On perturbations in Horndeski theories

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Horndeski theory

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X) \Box \pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X) R + 2 G_{4X}(\pi, X) \left[(\Box \pi)^2 - \pi_{;\mu\nu} \pi^{;\mu\nu} \right],$$

$$\mathcal{L}_5 = G_5(\pi, X) G^{\mu\nu} \pi_{;\mu\nu} + \frac{1}{3} G_{5X} \left[(\Box \pi)^3 - 3 \Box \pi \pi_{;\mu\nu} \pi^{;\mu\nu} + 2 \pi_{;\mu\nu} \pi^{;\mu\rho} \pi_{;\rho}{}^{\nu} \right]$$

where π is the Galileon field, $X = g^{\mu\nu} \pi_{,\mu} \pi_{,\nu}$, $\pi_{,\mu} = \partial_\mu \pi$, $\pi_{;\mu\nu} = \nabla_\nu \nabla_\mu \pi$, $\Box \pi = g^{\mu\nu} \nabla_\nu \nabla_\mu \pi$, $G_{4X} = \partial G_4 / \partial X$

Stability issue

The pathologies if any show up in the behaviour of small perturbations about the background π_0

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The quadratic Lagrangian in Minkowski space, that leads to a second order field equation for χ

$$L_\chi^{(2)} = \frac{1}{2} U \dot{\chi}^2 - \frac{1}{2} V (\partial_i \chi)^2 - \frac{1}{2} W \chi^2 \qquad (2)$$

where U, V, W depend on time.

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$$L_\chi^{(2)} = \frac{1}{2}U\dot{\chi}^2 - \frac{1}{2}V(\partial_i\chi)^2 - \frac{1}{2}W\chi^2 \quad (3)$$

We will consider only high momentum regime.

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So U, V, W may be considered constant.

Dispersion relation and energy density for χ read:

$$U\omega^2 = V\mathbf{p}^2 + W \ , \quad (4)$$

$$T_{00}^{(2)} = \frac{1}{2}U\dot{\chi}^2 + \frac{1}{2}V(\partial_i\chi)^2 + \frac{1}{2}W\chi^2 \quad (5)$$

Possible types of pathologies

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(a) **Gradient instability**:

$$U > 0 \ , \quad V < 0 \ , \quad \text{or} \quad U < 0 \ , \quad V > 0 \ .$$

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$\longrightarrow$ perturbations grow arbitrarily fast

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BUT *quantum-mechanically* unstable.

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If time scale $|W|^{-1/2}$ is shorter than the characteristic scale of the background π_c, it is a problem

Stable background case

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Different regimes of propagation of $\chi-$waves:

$V > U$ superluminal propagation

$V = U$ propagation at the speed of light

$V < U$ subluminal propagation

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In order to perform a stability analysis at the high momentum regime, one needs to obtain the quadratic Lagrangian for perturbations.

KYY approach

The method adopted in

T. Kobayashi, M. Yamaguchi and J. Yokoyama, 1105.5723

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3) Integrate out all non-dynamical degrees of freedom

DPSV approach

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Corresponding Galileon field equation (terms without second derivatives are omitted):

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\end{aligned}$$

Quadratic action for perturbations in $\mathcal{L}_3$

$$\mathcal{L}_3 = -\frac{1}{2\kappa}R + F(X,\pi) + K(X,\pi)\Box\pi$$

in FLRW background

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2.$$

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Parametrisation of metric perturbations (gauge partially fixed):

$$h_{00} = 2\alpha, \quad h_{0i} = -\partial_i\beta, \quad h_{ij} = -a^2 \cdot 2\zeta\delta_{ij}$$

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Galileon pertrubations about the homogeneous background

$$\pi \to \pi(t) + \chi.$$

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We are interested in high momentum and frequency modes, therefore we neglect terms in the action without derivatives of ζ and χ, as well as linear in $\dot{\zeta}$, $\dot{\chi}$. But we keep all terms that include α and $\partial_i\beta$.

The derivative part of quadratic action for metric and Galileon perturbations:

$$S^{(2)}_{gr+gal} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \Bigg(-\frac{3}{\kappa}\dot{\zeta}^2 + \frac{1}{\kappa}\frac{(\overrightarrow{\nabla}\zeta)^2}{a^2} + \Sigma\alpha^2 - 2\Theta\alpha\frac{\overrightarrow{\nabla}^2\beta}{a^2} + \frac{2}{\kappa}\dot{\zeta}\frac{\overrightarrow{\nabla}^2\beta}{a^2} +$$
$$+ 6\Theta\alpha\dot{\zeta} - \frac{2}{\kappa}\alpha\frac{\overrightarrow{\nabla}^2\zeta}{a^2} + 2\alpha\frac{\overrightarrow{\nabla}^2\chi}{a^2}K_X\dot{\pi}^2 - 2\dot{\chi}\frac{\overrightarrow{\nabla}^2\beta}{a^2}K_X\dot{\pi}^2 -$$
$$- 6\chi\ddot{\zeta}K_X\dot{\pi}^2 + 2\Gamma\alpha\dot{\chi} + 2\Lambda\frac{\overrightarrow{\nabla}^2\beta}{a^2}\chi + \mathcal{A}\dot{\chi}^2 - \mathcal{B}\frac{(\overrightarrow{\nabla}\chi)^2}{a^2}\Bigg),$$

where

$\mathcal{A}, \mathcal{B}, \Sigma, \Theta, \Gamma, \Lambda$ are some expressions of Galileon functions.

Constraint equations for non-dynamical degrees of freedom:

$$\frac{\overrightarrow{\nabla}^2\beta}{a^2} : \ \alpha = \frac{1}{\Theta}\left(\frac{1}{\kappa}\dot{\zeta} - \dot{\chi}K_X\dot{\pi}^2 + \Lambda\chi\right),$$

$$\alpha : \ \frac{\overrightarrow{\nabla}^2\beta}{a^2} = -\frac{1}{\kappa}\frac{\overrightarrow{\nabla}^2\zeta}{a^2} + \frac{\overrightarrow{\nabla}^2\chi}{a^2}K_X\dot{\pi}^2 + \frac{1}{\Theta}\left(\frac{\Sigma}{\Theta}\frac{\dot{\zeta}}{\kappa} - \frac{\Sigma}{\Theta}\dot{\chi}K_X\dot{\pi}^2 + \frac{\Sigma}{\Theta}\Lambda\chi + 3\Theta\dot{\zeta} + \Gamma\dot{\chi}\right).$$

According to constraints α is linear in derivatives, while $\overrightarrow{\nabla}^2\beta$ is quadratic.

Integrating out α and $\overrightarrow{\nabla}^2\beta$ results in

$$S^{(2)}_{gr+gal} = \int \mathrm{d}t\,\mathrm{d}^3x\, a^3 \left[\frac{1}{\dot{\pi}^2}\left(\frac{1}{\kappa^2}\frac{\Sigma}{\Theta^2} + \frac{3}{\kappa}\right)\left(\frac{\dot{a}}{a}\dot{\chi} - \dot{\zeta}\dot{\pi}\right)^2 - \frac{1}{\dot{\pi}^2}\left(\frac{1}{a\cdot\kappa^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{\Theta}\right] - \frac{1}{\kappa}\right)\left(\frac{\dot{a}}{a}\frac{\overrightarrow{\nabla}\chi}{a} - \frac{\overrightarrow{\nabla}\zeta}{a}\dot{\pi}\right)^2 \right]. \quad (6)$$

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$$\left. -\frac{1}{\dot{\pi}^2}\left(\frac{1}{a\cdot\kappa^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{\Theta}\right]-\frac{1}{\kappa}\right)\left(\frac{\dot{a}}{a}\frac{\overrightarrow{\nabla}\chi}{a}-\frac{\overrightarrow{\nabla}\zeta}{a}\dot{\pi}\right)^2\right]. \tag{6}$$

The combination that enters here,

$$\frac{\dot{a}}{a}\chi - \dot{\pi}\zeta\ ,$$

is invariant under the residual gauge transformations

$$\chi \to \chi + \xi_0\dot{\pi}, \quad \zeta \to \zeta + \xi_0\frac{\dot{a}}{a}, \quad \alpha \to \alpha + \dot{\xi_0}, \quad \beta \to \beta - \xi_0$$

modulo the derivatives of the background terms, which are omitted. So the action (6) is gauge invariant.

DPSV approach analysis

The Galileon field equation for $\mathcal{L}_3$:

$$
\begin{aligned}
&-4F_{XX}\nabla_\mu\nabla_\nu\pi\partial^\mu\pi\partial^\nu\pi - 2F_X\Box\pi + 2K_\pi\Box\pi - 2K_{\pi X}\Box\pi\partial_\mu\pi\partial^\mu\pi \\
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There is a specific set of terms with second derivatives that may in principle arise in the perturbed Galileon equation:

$$\ddot{\chi},\ \ \overrightarrow{\nabla}^2\chi,\ \ \ddot{\zeta},\ \ \overrightarrow{\nabla}^2\zeta,\ \ \dot{\alpha},\ \ \overrightarrow{\nabla}^2\beta.$$

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The equation has to be invariant under gauge transformations

$$\chi \to \chi + \xi_0 \dot{\pi}, \quad \alpha \to \alpha + \dot{\xi_0}, \quad \beta \to \beta - \xi_0.$$

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The only possible gauge invariant combinations of χ, α and β, which are quadratic in derivatives

$$\mathcal{Q}\left(\ddot{\chi} - \dot{\alpha}\dot{\pi}\right) - \mathcal{P}\left(\overrightarrow{\nabla}^2\chi + \overrightarrow{\nabla}^2\beta\dot{\pi}\right) = 0,$$

where $\mathcal{Q}$ and $\mathcal{P}$ are expressions with Lagrangian functions (for $\mathcal{L}_3$ these are F and K).

This is the general form of the Galileon field equation after the trick.

Gauging out $\dot{\alpha}\dot{\pi}$ and $\overrightarrow{\nabla}^2\beta\dot{\pi}$

α and β in terms of ζ and χ:

$$\alpha = u\dot{\zeta} + v\dot{\chi}, \tag{7a}$$

$$\beta = w\zeta + z\chi, \tag{7b}$$

where u, v, w and z are some functions.

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Let us choose the following gauge fixing condition in the arbitrary form:

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Expressing ζ and α from (7) gives

$$\zeta = \frac{1}{w}\left(A - z\right)\chi,$$

$$\alpha = \left(\frac{u}{w}\left(A - z\right) + v\right)\dot{\chi}.$$

The general form of Galileon equation with α and β expressed in terms of χ

$$\mathcal{Q}\left(\ddot{\chi}-\dot{\alpha}\dot{\pi}\right)-\mathcal{P}\left(\overrightarrow{\nabla}^2\chi+\overrightarrow{\nabla}^2\beta\dot{\pi}\right)=0.$$

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$$\mathcal{Q}\left(\ddot{\chi} - \left(\frac{u}{w}(A - z) + v\right)\ddot{\chi}\dot{\pi}\right) - \mathcal{P}\left(\overrightarrow{\nabla}^2\chi + A\overrightarrow{\nabla}^2\chi\dot{\pi}\right) = 0.$$

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$$\frac{u}{w} = -1, \quad -z - v = 0 \quad \longrightarrow \quad \boxed{\alpha = -\dot{\beta}}$$

as

$$\alpha = u\dot{\zeta} + v\dot{\chi} \quad \text{and} \quad \beta = w\zeta + z\chi.$$

$$\mathcal{Q}\left(\ddot{\chi} - \dot{\alpha}\dot{\pi}\right) - \mathcal{P}\left(\overrightarrow{\nabla}^2\chi + \overrightarrow{\nabla}^2\beta\dot{\pi}\right) = 0.$$

Now we can fix the gauge taking $A = 0$:

$$\beta = w\zeta + z\chi = A\chi \to 0,$$

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The only degree of freedom that is left is χ.

Equivalence of KYY and DPSV approaches

Gauge invariant action:

$$S^{(2)}_{gr+gal} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left[\frac{1}{\dot\pi^2} \left(\frac{1}{\kappa^2}\frac{\Sigma}{\Theta^2} + \frac{3}{\kappa} \right) \left(\frac{\dot a}{a}\dot\chi - \dot\zeta\dot\pi \right)^2 \right.$$

$$\left. - \frac{1}{\dot\pi^2} \left(\frac{1}{a\cdot\kappa^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{\Theta}\right] - \frac{1}{\kappa} \right) \left(\frac{\dot a}{a}\frac{\overrightarrow{\nabla}\chi}{a} - \frac{\overrightarrow{\nabla}\zeta}{a}\dot\pi \right)^2 \right].$$

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Both methods correspond to choosing a specific gauge in the gauge invariant action.

DPSV trick for $\mathcal{L}_4$

$$\mathcal{L}_3 = F(X,\pi) + K(X,\pi)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi,X)R + 2G_{4X}(\pi,X)\left[(\Box\pi)^2 - \nabla_\mu\nabla_\nu\pi\ \nabla^\mu\nabla^\nu\pi\right].$$

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Extra terms with second derivatives of metric in the Galileon field equation:

$$2R_{\mu\nu}\nabla^\mu\pi\nabla^\nu\pi\ K_X + 2R\nabla_\mu\nabla^\mu\pi\ G_{4X} - 4R_{\mu\nu}\nabla^\mu\nabla^\nu\pi\ G_{4X} +$$

$$+4R\nabla^\mu\pi\nabla_\mu\nabla_\nu\pi\nabla^\nu\pi\ G_{4XX} - 16R_{\mu\nu}\nabla^\mu\pi\nabla^\nu\nabla_\rho\pi\nabla^\rho\pi\ G_{4XX} +$$

$$+ 8R_{\mu\nu}\nabla^\mu\pi\nabla^\nu\pi\nabla_\rho\nabla^\rho\pi\ G_{4XX} - R\ G_{4\pi} + 2R\nabla_\mu\pi\nabla^\mu\pi\ G_{4\pi X} -$$

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and Einstein equations

$$2G_{\mu\nu}\,G_4 + 2R\nabla_\mu\pi\nabla_\nu\pi\,G_{4X} - 4R_{\nu\rho}\nabla_\mu\pi\nabla^\rho\pi\,G_{4X}-$$
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$$-4R_{\mu\rho\nu\sigma}\nabla^\rho\pi\nabla^\sigma\pi\,G_{4X} + \dots = 0.$$

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2. If DPSV trick works, inevitably $\alpha = -\dot{\beta}$.
3. DPSV trick applies to $\mathcal{L}_4$ case.

Thank you for your attention!