A massive star \((M > 8M_\odot)\) forms an iron core; it cannot produce energy via fusion and contraction ensues.

As the density increases, electron capture becomes favorable and neutrinos are produced:

\[
p^+ + e^- \rightarrow n + \nu_e
\]

When the Fermi energy of neutrinos gets large enough, they become trapped and \(\beta\)-equilibrium is achieved.

At this point the entropy per nucleon is \(S \approx 1\). Dripped neutrons require \(S \approx 8\) thus nuclei persist until the core contracts to \(n \sim n_o\). Then nucleonic matter emerges.

Nucleons are compressed to \(n \sim 3n_o\) where the repulsive core of the strong interaction dominates their attraction and inhibits further contraction.

The core rebounds and creates a shock wave that disrupts the star.
Matter in $\beta$-equilibrium supported against gravitational collapse by neutron degeneracy.

Structure determined by simultaneous solution of:

- **Interior mass,**

$$m(r) = 4\pi \int_0^r \epsilon(r') r'^2 dr'$$

- **Hydrostatic equilibrium,**

$$\frac{dp}{dr} = - \frac{(\epsilon + p)[Gm(r) + 4\pi Gr^3 p]}{r[r - 2Gm(r)]}$$

- **EOS,**

$$p = p(\epsilon)$$
Relativistic binaries not in equilibrium: Gravitational wave (GW) emission leads to orbital decay.

Early stage: only gravitational interactions, GW signal contains information for the masses of the components.

Coalescence stage: Tidal disruption of the lower-mass star, mass transfer onto the more massive one. Mass transfer rate depends on $C = M_{NS}/R_{NS}$ and reflected in GW signal. Ejected matter is very neutron-rich and can lead to heavy element formation via the r-process.

Late stage: Black hole or hypermassive neutron star formation.
Matter in Astrophysical Phenomena

<table>
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<th>Core-collapse supernovae</th>
<th>Proto-neutron stars</th>
<th>Mergers of compact binary stars</th>
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<tr>
<td>Baryon Density ($n_0$)</td>
<td>$10^{-8} - 10$</td>
<td>$10^{-8} - 10$</td>
<td>$10^{-8} - 10$</td>
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<tr>
<td>Temperature (MeV)</td>
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<td>Entropy ($k_B$)</td>
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<tr>
<td>Proton Fraction</td>
<td>$0.35 - 0.45$</td>
<td>$0.01 - 0.3$</td>
<td>$0.01 - 0.6$</td>
</tr>
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</table>

$n_0 \simeq 0.16$ fm$^{-3}$ (equilibrium density of symmetric nuclear matter)

- Objective: Construction of EOS based on the best nuclear physics input for use in hydrodynamic simulations of core-collapse supernova explosions and binary mergers as well as the thermal evolution of proto-neutron stars.
Nuclear Matter Phases

Phase I— Low Density Regime

Phase II— Mixed Phase

Phase III— $n > n_s$ Dissociated
Matter near the Equilibrium Point \((n = n_0, \alpha = 0)\) of SNM

- About \(\alpha = 0\),
  \[ E(n, \alpha) \simeq E_0(n) + S_2(n)\alpha^2 + \mathcal{O}(\alpha^4) \]

- About \(n = n_0\),
  \[ E_0(n) \simeq \mathcal{E}_0 + \frac{1}{2} K_0 \left( \frac{n-n_0}{3n_0} \right)^2 + \ldots \]
  \[ S_2(n) \simeq S_v + L \left( \frac{n-n_0}{3n_0} \right) + \ldots \]

- \(P(n, \alpha) = n^2 \frac{\partial E}{\partial n} \simeq n^2 \left[ \frac{K_0}{3n_0} \left( \frac{n-n_0}{3n_0} \right) + \frac{L}{3n_0} \alpha^2 \right] + \ldots \)

- \(\hat{\mu} = \mu_n - \mu_p = 2 \frac{\partial E}{\partial \alpha} \simeq 4\alpha S_2(n)\)
Saturation density, \( n_0 = 0.16 \pm 0.01 \text{ fm}^{-3} \)
High-energy electron scattering: \( r_0 \propto \pi/qR, \ n_0 = \left( \frac{4}{3} \pi r_0^3 \right)^{-1} \)

Energy per particle, \( \varepsilon_0 = -16 \pm 1 \text{ MeV} \)
Fits to masses of atomic nuclei:
\[
B(N, Z) = \varepsilon_0 A - b_{\text{surf}} A^{2/3} - S_v \frac{(N-Z)^2}{A} - b_{\text{Coul}} Z^2 A^{-1/3}
\]

Symmetry energy, \( S_v = 30 - 35 \text{ MeV} \)
(fits to masses of atomic nuclei)

Slope of \( S_2 \), \( L = 40 - 70 \text{ MeV} \)
(variety of experiments)

Compression modulus, \( K_0 = 240 \pm 30 \text{ MeV} \)
Giant monopole resonances: \( E_{\text{GMR}} = \left( \frac{K_A}{m<r^2>} \right)^{1/2} \)
\[
K_A = K_0 + K_{\text{surf}} A^{-1/3} + K_\tau \frac{(N-Z)^2}{A^2} + K_{\text{Coul}} \frac{Z^2}{A^{4/3}}
\]

Effective mass, \( M^*/M = 0.8 \pm 1 \)
Neutron evaporation spectra: \( N(E_n) \propto a, \ a_{\text{FermiGas}} = \frac{\pi^2 m^*}{2k_F^2} \).
EOS Constraints from Neutron Stars

- Largest observed mass, $M = 2.01 M_{\odot}$ (binaries)

- Largest observed frequency, $\Omega = 114 \text{ rad/s}$ (pulsars)

- Inferred radius range, $9 \text{ km} \leq R \leq 15 \text{ km}$ (photospheric emission, thermal spectra)

- ... 

Solution of Tolman-Oppenheimer-Volkoff (TOV) equations and EOS predicts $M_{\text{max}}, R_{\text{max}}, I_{\text{max}}, \Omega_{\text{max}}$, etc.
Supernovae:

- EOS controls electron capture rates (via $\hat{\mu}$) and therefore the neutrino signal.
- GW amplitude related to PNS compactness and thus the high-density properties of the EOS.

Binary mergers:

- Tidal disruption of NS during coalescence of BH-NS binary depends on the stiffness of the EOS. GW frequency sensitive to orbital frequency at disruption.
- Short gamma-ray bursts may be produced in BH-NS mergers. The luminosity and the lifetime of the (metastable) remnant are related to the thermal and high-density properties of the EOS respectively.
- $r$-process production rates and final abundances depend on the composition of the ejecta and thus the EOS.
- NS radii: tidal deformity, $\lambda \propto R_{NS}^5$. 

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Approaches to the Nucleon-Nucleon Interaction

- High-precision interactions fitted to NN scattering data
  - meson-exchange models
    e.g. Nijmegen, Paris, Juelich-Bonn
  - sums of local operators
    e.g. Urbana, Argonne

- Interactions from chiral EFT

- RG-evolved potentials

Extension of the above to bulk matter by a variety of techniques: SCGF, BHF, variational, etc.

- Phenomenological approaches: Skyrme, Gogny, RMFT
  Typically treated at the HF level, fitted to extracted quantities.
Skyrme Interactions

- \( \hat{V}_{NN} = \sum_{i<j} \hat{V}_{ij} + \sum_{i<j<k} \hat{V}_{ijk} \), zero-range

- Evaluating \( \hat{H} = \hat{T} + \hat{V}_{NN} \) in the HF approximation gives

\[
\mathcal{H}_{Skyrme} = \frac{\hbar^2}{2m_n} \tau_n + \frac{\hbar^2}{2m_p} \tau_p \\
+ n(\tau_n + \tau_p) \left[ \frac{t_1}{4} \left( 1 + \frac{x_1}{2} \right) + \frac{t_2}{4} \left( 1 + \frac{x_2}{2} \right) \right] \\
+ (\tau_n n_n + \tau_p n_p) \left[ \frac{t_2}{4} \left( \frac{1}{2} + x_2 \right) - \frac{t_1}{4} \left( \frac{1}{2} + x_1 \right) \right] \\
+ \frac{t_o}{2} \left( 1 + \frac{x_o}{2} \right) n^2 - \frac{t_o}{2} \left( \frac{1}{2} + x_o \right) (n_n^2 + n_p^2) \\
\left[ \frac{t_3}{12} \left( 1 + \frac{x_3}{2} \right) n^2 - \frac{t_3}{12} \left( \frac{1}{2} + x_3 \right) (n_n^2 + n_p^2) \right] n^e
\]
Thermodynamics

▶ Single-particle energy spectrum:

\[ \varepsilon_i = k_i^2 \frac{\partial H}{\partial \tau_i} + \frac{\partial H}{\partial n_i} \equiv \varepsilon_k + V_i \]

\[ n_i = \frac{1}{2\pi^2} \left( \frac{2m_i^* T}{\hbar^2} \right)^{3/2} F_{1/2i} \]

\[ \tau_i = \frac{1}{2\pi^2} \left( \frac{2m_i^* T}{\hbar^2} \right)^{5/2} F_{3/2i} \]

\[ F_{\alpha i} = \int_{0}^{\infty} \frac{x_i^\alpha}{e^{-\psi_i} e^{x_i} + 1} dx_i \]

\[ x_i = \frac{\varepsilon_{ki}}{T}, \quad \psi_i = \frac{\mu_i - V_i}{T} = \frac{\nu_i}{T} \]

▶ Rest of state variables:

▶ Energy density

\[ \varepsilon = \frac{\hbar^2}{2m^*_n} \tau_n + \frac{\hbar^2}{2m^*_p} \tau_p + U(n) \]

▶ Chemical potentials

\[ \mu_i = T\psi_i + V_i \]

▶ Entropy density

\[ s_i = \frac{1}{T} \left[ \frac{5}{3} \frac{\hbar^2}{2m_i^*} \tau_i + n_i (V_i - \mu_i) \right] \]

▶ Pressure

\[ P = T(s_n + s_p) + \mu_n n_n + \mu_p n_p - \varepsilon \]

▶ Free energy density

\[ F = \varepsilon - Ts \]
Nucleons, $\Psi$, coupled to $\sigma$, $\omega$, and $\vec{\rho}$ mesons:

$$\mathcal{L} = \bar{\Psi} \left[ \gamma_\mu \left( i \partial^\mu - g_\omega \omega^\mu - \frac{g_\rho}{2} \vec{\rho}^\mu \cdot \vec{\tau} \right) - (M - g_\sigma \sigma) \right] \Psi$$

$$+ \frac{1}{2} \left[ \partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2 - \frac{\kappa}{3} (g_\sigma \sigma)^3 - \frac{\lambda}{12} (g_\sigma \sigma)^4 \right]$$

$$+ \frac{1}{2} \left[ - \frac{1}{2} f_{\mu\nu} f^{\mu\nu} + m_\omega^2 \omega^\mu \omega_\mu \right]$$

$$+ \frac{1}{2} \left[ - \frac{1}{2} \tilde{B}_{\mu\nu} \tilde{B}^{\mu\nu} + m_\rho^2 \vec{\rho}^\mu \vec{\rho}_\mu \right]$$

Extensions: Hyperons, scalar-isovector $\delta$-meson, density-dependent couplings, derivative couplings, nucleon form factors, etc.
Negligible meson-field fluctuations, uniform and static system \[ \Rightarrow \]
\[
\sigma_0 = \frac{g_\sigma}{m_\sigma^2} < \bar{\Psi}\Psi > - \frac{1}{m_\sigma^2} \left( \frac{\kappa}{2} g_\sigma^3 \sigma_0^2 + \frac{\lambda}{6} g_\sigma^4 \sigma_0^3 \right)
\]
\[
= \frac{g_\sigma}{m_\sigma^2} n_s - \frac{1}{m_\sigma^2} \left( \frac{\kappa}{2} g_\sigma^3 \sigma_0^2 + \frac{\lambda}{6} g_\sigma^4 \sigma_0^3 \right)
\]
\[
\omega_0 = \frac{g_\omega}{m_\omega^2} < \bar{\Psi}\gamma^0\Psi > = \frac{g_\omega}{m_\omega^2} n
\]
\[
\rho_0 = \frac{g_\rho}{2m_\rho^2} < \bar{\Psi}\gamma^0\tau_3\Psi > = -\frac{g_\rho}{2m_\rho^2} n(1 - 2x)
\]

**Spectrum:** \[ \epsilon_{i\pm} = \pm \left( p_i^2 + M^*^2 \right)^{1/2} + \frac{g_\omega^2}{m_\omega^2} n + \frac{g_\rho^2}{4m_\rho^2} (n_i - n_j) \]

**Diagonal elements of the stress-energy tensor**
\[ T_{\mu\nu} = \frac{\partial L}{\partial (\partial_{\mu} q_i)} \partial_{\nu} q_i - g_{\mu\nu} L, \] give:

- **Energy density,** \( \varepsilon = < T_{00} > \)
- **Pressure,** \( P = \frac{1}{3} < T_{ii} > \)

**Dirac mass,** \( M^* \), derived from the requirement \[ \frac{\delta \Omega}{\delta \sigma} = 0 \]
Finite-Range Interactions: Deficiencies of Skyrme models and MFT’s

- Single particle potential, \( U(\rho, p) \) in both Skyrme and MFT models grows monotonically with momentum; inconsistent with optical model fits to nucleon-nucleus reaction data.

- Microscopic calculations (RBHF, variational calculations, etc.) show distinctly different behaviors in their momentum dependence, consistent with optical model fits.

- The above features were found necessary to account for heavy-ion data on transverse momentum and energy flow (in conjunction with \( K \sim 240 \text{ MeV} \)).

\[ \text{SkO'}: P. G. Reinhard \text{ et al., PRC 60, 014316 (1999)} \]
\[ \text{BHF-TBF: W. Zuo et al. PRC 74, 014317 (2006)} \]
\[ \text{UV14-TNI: R. B. Wiringa PRC 38, 2967 (1988)} \]
Degenerate Limit

Landau Fermi Liquid Theory

- Interaction switched-on adiabatically
- Entropy density and number density maintain their free Fermi-gas forms:

\[ s_i = \frac{1}{V} \sum_{k_i} \left[ f_{k_i} \ln f_{k_i} + (1 - f_{k_i}) \ln(1 - f_{k_i}) \right] \]

\[ n_i = \frac{1}{V} \sum_{k} f_{k_i}(T) \]

- \[ \int d\varepsilon \frac{\delta s}{\delta T} \Rightarrow s_i = 2a_in_iT \]

\[ a_i = \frac{\pi^2 m^*_i}{2 k^2_{Fi}} \] \quad \text{level density parameter}

- Other thermodynamics via Maxwell’s relations:

\[ \frac{d\varepsilon}{ds} = T, \quad \frac{dp}{dT} = -n^2 \frac{d(s/n)}{dn}, \]

\[ \frac{d\mu}{dT} = -\frac{ds}{dn}, \quad \frac{dF}{dT} = -s, \text{ etc.} \]
Effective masses

\[ m^* = \frac{m}{1 + \beta(x)n} \]

For \( n \) large, \( \frac{dm^*}{dn} \approx 0 \)

\[ m^* = E_F^* = \left( p_F^2 + M^*^2 \right)^{1/2} \]

\[ \frac{\rho_F}{M^*} + \frac{dM^*}{dp_F} = 0 \]
The three models produce quantitatively similar results.

Agreement with exact results extended to $n \simeq 0.1 \text{ fm}^{-3}$.

Better agreement for PNM than for SNM.
Results: $P_{th}$ and $\mu_{th}$

- Model dependence is evident due to $\frac{d m^*}{d n}$.
- Agreement with exact results extended to $n \simeq 0.1 \text{ fm}^{-3}$.
- Better agreement for PNM than for SNM.
The MDI and MFT $C_V$ exceed the classical value of 1.5 in the nondegenerate limit. In this regime the $T$-dependence of the spectrum becomes important.

The peaks in $C_P$ are due to the proximity to the nuclear liquid-gas phase transition.
Results: Specific Heats

\( C_v (k_B) \)

(a) \( T = 20 \text{ MeV} \)
(b) \( T = 50 \text{ MeV} \)

\( x = 0 \)
MDI(A)
SkO'

\( T = 20 \text{ MeV} \)
\( T = 60 \text{ MeV} \)
\( T = 80 \text{ MeV} \)
\( T = 40 \text{ MeV} \)
\( T = 20 \text{ MeV} \)

SkM* MFT

\( C_v \) vs. \( n_{B} [\text{fm}^{-3}] \)

\( x = 0.5 \)
\[ \Gamma_{th} = 1 + \frac{P_{th}}{\varepsilon_{th}} \]

- Weak composition and temperature dependence
- Finite-range effects suppress density dependence
The critical temperature and density of the transition for a given composition are obtained by the condition
\[
\left. \frac{dP}{dn} \right|_{n_c, T_c} = \left. \frac{d^2P}{dn^2} \right|_{n_c, T_c} = 0.
\]

\((n_c, T_c)\) is the termination point of the phase boundary separating the homogeneous and the inhomogeneous phases of nuclear matter.
Approaches to the Subnuclear Region

- **Nuclear statistical equilibrium**
  - Statistical ensemble of nucleons and nuclei in thermodynamic equilibrium
  - Chemical potentials of nuclei, \( \mu_a = N_a \mu_n + Z_a \mu_p \)
  - Maxwell-Boltzmann statistics
  - Abundances determined by the Saha equation
  - Requires nuclear binding energies as input

- **Lattimer-Swesty**
  - Single representative species of nucleus described by the liquid-drop model
  - Light nuclei represented by \( \alpha \)-particles
  - Nucleons treated by the same model used in the homogeneous phase
  - Hard-sphere interactions
  - Equilibrium obtained by minimizing the free energy of the system wrt its internal variables.

- **Virial expansion**
  - Nondegenerate limit expansion of the grand potential in small fugacity: \( z = \exp[(\mu - m)/T] \ll 1 \)
  - Coefficients depend on scattering phase shifts corresponding to the interaction
  - Supplements NSE

- **Nucleons-in-cell:** Molecular dynamics, Thomas-Fermi, Hartree-Fock ...
Results using the Lattimer-Swesty approach

**Pressure**

- **$P_{\text{total}}$ (MeV fm$^{-3}$)**
  - $10^{-6}$
  - $10^{-5}$
  - $10^{-4}$
  - $10^{-3}$
  - $10^{-2}$
  - $10^{-1}$
  - $10^{0}$

- **$n$ (fm$^{-3}$)**
  - $10^{-6}$
  - $10^{-5}$
  - $10^{-4}$
  - $10^{-3}$
  - $10^{-2}$
  - $10^{-1}$
  - $10^{0}$

**Nuclear size**

- **$A$ (none)**
  - $10^{-6}$
  - $10^{-5}$
  - $10^{-4}$
  - $10^{-3}$
  - $10^{-2}$
  - $10^{-1}$
  - $10^{0}$

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Conclusions

- Matter for astrophysical applications covers a wide range in \((n, x, T)\)-space; much of which cannot be accessed by terrestrial experiments.

- Upcoming astrophysical applications and laboratory experiments involving heavy ions/rare isotopes will probe higher densities and temperatures allowing a tighter grip on the EOS.

- The structure of the interaction, and thus \(m^*\), is crucial in the determination of thermal effects. These, in turn, are important in neutrino and gravitational radiation emission.

- Open questions:
  - Neutron star \(M_{\text{max}}\)
  - Emergence of non-nucleonic DoF.
  - Treatment of the inhomogeneous phase.
  - Phase transitions