Pedal coordinates, Dark Kepler and other force problems

Petr Blaschke

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Force problems

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- Bertrand theorem (1873) : *No other central force problem has the property that all bounded trajectories are also closed.*

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(Mahomed and Vawda in 2000)

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$$\varphi \to \varphi + \omega \int r \mathrm{d}\varphi.$$

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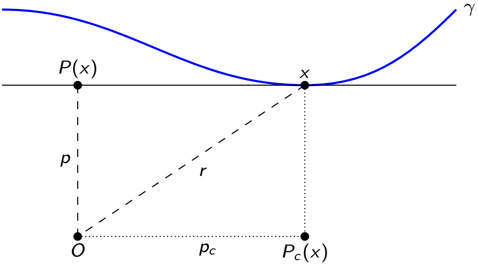
If not the transform $E_\alpha^\star B_\alpha$ can be used to convert the problem into the previous case.

Pedal coordinates



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$$\kappa := \frac{r^2 + 2{r'_\varphi}^2 - rr''_\varphi}{\left(r^2 + {r'_\varphi}^2\right)^{\frac{3}{2}}}, \qquad \Rightarrow \qquad \kappa = \frac{1}{rr'}.$$

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f\left(\frac{1}{p^2}, r\right) = 0 &\quad \xrightarrow{E_c^\star := D_1 E_c D_1} \quad f\left(\frac{1}{p^2} - \frac{2c}{r} + c^2, \frac{r}{1 - cr}\right) = 0,
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Selected transforms

$$f(p,r)=0 \quad \xrightarrow{P} \quad f\left(r,\frac{r^2}{p}\right)=0,$$

$$f(p,r)=0 \quad \xrightarrow{S_\alpha} \quad f(\alpha p,\alpha r,)=0,$$

$$f(p,r)=0 \quad \xrightarrow{I_R} \quad f\left(\frac{Rp}{r^2},\frac{R}{r}\right)=0,$$

$$f\left(p,r^2\right)=0 \quad \xrightarrow{E_c} \quad f\left(p-c,r^2-2pc+c^2\right)=0,$$

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$$\xrightarrow{C:=ES_{\frac{1}{2}}P}$$

Main theorem

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Cassini oval from focus (Geometer's Sketchpad)

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Kepler problem in General relativity

$$r'_{\varphi}{}^2 = \frac{r^4}{b^2} - \left(1 - \frac{r_s}{r}\right)\left(\frac{r^4}{a^2} + a^2\right), \tag{4}$$

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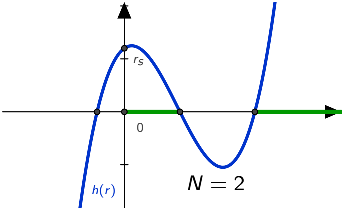
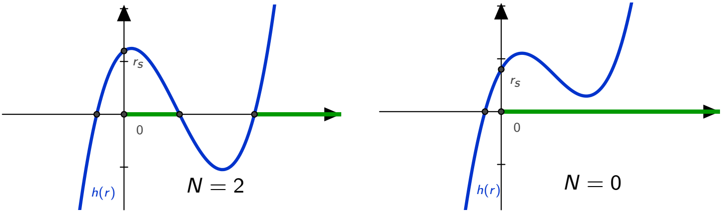
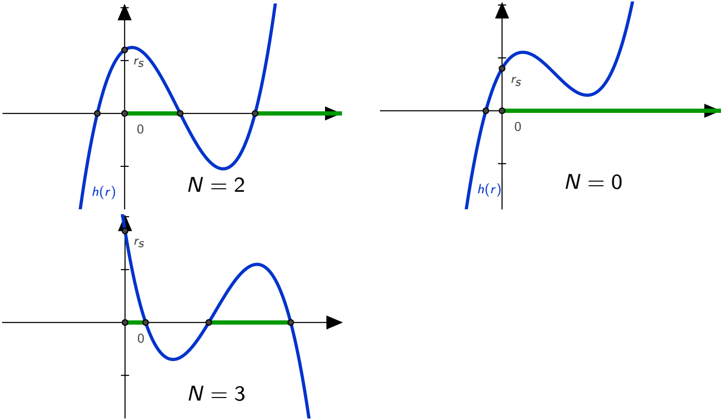
or

$$0 \le dr^3 + \frac{r_s}{a^2} r^2 - r + r_s =: h(r).$$


r_s
0
$h(r)$
$N = 2$


r_s
0
$h(r)$
$N = 2$
r_s
0
$h(r)$
$N = 0$


r_s
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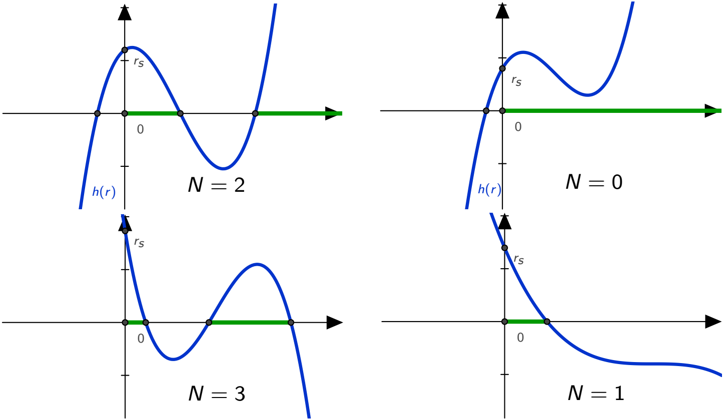
$h(r)$
$N = 2$
r_s
0
$h(r)$
$N = 0$
r_s
0
$N = 3$


r_S
0
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r_S
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r_S
0
$N = 3$
r_S
0
$N = 1$

P. Blaschke: *Pedal coordinates, Dark Kepler and other force problems*, Journal of Mathematical Physics, DOI: 10.1063/1.4984905

arXiv:1704.00897