Pure state post-selection is universal

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Pure state pre- and postselection

Logical options:

$\mathcal{H}_\mathrm{i}, \mathcal{H}_\mathrm{f}, \mathcal{H}$ are factors of $\mathcal{H}_\mathrm{tot}$
$|\Psi\rangle \in \mathcal{H}_\mathrm{tot}$ total inital state
$|\mathrm{i}\rangle \in \mathcal{H}_\mathrm{i}$ preselection state (projection by $|\mathrm{i}\rangle\langle\mathrm{i}|$)
$|\mathrm{f}\rangle \in \mathcal{H}_\mathrm{f}$ postselection state (projection by $|\mathrm{f}\rangle\langle\mathrm{f}|$)
$\mathcal{H}$ observed state space

Special cases:

1) $\mathcal{H}_\mathrm{tot} = \mathcal{H}_\mathrm{i} = \mathcal{H}_\mathrm{f} = \mathcal{H}$
2) $\mathcal{H}_\mathrm{tot} = \mathcal{H}_\mathrm{i} = \mathcal{H}_\mathrm{f} = \mathcal{H} \otimes \mathcal{H}'$ ($\mathcal{H}'$: ancilla space)

SilvaGuryanovaBrunnerLindenShortPopescu PRA**89**,012121(2014)

2-time state

$$|\mathrm{i}\rangle, |\mathrm{f}\rangle \in \mathcal{H}; \quad \sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{1} \quad \Longrightarrow \quad p(\mu, succ)$$

$$p(succ) = \sum_\mu p(\mu, succ), \qquad p(\mu|succ) = \frac{p(\mu, succ)}{\sum_\mu p(\mu, succ)}$$

$$\begin{aligned} p(\mu, succ) &= |\langle \mathrm{f}|\hat{A}_\mu|\mathrm{i}\rangle|^2 = \mathbf{tr}\,(|\mathrm{f}\rangle\langle\mathrm{f}|)(\hat{A}_\mu|\mathrm{i}\rangle\langle\mathrm{i}|\hat{A}_\mu^\dagger) \\ &\qquad \equiv \mathbf{tr}\,(\hat{A}_\mu \otimes \hat{A}_\mu^\dagger)\widehat{\rho}_{\mathrm{if}} \\ 2\text{–}time\ density:\quad \widehat{\rho}_{\mathrm{if}} &= |\mathrm{i}\rangle\langle\mathrm{f}| \otimes |\mathrm{f}\rangle\langle\mathrm{i}| \\ &\equiv \hat{\Psi}_{\mathrm{if}} \otimes \hat{\Psi}_{\mathrm{if}}^\dagger \\ 2\text{–}time\ pure\ state:\quad \hat{\Psi}_{\mathrm{if}} &= |\mathrm{i}\rangle\langle\mathrm{f}| \end{aligned}$$

So far $\hat{\Psi}_{\mathrm{if}} = |\mathrm{i}\rangle\langle\mathrm{f}|$ is unentangled, $\widehat{\rho}_{\mathrm{if}} = \hat{\Psi}_{\mathrm{if}} \otimes \hat{\Psi}_{\mathrm{if}}^\dagger$ is not mixed (pure).

General 2-time state

$$p(\mu, succ) = \mathbf{tr}\ (\hat{A}_\mu \otimes \hat{A}_\mu^\dagger)\widehat{\rho}_{\rm if}$$

Success becomes independent of measurement if it is weak:

$$p_{\rm WM}(succ) = \mathbf{tr}\widehat{\rho}_{\rm if}$$

Definitive math conditions for $\widehat{\rho}_{\rm if}$:

$$1)\ \mathbf{tr}(\hat{V} \otimes \hat{V}^\dagger)\widehat{\rho}_{\rm if} \geq 0,\quad \forall \hat{V}; \qquad 2)\ \mathbf{tr}\widehat{\rho}_{\rm if} \leq 1$$

Standard form:

$$\widehat{\rho}_{\rm if} = \sum_r \hat{\Psi}_{\rm if}^r \otimes \hat{\Psi}_{\rm if}^{r\dagger}, \quad \mathbf{tr}\hat{\Psi}_{\rm if}^{r\dagger}\hat{\Psi}_{\rm if}^s = 0\ \ (r \neq s)$$

Can we prepare all $\widehat{\rho}_{\rm if}$? Yes, upto a prefactor since

$$\mathbf{tr}\widehat{\rho}_{\rm if} = p_{\rm WM}(succ)$$

Preparing entangled 2-time pure state

$|\mathrm{i}\rangle, |\mathrm{f}\rangle \in \mathcal{H} \otimes \mathcal{H}'$

$$
\begin{aligned}
p(\mu, succ) &= \mathbf{tr}\ (|\mathrm{f}\rangle\langle\mathrm{f}|)(\hat{A}_\mu \otimes \hat{1}')|\mathrm{i}\rangle\langle\mathrm{i}|(\hat{A}_\mu^\dagger \otimes \hat{1}') \equiv \mathbf{tr}\ (\hat{A}_\mu \otimes \hat{A}_\mu^\dagger)\widehat{\rho}_{\mathrm{if}} \\
\widehat{\rho}_{\mathrm{if}} &= (\mathbf{tr}'|\mathrm{i}\rangle\langle\mathrm{f}|) \otimes (\mathbf{tr}'|\mathrm{f}\rangle\langle\mathrm{i}|) \equiv \hat{\Psi}_{\mathrm{if}} \otimes \hat{\Psi}_{\mathrm{if}}^\dagger \\
\hat{\Psi}_{\mathrm{if}} &= \mathbf{tr}'|\mathrm{i}\rangle\langle\mathrm{f}|
\end{aligned}
$$

In coordinates:

$|\mathrm{i}\rangle = \sum_{k,r} c_{kr}^{\mathrm{i}}\, |k\rangle \otimes |r\rangle'\,, \qquad |\mathrm{f}\rangle = \sum_{k,r} c_{kr}^{\mathrm{f}}\, |k\rangle \otimes |r\rangle'$

$$
\hat{\Psi}_{\mathrm{if}} = \mathbf{tr}'|\mathrm{i}\rangle\langle\mathrm{f}| = \mathbf{tr}' \sum_{k,r} c_{kr}^{\mathrm{i}}\, |k\rangle \otimes |r\rangle' \sum_{l,s} c_{ls}^{\mathrm{f}\star}\, \langle l| \otimes \langle s|'
$$

$$
= \sum_{kl}\ {}_k[c^{\mathrm{i}}][c^{\mathrm{f}}]_l^\dagger\, |k\rangle\langle l| \equiv \sum_{kl} c_{kl}^{\mathrm{if}}\, |k\rangle\langle l|
$$

Can we reach any *2-time amplitudes* c_{kl}^{if}?

Can we prepare any 2-time entangled pure state?

From previous slide:

$|\mathrm{i}\rangle = \sum_{k,r} c^{\mathrm{i}}_{kr} |k\rangle \otimes |r\rangle'$, $\quad |\mathrm{f}\rangle = \sum_{k,r} c^{\mathrm{f}}_{kr} |k\rangle \otimes |r\rangle' \quad \Longrightarrow \hat{\Psi}_{\mathrm{if}} = \sum_{kl} c^{\mathrm{if}}_{kl} |k\rangle\langle l|$

$p_{\mathrm{WM}}(succ) = |\mathbf{tr}\hat{\Psi}_{\mathrm{if}}|^2 = |\mathbf{tr} c^{\mathrm{if}}|^2$

$c^{\mathrm{if}} = c^{\mathrm{i}} c^{\mathrm{f}\dagger}$ (recall: $\mathbf{tr} c^{\mathrm{i}\dagger} c^{\mathrm{i}} = \mathbf{tr} c^{\mathrm{f}\dagger} c^{\mathrm{f}} = 1$)

2-time amplitude c^{if} is subnormalized unless $c^{\mathrm{i}} = c^{\mathrm{f}}$ (upto a phase).

$* * * *$

Silva et al. prepare any $\hat{\Psi}$ upto a factor: $c^{\mathrm{f}} = 1/\sqrt{d}$ yields $c^{\mathrm{if}} = c^{\mathrm{i}}/\sqrt{d}$

$p_{\mathrm{WM}}(succ) = |\mathbf{tr} c^{\mathrm{i}}|^2/d$ can be suboptimal.

If $c^{\mathrm{if}} \geq 0$, then we can choose $c^{\mathrm{i}} = c^{\mathrm{f}} = \sqrt{c^{\mathrm{if}}/\mathbf{tr} c^{\mathrm{if}}}$ at

$p_{\mathrm{WM}}(succ) = 1$.

Open issue: To prepare a given $\hat{\Psi}$ what are the 'closest' $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$ to guarantee the highest $p_{\mathrm{WM}}(succ)$?

Mixed 2-time state

$$|\mathrm{i}\rangle, |\mathrm{f}\rangle \in \mathcal{H}\otimes\mathcal{H}', \quad \sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{1}, \quad \sum_\mu \hat{A}_r'^\dagger \hat{A}_r' = \hat{1}'$$

$$\begin{aligned} p(\mu, r, succ) &= \mathbf{tr}\ (|\mathrm{f}\rangle\langle\mathrm{f}|)(\hat{A}_\mu\otimes\hat{A}_r')|\mathrm{i}\rangle\langle\mathrm{i}|(\hat{A}_\mu^\dagger\otimes\hat{A}_r'^\dagger) \\ p(\mu, succ) &= \sum_r p(\mu, r, succ) \equiv \mathbf{tr}(\hat{A}_\mu\otimes\hat{A}_\mu^\dagger)\widehat{\rho}_{\mathrm{if}} \\ \widehat{\rho}_{\mathrm{if}} &= \sum_r \left\{ \left(\mathbf{tr}'(\hat{1}\otimes\hat{A}_r')|\mathrm{i}\rangle\langle\mathrm{f}|\right)\otimes\left(\mathbf{tr}'(\hat{1}\otimes\hat{A}_r'^\dagger)|\mathrm{f}\rangle\langle\mathrm{i}|\right)\right\} \\ &\equiv \sum_r \hat{\Psi}_{\mathrm{if}}^r \otimes \hat{\Psi}_{\mathrm{if}}^{r\dagger} \end{aligned}$$

Example:

$$|\mathrm{i}\rangle = \sum_r \sqrt{p_r}\,|\mathrm{i}; r\rangle\otimes|r\rangle', \quad |\mathrm{f}\rangle = \frac{1}{\sqrt{d'}}\sum_r |\mathrm{f}; r\rangle\otimes|r\rangle', \quad \hat{A}_r' = |r\rangle'\langle r|'$$

$$\widehat{\rho}_{\mathrm{if}} = \frac{1}{d'}\sum_r p_r\,|\mathrm{i}; r\rangle\langle\mathrm{f}; r|\otimes|\mathrm{f}; r\rangle\langle\mathrm{i}; r|$$

Issue: Pure state pre/postselection has $1/d'$-times smaller $p_{\mathrm{WM}}(succ)$ vs Silva et al.

Inferring success rate without tomography

Silva et al.: For mixed 2-states romography, projective and WM's are insufficient, generalized measurements are needed, they constructed one.

Assume pure 2-states! Both projective and WM's remain insufficient. What WM's are sufficient for?
WMs yield $p_{\mathrm{WM}}(succ)$ without tomography.

Example: AAV spin weak value
$\hat\Psi = |\mathrm{i}\rangle\langle\mathrm{f}| = |\vec n_\mathrm{i}\rangle\langle\vec n_\mathrm{f}|\,,\qquad p_{\mathrm{WM}}(succ) = |\mathbf{tr}\hat\Psi|^2 = \frac{1}{2}(1+\vec n_\mathrm{i}\vec n_\mathrm{f})$

$$\vec\sigma_W = \mathrm{Re}\frac{\mathbf{tr}\hat{\vec\sigma}\hat\Psi}{\mathbf{tr}\hat\Psi} = \frac{\vec n_\mathrm{i}+\vec n_\mathrm{f}}{1+\vec n_\mathrm{i}\vec n_\mathrm{f}},\qquad |\vec\sigma_W|^2 = \frac{1}{p_{\mathrm{WM}}(succ)}$$

$$p_{\mathrm{WM}}(succ) = \frac{1}{(\sigma_{xW})^2+(\sigma_{xW})^2+(\sigma_{zW})^2}$$

Summary

- 2-time density $\widehat{\rho}_{\mathrm{if}}$ introduced for $p(\mu, succ) = \mathbf{tr}(\hat{A}_\mu \otimes \hat{A}_\mu^\dagger)\widehat{\rho}_{\mathrm{if}}$

$$p_{\mathrm{WM}}(succ) = \mathbf{tr}\widehat{\rho}_{\mathrm{if}}$$

  All $\widehat{\rho}_{\mathrm{if}}$ are, upto normalization $p_{\mathrm{WM}}(succ)$, preparable via pure state pre- and postselection.
- $p_{\mathrm{WM}}(succ)$ is a figure of merit of preparation. Optimum preparartion protocols remain to be found.
- In the simple VVA case, weak measurements provide $p_{\mathrm{WM}}(succ)$ without tomography:

$$p_{\mathrm{WM}}(succ) = \frac{1}{(\sigma_{xW})^2 + (\sigma_{xW})^2 + (\sigma_{zW})^2}$$