabla \mathrm{d}x^1 = (\mathrm{d} \otimes \mathrm{id} - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \nabla))\nabla \mathrm{d}x^1 = 0.$
Similarly, one can check for all the other bimodule connections.

Classification for $n = 3$

Inner noncommutative geometries on $\mathbb{F}_2[x^0, x^1, x^2]$.

In each case x^0, x^1, x^2 are basis and we have
$x^0 \circ x^0 = x^0, x^0 \circ x^1 = x^1 = x^1 \circ x^0, x^0 \circ x^2 = x^2 = x^2 \circ x^0$,
with the remaining relations as:

A: $x^1 \circ x^2 = 0 = x^2 \circ x^1, \quad x^1 \circ x^1 = 0 = x^2 \circ x^2$,

B: $x^1 \circ x^2 = 0 = x^2 \circ x^1, \quad x^1 \circ x^1 = x^1, \quad x^2 \circ x^2 = x^2$,

C: $x^1 \circ x^2 = 0 = x^2 \circ x^1, \quad x^1 \circ x^1 = x^1, \quad x^2 \circ x^2 = 0$,

D: $x^1 \circ x^2 = x^1 + x^2 = x^2 \circ x^1, \quad x^1 \circ x^1 = x^2, \quad x^2 \circ x^2 = x^1$,

E: $x^1 \circ x^2 = 0 = x^2 \circ x^1, \quad x^1 \circ x^1 = x^2, \quad x^2 \circ x^2 = 0$,

F:
$x^1 \circ x^2 = x^1 + x^2 = x^2 \circ x^1, \quad x^1 \circ x^1 = x^0 + x^1 + x^2, \quad x^2 \circ x^2 = x^1$.

These six inequivalent **commutative (pre-Lie) unital algebras** imply six **noncommutative differential calculi**.

Quantum geometries on $\mathbb{F}_2[x^0, x^1, x^2]$

For all six families we have: $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0$, $[\mathrm{d}x^0, x^1] = \mathrm{d}x^1 = [\mathrm{d}x^1, x^0]$, $[\mathrm{d}x^0, x^2] = \mathrm{d}x^2 = [\mathrm{d}x^2, x^0]$

Only non-zero commutators	# Q. metrics	# Nonzero QLC	$R_\nabla \neq 0$
A	0	-	-
B : $[\mathrm{d}x^1, x^1] = \mathrm{d}x^1$, $[\mathrm{d}x^2, x^2] = \mathrm{d}x^2$	1	3	0
C : $[\mathrm{d}x^1, x^1] = \mathrm{d}x^1$	2	13 for each g_C	2 for $g_{C.I}$! 3 for $g_{C.II}$!
D : $[\mathrm{d}x^1, x^1] = \mathrm{d}x^2$, $[\mathrm{d}x^2, x^2] = \mathrm{d}x^1$ $[\mathrm{d}x^1, x^2] = \mathrm{d}x + \mathrm{d}x^2 = [\mathrm{d}x^2, x^1]$	3	3 for each g_D	1 for $g_{D.I}$! 0 for $g_{D.II}$ 1 for $g_{D.III}$!
E : $[\mathrm{d}x^1, x^1] = \mathrm{d}x^2$	4	13 for each g_E	2 for $g_{E.I}$ 3 for $g_{E.II}$! 5 for $g_{E.III}$ 4 for $g_{E.IV}$
F : $[\mathrm{d}x^1, x^1] = \mathrm{d}x^1 + \mathrm{d}x^2 + \mathrm{d}x^0$ $[\mathrm{d}x^2, x^2] = \mathrm{d}x^1$ $[\mathrm{d}x^1, x^2] = \mathrm{d}x^0 + \mathrm{d}x^1 = [\mathrm{d}x^2, x^1]$	7	3 for each g_F	2 for each g_F ! except 0 for $g_{F.II}$

Purely quantum phenomenon

Family D

- $\left[\mathrm{d}x^1, x^2\right] = \mathrm{d}x^1 + \mathrm{d}x^2 = \left[\mathrm{d}x^2, x^1\right], \left[\mathrm{d}x^1, x^1\right] = \mathrm{d}x^2, \left[\mathrm{d}x^2, x^2\right] = \mathrm{d}x^1$
- Quantum metric (one of the three)

  $$g_{D.I} = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^1 + \mathrm{d}x^1 \otimes \mathrm{d}x^0 + \mathrm{d}x^0 \otimes \mathrm{d}x^2 + \mathrm{d}x^2 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1$$

  admits 3 ∇s

Purely quantum phenomenon

Family D

- $[\mathrm{d}x^1, x^2] = \mathrm{d}x^1 + \mathrm{d}x^2 = [\mathrm{d}x^2, x^1]$, $[\mathrm{d}x^1, x^1] = \mathrm{d}x^2$, $[\mathrm{d}x^2, x^2] = \mathrm{d}x^1$
- Quantum metric (one of the three)

  $$g_{D.I} = \mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^0\otimes\mathrm{d}x^1+\mathrm{d}x^1\otimes\mathrm{d}x^0+\mathrm{d}x^0\otimes\mathrm{d}x^2+\mathrm{d}x^2\otimes\mathrm{d}x^0+\mathrm{d}x^1\otimes\mathrm{d}x^1$$

  admits 3 ∇s
- only one of them has non-zero curvature: (D.I.3)
  $\nabla\mathrm{d}x^0 = \mathrm{d}x^0\otimes\mathrm{d}x^1+\mathrm{d}x^1\otimes\mathrm{d}x^0+\mathrm{d}x^0\otimes\mathrm{d}x^2+\mathrm{d}x^2\otimes\mathrm{d}x^0+\mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^2\otimes\mathrm{d}x^2$,
  $\nabla\mathrm{d}x^1 = \mathrm{d}x^0\otimes\mathrm{d}x^1+\mathrm{d}x^1\otimes\mathrm{d}x^0+\mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^2\otimes\mathrm{d}x^2$,
  $\nabla\mathrm{d}x^2 = \mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^1\otimes\mathrm{d}x^2+\mathrm{d}x^2\otimes\mathrm{d}x^1+\mathrm{d}x^2\otimes\mathrm{d}x^2$,

$$R_\nabla\mathrm{d}x^0 = \mathrm{d}x^1\wedge\mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^0\wedge\mathrm{d}x^1\otimes\mathrm{d}x^1+\mathrm{d}x^2\wedge\mathrm{d}x^0\otimes\mathrm{d}x^0,$$
$$R_\nabla\mathrm{d}x^1 = \mathrm{d}x^0\wedge\mathrm{d}x^1\otimes\mathrm{d}x^1+\mathrm{d}x^1\wedge\mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^1\wedge\mathrm{d}x^2\otimes\mathrm{d}x^0+\mathrm{d}x^1\wedge\mathrm{d}x^0\otimes\mathrm{d}x^2,$$
$$R_\nabla\mathrm{d}x^2 = \mathrm{d}x^0\wedge\mathrm{d}x^2\otimes\mathrm{d}x^2+\mathrm{d}x^2\wedge\mathrm{d}x^0\otimes\mathrm{d}x^0+\mathrm{d}x^2\wedge\mathrm{d}x^0\otimes\mathrm{d}x^1$$
$$+\mathrm{d}x^2\wedge\mathrm{d}x^1\otimes\mathrm{d}x^1+\mathrm{d}x^2\wedge\mathrm{d}x^1\otimes\mathrm{d}x^0$$

Classification for $n = 4$

There are 16 families in 4 dimensions (up to isomorphism).
Only 3 do not admit quantum metrics.

Classification method I

In 2-dimensions *a priori* the noncommutative geometry of interest is that of $\mathbb{F}_2[x^0, x^1]$, defined as **the universal enveloping algebra of Abelian Lie algebra** generated by basis elements x^0, x^1, the commutative algebra product $x^\mu \circ x^\nu = V^{\mu\nu}{}_\rho x^\rho$ induces the differential calculus.

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- in 2 dimensions with variables x^0 and x^1 we have three possibilities of inner calculi with θ as the differential of an element of the pre-Lie algebra and they are $\theta = \mathrm{d}x^0$ or $\mathrm{d}x^1$ or $\mathrm{d}x^0 + \mathrm{d}x^1$.

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- in 2 dimensions with variables x^0 and x^1 we have three possibilities of inner calculi with θ as the differential of an element of the pre-Lie algebra and they are $\theta = \mathrm{d}x^0$ or $\mathrm{d}x^1$ or $\mathrm{d}x^0 + \mathrm{d}x^1$.
- We find all (12) the possible solutions of the commutative pre-Lie algebra structure in 2 dimensions which induces inner differential calculus.

Classification method II

Let $S = \{s_1, ..., s_{12}\}$ be the set of all solutions and they can be grouped as follows:

• 4 cases of inner calculus with $\theta = \mathrm{d}x^0$, all have the following commutation relations $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0$, $[\mathrm{d}x^0, x^1] = \mathrm{d}x^1 = [\mathrm{d}x^1, x^0]$ and the remaining commutators we order as follows:

$s_1 : [\mathrm{d}x^1, x^1] = 0$, $s_2 : [\mathrm{d}x^1, x^1] = \mathrm{d}x^1$, $s_3 : [\mathrm{d}x^1, x^1] = \mathrm{d}x^0 + \mathrm{d}x^1$, $s_4 : [\mathrm{d}x^1, x^1] = \mathrm{d}x^0$;

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$s_4 : [\mathrm{d}x^1, x^1] = \mathrm{d}x^0$;

• 4 cases of inner calculus with $\theta = \mathrm{d}x^1$ and $[\mathrm{d}x^1, x^i] = \mathrm{d}x^i = [\mathrm{d}x^i, x^1]$ and
$s_5 : [\mathrm{d}x^0, x^0] = 0$, $s_6 : [\mathrm{d}x^0, x^0] = \mathrm{d}x^1$, $s_7 : [\mathrm{d}x^0, x^0] = \mathrm{d}x^0$,
$s_8 : [\mathrm{d}x^0, x^0] = \mathrm{d}x^0 + \mathrm{d}x^1$;

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s_4 : $[\mathrm{d}x^1, x^1] = \mathrm{d}x^0$;

• 4 cases of inner calculus with $\theta = \mathrm{d}x^1$ and $[\mathrm{d}x^1, x^i] = \mathrm{d}x^i = [\mathrm{d}x^i, x^1]$ and
s_5 : $[\mathrm{d}x^0, x^0] = 0$, s_6 : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^1$, s_7 : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0$,
s_8 : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0 + \mathrm{d}x^1$;

• 4 cases of inner calculus with $\theta = \mathrm{d}x^0 + \mathrm{d}x^1$, such that $[\mathrm{d}x^0 + \mathrm{d}x^1, x^i] = \mathrm{d}x^i$
s_9 : $[\mathrm{d}x^0, x^0] = 0$, $[\mathrm{d}x^0, x^1] = \mathrm{d}x^0 = [\mathrm{d}x^1, x^0]$, $[\mathrm{d}x^1, x^1] = \mathrm{d}x^0 + \mathrm{d}x^1$,
s_{10} : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^1$, $[\mathrm{d}x^0, x^1] = \mathrm{d}x^0 + \mathrm{d}x^1 = [\mathrm{d}x^1, x^0]$, $[\mathrm{d}x^1, x^1] = \mathrm{d}x^0$,
s_{11} : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0$, $[\mathrm{d}x^0, x^1] = 0 = [\mathrm{d}x^1, x^0]$, $[\mathrm{d}x^1, x^1] = \mathrm{d}x^1$,
s_{12} : $[\mathrm{d}x^0, x^0] = \mathrm{d}x^0 + \mathrm{d}x^1$, $[\mathrm{d}x^0, x^1] = \mathrm{d}x^1 = [\mathrm{d}x^1, x^0]$, $[\mathrm{d}x^1, x^1] = 0$.

Classification method III

Due to the action of the group of isomorphisms G on the set S of the solutions we get only three inequivalent families, corresponding to the orbits of the action of the group.

- The group of isomorphisms in 2 dimensions over $\mathbb{F}_2$ is $G = SL(2,2) = PSL(2,2) = S_3$ (of order 6)
- Its action on the set of solutions S results in the change of variables
- Already the set of the first 4 solutions (for inner calculi with $\mathrm{d}x^0$,i.e. $S_1 = \{s_1, s_2, s_3, s_4\}$ splits into the three orbits under the action of the group G.

Classification method IV

If G acts on a set S the orbits of this action are the sets

$$O_s = \{s' \in S \quad | \quad g \cdot s = s' \;\; \text{for} \;\; g \in G\}.$$

We obtain the following:

- For the calculus A the orbit consist of the elements: $O_{s_1} = \{s_1, s_4, s_5, s_6, s_9, s_{12}\}, |O_{s_1}| = 6$ and the isotropy group of element $Hs_1 = \{\imath\}$.
- For the calculus B: $O_{s_2} = \{s_2, s_7, s_{11}\}, |O_{s_2}| = 2$ and the isotropy group of the element $Hs_2 = \{\imath, u\}$.
- For the calculus C: $O_{s_3} = \{s_3, s_8, s_{10}\}, |O_{s_3}| = 2$ and the isotropy group of the element $Hs_3 = \{\imath, v\}$.

Where

$$u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

And, e.g. the action of the element u corresponds to the change

of variables

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

Classification method V

The same method was used in $n = 3, 4$*.

- In $n = 3$ for $\mathbb{F}_2[x^0, x^1, x^2]$ there is 7 possibilities for element θ.
  The set of solutions has 7×64 elements.
  The group of isomorphisms over
  $\mathbb{F}_2 : G = GL(3, \mathbb{F}_2) = PSL(2, 7)$.
  The set of the solutions splits into **six** orbits.

- In $n = 4$ for $\mathbb{F}_2[x^0, x^1, x^2, x^3]$ we get, only for $\theta = dx^0$, 5216 solutions, and similarly for other choices of θ.
  The isomorphisms group has the order $|G| = 20160$.
  The set of the solutions splits into **sixteen** orbits.

* with an extensive use of 'Mathematica' and 'R'.

Conclusions & Perspectives

- We were interested in inner differential calculi and we classified all noncommutative (digital) Riemannian geometries on $\mathbb{F}_2[x^0, ..., x^{n-1}]$ up to $n = 3$, with some partial results for $n = 4$.

- **Inner case** - useful restriction, but similar analysis can be done without this requirement.

- Our 'coordinate algebra' $A = \mathbb{F}_2[x^0, ..., x^{n-1}]$ has been **classical**, but the same formulae for differential geometries hold if we have commutation relations of the **Heisenberg type**.

Moyal space

- Main limitation - constant coefficients.
- Some of the cases have non-zero curvature - purely 'quantum' phenomenon.

Conclusions & Perspectives

- Quantum gravity is normally seen as a weighted 'sum' over all possible metrics but our results are a step towards a deeper approach in which we must also 'sum' over differential structures.

- Additionally, one can translate the algebra over $\mathbb{F}_2$ into digital electronics, and perhaps quantum information - justifying our terminology - as 'digital geometry'.

- Possible applications in other contexts such as 'digital' models of quantum mechanics phase spaces or other 'geometric' applications in engineering.

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Thank you!