Noncommutative Spherically Symmetric Spacetimes

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Quantum Spacetime Hypothesis

Quantum Gravity

$\Downarrow$

Quantum Geometry

$\Downarrow$

Classical Geometry

Outline

- Differential graded algebra $\rightarrow$ quantum metric
- Geometry at the semi-classical level
- Application to spherically symmetric spacetimes

Quantum Differential Algebra

Quantum Algebra

- $(\mathcal{A}, \cdot)$ unital, associative etc.
- Not necessarily commutative: $ab \neq ba$

Quantum Differential Algebra

- Ω^1 is a bimodule of $\mathcal{A}$ so $a((db)c) = (a(db))c$
- $d : \mathcal{A} \to \Omega^1$ so $d(ab) = (da)b + a(db)$
- $\{adb\}$ span Ω^1
- $\ker d = k$

Extends to Differential Graded Algebra $\Omega = \bigoplus_i \Omega^i$

- $d^2 = 0$
- Product $\wedge : \Omega^n(\mathcal{A}) \otimes \Omega^m(\mathcal{A}) \to \Omega^{n+m}$

Quantum metric

$$g \in \Omega^1 \otimes \Omega^1 \text{ with quantum symmetry } \wedge(g) = 0$$

Has an inverse:

$$(\ ,\ ) : \Omega^1 \otimes \Omega^1 \to \mathcal{A}$$

Bimodule requirement forces g to be central!
Also require a compatible bimodule connection

$$\nabla : \Omega^1 \to \Omega^1 \otimes \Omega^1, \quad \sigma : \Omega^1 \otimes \Omega^1 \to \Omega^1 \otimes \Omega^1$$

- Metric compatible $\nabla(g) = 0$
- Torsion free $\wedge \nabla = d$

Quantum Levi-Civita connection!

Example

S. Majid, W. Tao; Phys. Rev. D 91 (2015)

Take Majid-Ruegg model: $[x^i, t] = i\lambda_P x^i$ & $[t, dt] = i\lambda_P \alpha dt$

$$g = a_{ij} dx^i \otimes dx^j + b_i (dx^i \otimes dt + dt \otimes dx^i) + c dt \otimes dt$$

$$[g, f] = 0 \quad \forall f \in \mathcal{A}, \quad \wedge(g) = 0$$

$$\Downarrow$$

$$g = \delta^{-1} d\Omega^2 + a r^{-2} dr \otimes dr + b r^{\alpha-1}(dr \otimes dt + dt \otimes dr) + c r^{2\alpha} dt \otimes dt$$

$$\bar{\delta} = \frac{c\alpha^2}{b^2 - ac} > 0 \quad a, b, c, \delta \in \mathbf{R}$$

This is the Bertotti-Robinson metric.
Algebra forces the metric!

Semi-Classical Quantum Gravity

E. Beggs, S.Majid; Class.Quant.Grav. 31 (2014)

Interested in quantization to $\mathcal{O}(\lambda) \to$ Semiquantization

- Controlled by Poisson bracket $(\{\,,\,\}, C^\infty(\mathcal{M}))$
- Have $[a,b] = \lambda\{a,b\} + \mathcal{O}(\lambda^2)$
- $\{\,,\,\} \leftrightarrow \omega^{ij}$

For differential structure have

- For $\eta \in \Omega^1$ have $[a,\eta] = \lambda \nabla_{\hat{a}}\eta + \mathcal{O}(\lambda^2)$
- $\hat{a} = \{a,\ \}$ and $\nabla_{\hat{a}}$ is a Poisson (pre)connection defined along a Hamiltonian vector field
- $d\{a,b\} = \nabla_{\hat{a}} b - \nabla_{\hat{a}} b$

Look at associativity $\longrightarrow$ Jacobi identity

$$[a,[db,c]]+[c,[a,db]]+[db,[c,a]] \sim \nabla_{\hat{a}}\nabla_{\hat{b}}db - \nabla_{\hat{a}}\nabla_{\hat{c}}db - \nabla_{\widehat{\{a,c\}}}db$$

Nonflat connection $\longrightarrow$ nonassociative calculus at $\mathcal{O}(\lambda^2)$ (but associative functions)

Semiquantization

E. Beggs, S.Majid; J.Geom.Phys. 114 (2017)

- Inverse problem: Can we construct a quantization that will produce a particular metric g?
- Answer: Yes, but only to first order in λ: Semiquantization
- We map geometric data to algebraic data

$$\text{Q:(vector bundles and connections)} \longrightarrow \text{(bimodules over a deformed algebra)}$$

From the standpoint of physics can view λ as the effective scale of the theory (e.g. Planck scale), so it is reasonable to work only to first order.

Semiquantization

What do we need? Have

- Metric g
- Poisson bracket $\{\,,\,\} \Leftrightarrow \sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu}\omega^{\beta\gamma}{}_{,\mu} = 0$
- Poisson connection ∇
- Levi-Civita connection: $\widehat{\nabla} = \nabla + S$
  $S^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(T_{\beta\mu\nu} + T_{\mu\beta\nu} + T_{\nu\beta\mu}\right)$

And want

- Metric: $g \longrightarrow g_1$
- Connection: $\nabla \longrightarrow \nabla_1$
- Tensor product: $E \otimes F \longrightarrow Q(E) \otimes_1 Q(F)$
- Wedge product: $\eta \wedge \xi \longrightarrow Q(\eta) \wedge_1 Q(\xi)$

Semiquantization

We take $\mathcal{A} = C^\infty(\mathcal{M})$ and a deformed product $\bullet$. Now have

$$a \bullet b = ab + \frac{\lambda}{2}\omega^{\mu\nu}\partial_\mu a \partial_\nu b$$

Extend this to Ω

$$a \bullet dx = adx + \frac{\lambda}{2}\omega^{\mu\nu}a_{,\mu}\nabla_\nu dx$$

$$dx \bullet a = (dx)a + \frac{\lambda}{2}\omega^{\mu\nu}a_{,\nu}\nabla_\mu dx$$

Defines a Bimodule structure $E \otimes_1 A_1 \to E$

This is the action of Q!

Semiquantization

Quantum metric

$$g_1 = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2}\omega^{\alpha\beta}\Gamma_{\mu\alpha\kappa}\Gamma^\kappa{}_{\beta\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2}\mathcal{R}_{\mu\nu} dx^\mu \otimes_1 dx^\nu$$

Quantum Connection

$$\begin{aligned}\nabla_1 \mathrm{d}x^\iota = -\Big[\widehat{\Gamma}^\iota{}_{\mu\nu} + \frac{\lambda}{2}\omega^{\alpha\beta}\Big(\widehat{\Gamma}^\iota{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu} - \widehat{\Gamma}^\iota{}_{\kappa\tau}\Gamma^\kappa{}_{\alpha\mu}\Gamma^\tau{}_{\beta\nu} \\ + \widehat{\Gamma}^\iota{}_{\alpha\kappa}(R^\kappa{}_{\nu\mu\beta} + \nabla_\beta S^\kappa{}_{\mu\nu})\Big)\Big] \mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu\end{aligned}$$

Quantum Laplace operator

$$\Box_1 f := (\ ,\ )_1 \nabla_1 \mathrm{d}f = \Box f + \frac{\lambda}{2}\omega^{\alpha\beta}(\mathrm{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha})(\widehat{\nabla}_\beta \mathrm{d}f)_\gamma$$

Semiquantization

Generalized Ricci two form

$$\mathcal{R} = g_{\alpha\beta}\omega^{\alpha\gamma}(\nabla_\gamma S^\beta{}_{\mu\nu} - R^\beta{}_{\mu\nu\gamma})dx^\nu \wedge dx^\mu$$

Then for $\widehat{\nabla} = \nabla + S$ have the conditions

- Poisson Compatibility $\widehat{\nabla}_\gamma \omega^{\alpha\beta} + S^\alpha{}_{\delta\gamma}\omega^{\delta\beta} + S^\beta{}_{\delta\gamma}\omega^{\alpha\delta} = 0$
- Metric compatibility $\nabla g = 0$
- Quantum Levi-Civita condition

$$\widehat{\nabla}\mathcal{R} + \omega^{\alpha\beta}\, g_{\rho\sigma}\, S^\sigma{}_{\beta\nu}(R^\rho{}_{\mu\gamma\alpha} + \nabla_\alpha S^\rho{}_{\gamma\mu})\, \mathrm{d}x^\gamma \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0$$

Application

Take generic spherically symmetric metric

$$g = a^2(r,t)\mathrm{d}t\otimes\mathrm{d}t + b^2(r,t)\mathrm{d}r\otimes\mathrm{d}r + c^2(r,t)(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)$$

Poisson tensor? Ansatz: spherical symmetry

$$\omega^{23} = \frac{f(r,t)}{\sin(\theta)} \quad \omega^{01} = g(r,t)$$

So have Levi-Civita connection $\widehat{\nabla}$.

- Need ω Poisson: $g\partial_t f = g\partial_r f = 0$
- Find Poisson connection $\nabla = \widehat{\nabla} - S$
- Quantum L.C. condition

Uniqueness Theorem

Find $f = k$ and $g = 0$ and

$$S_{022} = c\partial_t c,\ S_{122} = c\partial_r c,\ S_{033} = c\partial_t c \sin^2(\theta),\ S_{133} = c\partial_r c \sin^2(\theta)$$

$$S_{120} = S_{123} = S_{223} = S_{320} = S_{130} = S_{132} = S_{230} = S_{233} = 0$$

Uniquely!
Unique quantization uo to $\mathcal{O}(\lambda^2)$ with algebra

$$[z^i, z^j] = \lambda \epsilon^{ij}{}_k z^k, \quad [z^i, dz^j] = \lambda z^i \epsilon^j{}_{mn} z^m dz^n$$

$$\sum (z^i)^2 = 1, \quad \sum z^i dz^i = 0$$

Nonassociative fuzzy sphere.
Central (classical) t, r, dt, dr

Quantum Geometry

Recall, we have g_1, $(\,,\,)_1$, ∇_1, $\wedge_1$. Can construct

- Quantum Riemann tensor

$$\mathrm{Riem}_1(dx^\mu) = (\mathrm{d} \otimes_1 \mathrm{id} - (\wedge_1 \otimes_1 \mathrm{id})(\mathrm{id} \otimes_1 \nabla_1))\nabla_1(dx^\mu)$$

- Quantum Ricci tensor

$$\mathrm{Ricci}_1 = ((\;,\;)_1 \otimes_1 \mathrm{id} \otimes_1 \otimes \mathrm{id})(\mathrm{id} \otimes_1 i_1 \otimes_1 \mathrm{id})(\mathrm{id} \otimes_1 \mathrm{Riem}_1)(g_1)$$

Where $\wedge \circ i = \mathrm{id}$ and $i : \Omega^2 \to \Omega \otimes \Omega$

- Quantum Ricci scalar

$$S_1 = (\;,\;)_1 \mathrm{Ricci}_1$$

- Quantum Laplace operator

$$\Box_1 f := (\;,\;)_1 \nabla_1 \mathrm{d} f = \Box f + \frac{\lambda}{2} \omega^{\alpha\beta} (\mathrm{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha})(\widehat{\nabla}_\beta \mathrm{d} f)_\gamma$$

Metric and Connection

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda c^2}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}t) = -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \frac{\lambda}{2(z^3)^2}\frac{c\partial_t c}{a^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}r) = -\widehat{\Gamma}^1{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2(z^3)^2}\frac{c\partial_r c}{b^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}z^a) = -\widehat{\Gamma}^i{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\left(\epsilon_{ijk}z^k z^a\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \frac{1}{(z^3)^2}\epsilon^a{}_{i3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^i\right)$$

Riemann Curvature tensor

$$\begin{aligned}\mathrm{Riem}_1(\mathrm{d}t) &= -\frac{1}{2}\widehat{R}^0{}_{\alpha\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\alpha-\frac{\lambda}{2(z^3)^2}\Big(\\ &\epsilon_{3ij}z^3(F_3\mathrm{d}t-F_4\mathrm{d}r)\wedge\mathrm{d}z^i\otimes_1\mathrm{d}z^j+\\ &+\epsilon_{3ij}z^i(F_3\mathrm{d}t-F_4\mathrm{d}r)\wedge\mathrm{d}z^j\otimes_1\mathrm{d}z^3\Big)\end{aligned}$$

$$\begin{aligned}\mathrm{Riem}_1(\mathrm{d}r) &= -\frac{1}{2}\widehat{R}^1{}_{\alpha\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\alpha+\frac{\lambda}{2(z^3)^2}\Big(\\ &+\epsilon_{3ij}z^3(F_5\mathrm{d}t-F_3\mathrm{d}r)\wedge\mathrm{d}z^i\otimes_1\mathrm{d}z^j\\ &+\epsilon_{3ij}z^i(F_5\mathrm{d}t-F_3\mathrm{d}r)\wedge\mathrm{d}z^j\otimes_1\mathrm{d}z^3\Big)\end{aligned}$$

$$\mathrm{Riem}_1(\mathrm{d}z^\mu)=-\frac{1}{2}\widehat{R}^\mu{}_{\alpha\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\alpha+\frac{\lambda F_6}{2(z^3)^2}(1+(z^3)^3)\mathrm{d}z^1\wedge\mathrm{d}z^2\otimes_1\mathrm{d}z^\mu$$

Ricci Curvature tensor and Laplace operator

Ask that $\wedge_1(\mathrm{Ricci}_1) = 0$ then

$$\begin{aligned}\mathrm{Ricci}_1 &= -\frac{1}{2}\widehat{R}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu \\ &\quad -\frac{\lambda}{4(z^3)^2}(F_6 + F_5 - F_4)\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right) \\ &\quad -\frac{3\lambda}{4z^3}F_6\epsilon_{3ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j\end{aligned}$$

and

$$S_1 = -\frac{1}{2}\widehat{S}$$

For Laplace operator

$$\Box_1 f = g^{\alpha\beta}\left(f_{,\alpha\beta} + f_{,\gamma}\widehat{\Gamma}^\gamma{}_{\alpha\beta}\right)$$

Any Questions?

$$
\begin{aligned}
\mathrm{Riem}(dx^\mu) &= (d\otimes \mathrm{id} - (\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\nabla))(-\Gamma^\mu{}_{\alpha\beta}dx^\alpha\otimes dx^\beta) \\
&= -\Gamma^\mu{}_{\alpha\beta,\gamma}dx^\gamma\wedge dx^\alpha\otimes dx^\beta + \Gamma^\mu{}_{\alpha\beta}\Gamma^\beta{}_{\gamma\delta}dx^\alpha\wedge dx^\gamma\otimes dx^\delta \\
&= -(\Gamma^\mu{}_{\beta\gamma,\alpha} - \Gamma^\mu{}_{\alpha\delta}\Gamma^\delta{}_{\beta\gamma})dx^\alpha\wedge dx^\beta\otimes dx^\gamma \\
&= -\frac{1}{2}R^\mu{}_{\gamma\alpha\beta}dx^\alpha\wedge dx^\beta\otimes dx^\gamma
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ricci} &= ((\,,\,)\otimes\mathrm{id})(\mathrm{id}\otimes i\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{Riem})g_{\mu\nu}dx^\mu\otimes dx^\nu \\
&= -\frac{1}{2}((\,,\,)\otimes\mathrm{id})(\mathrm{id}\otimes i\otimes\mathrm{id})g_{\mu\kappa}R^\kappa{}_{\gamma\alpha\beta}dx^\mu\otimes dx^\alpha\wedge dx^\beta\otimes dx^\gamma \\
&= -\frac{1}{2}((\,,\,)\otimes\mathrm{id})g_{\mu\kappa}R^\kappa{}_{\gamma\alpha\beta}dx^\mu\otimes dx^\alpha\otimes dx^\beta\otimes dx^\gamma \\
&= -\frac{1}{2}g_{\mu\kappa}g^{\mu\alpha}R^\kappa{}_{\gamma\alpha\beta}dx^\beta\otimes dx^\gamma \\
&= -\frac{1}{2}R_{\gamma\beta}dx^\beta\otimes dx^\gamma
\end{aligned}
$$