Pontryagin trace anomaly

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Introduction: quantum anomalies

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- Harmful anomalies: destroying theories
  - helpful in model building
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Being genuine quantum properties, unveiling the essence of QFT!

("What is QFT?" is still a tantalising question.)

From Weyl symmetry to trace anomaly

Symmetry: (local) Weyl rescaling

$$g_{\mu\nu}(x) \to e^{\omega(x)}\, g_{\mu\nu}(x) \qquad , \qquad \psi_j(x) \to e^{\lambda_j\, \omega(x)}\, \psi_j(x)$$

Invariance of classical action for matter $\Rightarrow$ traceless EM tensor:

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However, perturbative QFT calculations found anomaly [Capper, Duff 1975]:

$$\langle\langle T^\mu_\mu \rangle\rangle = a\, E_2 + c\, (W_{\mu\nu\rho\sigma})^2 \qquad , \qquad D = 4$$

E_2 is 2nd Euler (Gauss-Bonnet) density, $W_{\mu\nu\rho\sigma}$ is Weyl tensor.

- The form is universal, coefficients a and c theory dependent.
- coefficients a and c not 1-loop saturated
- At the beginning, community did not believe it!

General form of trace anomalies in $D = 4$

$$\langle\langle T^{\mu}_{\mu}(x) \rangle\rangle = \mathcal{A}_W(x)$$

WZ consistency conditions connect trace and diff- anomalies.

General (nontrivial) solution: no (pure gravity) diff-anomalies and

$$\mathcal{A}_W = a\, E_2 + c\, (W_{\mu\nu\rho\sigma})^2 + e\, P_4$$ [Bonora, Pasti, Tonin]

where P_4 is Pontryagin (topological) tensity

$$P_4 = \frac{1}{32\,\pi^2}\, \varepsilon^{\mu\nu\rho\sigma}\, R^{\mu\nu}{}_{\alpha\beta}\, R_{\rho\sigma\alpha\beta}$$

- Pontryagin anomaly, being P and CP odd, is exceptional,
- but, with rare exceptions ([Nakayama 2012]), neglected $\Rightarrow$ Why?

Status of $4D$ Pontryagin anomaly before 2014

All against it:

- No CP violating free relativistic QFT's
- For Weyl fermion: in $4D$ classical 1-1 mapping with massless Majorana fermion, and P-odd anomaly not expected for Majorana
  - No Pontryagin anomaly for Majorana fermion [Bastianelli 2016]
  - Careful: mapping is not linear unitary!
- Realisations having unitarity problems [Nakayama 2012]
- Non-compatibility with SUSY [Bonora, Giaccari 2013]

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As a consequence, Pontryagin anomaly mainly neglected by the community!

Status of $4D$ Pontryagin anomaly in 2016

For free Weyl fermion in curved spacetime anomaly was found, with

$$e_W = \pm \frac{i}{48}$$

[Bonora, Giaccari, Lima de Souza 2014]

Independently obtained with heat kernel method and perturbatively (1-loop) using dimensional regularisation.

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- Note the strange-looking imaginary coefficient.
- Calculation somewhat incomplete:
  - Heat kernel calculation naive extrapolation of real fermion one
  - Simplifying field redefinition used, dim. regularisation subtleties

Result was received (mainly) with disbelief!

Main part of the talk

Due to a controversial character of the result, we are in need of more complete, and also independent calculations.

We focused on perturbative 1-loop calculation using dimensional regularisation, and did it in two ways:

1. Repeat [Bonora at al.] calculation in more pedantic way
   - without using field redefinitions
2. Working with Dirac fermion by introducing chiral metric (MAT gravity)
   - analogue of Bardeen method used for chiral gauge anomalies

[Bonora, Cvitan, D.P., D.Pereira, Giaccari, Štemberga, Eur. Phys. J. C **77** (2017) 511]

I. Calculation with free Weyl fermion: the theory

Action: $$S_L = \int \mathrm{d}^4x\,\sqrt{|g|}\,i\,\overline{\psi_L}\,\gamma^\mu\left(\nabla_\mu + \frac{\omega_\mu}{2}\right)\psi_L$$

EM tensor: $$T^{\mu\nu} = -\frac{i}{4}\overline{\psi_L}\gamma^\mu\overleftrightarrow{\nabla}^\nu\psi_L + (\mu\leftrightarrow\nu)$$

Perturbative expansion in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ up to h^3:

$$\begin{aligned}
S_{\rm int} \;=\; \int d^4x\,\Big[& -\frac{i}{4}\overline{\psi_L}h_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L + \frac{3i}{16}\overline{\psi_L}(h^2)_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L - \frac{5i}{32}\overline{\psi_L}(h^3)_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L \\
& -\frac{1}{16}\epsilon^{mabc}\overline{\psi_L}\gamma_c\gamma_5\psi_L\left(h_m^\sigma\partial_a h_{b\sigma} + (h^2)_m^\sigma\partial_b h_{a\sigma} - h_m^\rho h_a^\sigma\partial_\sigma h_{\rho b} - \frac{1}{2}h_m^\rho\partial_a h_{\rho\sigma}h_c^\sigma\right) \\
& +\frac{1}{2}(\mathrm{tr}\,h)\left(\frac{i}{2}\overline{\psi_L}\gamma^m\overleftrightarrow{\partial}_m\psi_L - \frac{i}{4}\overline{\psi_L}h_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L + \frac{3i}{16}\overline{\psi_L}(h^2)_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L\right. \\
& \qquad\left. -\frac{1}{16}\,\epsilon^{mabc}\overline{\psi_L}\gamma_c\gamma_5\psi_L h_m^\sigma\partial_a h_{b\sigma}\right) \\
& +\left(\frac{1}{8}(\mathrm{tr}\,h)^2 - \frac{1}{4}(\mathrm{tr}\,h^2)\right)\left(\frac{i}{2}\overline{\psi_L}\gamma^m\overleftrightarrow{\partial}_m\psi_L - \frac{i}{4}\overline{\psi_L}h_a^m\gamma^a\overleftrightarrow{\partial}_m\psi_L\right) \\
& +\left(-\frac{1}{8}(\mathrm{tr}\,h)(\mathrm{tr}\,h^2) + \frac{1}{48}(\mathrm{tr}\,h)^3 + \frac{1}{6}(\mathrm{tr}\,h^3)\right)\frac{i}{2}\overline{\psi_L}\gamma^m\overleftrightarrow{\partial}_m\psi_L + \dots\Big]
\end{aligned}$$

I. Calculation with free Weyl fermion: Weyl anomaly

Full perturbative calculation up to 3-point functions:

- no assumptions, no field redefinition

$$\langle\langle T^{\mu}_{\mu} \rangle\rangle_{\mathrm{P-odd}} = \frac{\mp i}{768\,\pi^2}\,\varepsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^\alpha_\nu\,\partial_\lambda\partial_\alpha h^\sigma_\rho - \partial_\mu\partial_\sigma h^\alpha_\nu\,\partial_\lambda\partial^\sigma h_{\alpha_\rho}\right)+O(h^3)$$

Covariantisation gives:

$$\langle\langle T^{\mu}_{\mu} \rangle\rangle_{\mathrm{P-odd}} = \mp\frac{1}{16}\,P_4$$

Opposite sign and three times larger then in [Bonora et al. 2014]

Inconsistency? Not yet. First one has to check diff-covariance.

I. Calculation with Weyl fermion: Diff-anomaly and result

We have to check the conservation of EM tensor

$$\nabla_\mu \langle\langle T^{\mu\nu} \rangle\rangle_{\mathrm{P-odd}} = \mathcal{A}_{\mathrm{diff}}$$

We obtain $\mathcal{A}_{\mathrm{diff}} \neq 0$, which can be canceled by the local counterterm:

$$\mathcal{C} = \frac{1}{6} \int \mathrm{d}^4 x \, h^\mu_\mu \, \mathcal{A}^{(0)}_W$$

But, this counterterm adds to trace anomaly $-\frac{4}{3}\mathcal{A}^{(0)}_W$, so the final result is:

$$\langle\langle T^\mu_\mu \rangle\rangle_{\mathrm{P-odd}} = \pm \frac{1}{48} P_4 \qquad , \qquad \nabla_\mu \langle\langle T^{\mu\nu} \rangle\rangle_{\mathrm{P-odd}} = 0$$

In full agreement with [Bonora et al. 2014, 2015]

II. Calculation with Dirac fermion. MAT gravity

Avoid subtleties with Weyl fermions by using Dirac field.

For this we have to invent chiral coupling to gravity $\Rightarrow$ MAT gravity

Metric-axial-tensor (MAT) gravity based on chiral metric:

$$G_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu} \qquad , \qquad f_{\mu\nu} \text{ is additional spin-2 field}$$

One defines chiral vierbein, connections, Riemann tensor, ... in the usual way from $G_{\mu\nu}$.

Two diff-symmetries and two Weyl-symmetries $\Rightarrow$ two conserved traceless spin-2 tensors.

(MAT gravity can be viewed as a bimetric gravity with $g^{(\pm)}_{\mu\nu} = g_{\mu\nu} \pm f_{\mu\nu}$.)

II. Calculation with Dirac fermion. The theory

We put Dirac fermion in MAT gravity background

$$S_D = \int \mathrm{d}^4x\, i\overline{\psi}\sqrt{|\bar{G}|}\gamma^a \hat{E}_a^\mu \left(\partial_\mu + \frac{1}{2}\Omega_\mu\right)\psi$$

We obtained (dim. regularisation) for the trace anomalies

$$\langle\langle T^\mu_\mu \rangle\rangle = \frac{i}{768\,\pi^2}\,\epsilon^{\mu\nu\lambda\rho}\,R^{(1)\sigma\tau}_{\mu\nu}\,R^{(2)}_{\lambda\rho\sigma\tau}$$

$$\langle\langle T_{5\,\mu}^{\mu} \rangle\rangle = \frac{i}{768\,\pi^2}\,\frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}\left(R^{(1)\sigma\tau}_{\mu\nu}\,R^{(1)}_{\lambda\rho\sigma\tau} + R^{(2)\sigma\tau}_{\mu\nu}\,R^{(2)}_{\lambda\rho\sigma\tau}\right)$$

where $\quad R^{(1,2)}_{\mu\nu\lambda\rho} = \frac{1}{2}\left(R^{(+)}_{\mu\nu\lambda\rho} \pm R^{(-)}_{\mu\nu\lambda\rho}\right)$

We are interested in the blue one, which is the standard trace anomaly.

II. Calculation with Dirac fermion. The results

(a) If we take $g^{(-)}_{\mu\nu} = \eta_{\mu\nu} \;\Rightarrow\; \omega^{(-)a}{}_{b\mu} = 0 = \Gamma^{(-)\mu}{}_{\nu\rho}$

- Dirac MAT action $\Rightarrow$ action in flat space for Ψ_R plus action in curved metric $g^{(+)}$ for ψ_L
- trace anomaly belongs to the left Weyl fermion, and is

$$\langle\langle T^\mu_\mu \rangle\rangle = \frac{i}{768\,\pi^2} \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} R^{(+)\sigma\tau}_{\mu\nu} R^{(+)}_{\lambda\rho\sigma\tau}$$

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(c) If we take $g^{(+)}_{\mu\nu} = g^{(-)}_{\mu\nu} = g_{\mu\nu}$ we obtain a standard Dirac field in curved space, and

$$\langle\langle T^\mu_\mu \rangle\rangle = 0$$

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Confirms previously obtained results

Conclusion

We have extended evidence that Weyl fermion in 4D has Pontryagin trace anomaly.

If true, this has important and strange consequences:

- Gravitational perturbative CP violation
- Weyl and massless Majorana not equivalent in curved spacetime.
- Strange imaginary coefficient, is unitarity violated?
- If yes, anomaly is harmful $\Rightarrow$ only Dirac and Majorana fermions

  $\Rightarrow$ minimal SM with Weyl fermions not consistent
- If harmless, leads to CP-violating imaginary term in effective action
  - CP-violating gravitational Schwinger mechanism?
  - No effect on 0th-order cosmology, could affect perturbations (GWs) [Mauro, Shapiro 2015]
  - could be important for black holes

Outlook

Assuring the anomaly is not a quirk of dimensional regularisation:

- Perturbative calculation using other schemes
  - Pauli-Villars in MAT-gravity
- Point-splitting and heat kernel

If sticks, understand its nature and further consequences.

If bogus, understand what happened with dimensional regularisation.

Result looks really strange, but we've seen stranger stuff in QM.