(Continuous Variables) Quantum Optics, Quantum Information and Relativistic Quantum Information

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Partially based on: Introductory Quantum Optics, C. C. Gerry and P. L. Knight.
1 Introduction
- High energy physics
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- Quantum Optics premises
- Quantum Optics implementations

3 Convariance matrix formalism
- Gaussian states
- Entanglement of bipartite Gaussian states
- Examples with Gaussian states

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- Quantum Information theory: teleportation
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Quantum Electro Dynamics (QED)

Relativistic and quantum Lagrangian $\mathcal{L}_{\text{QED}}$ for matter interacting with light

$$\mathcal{L}_{\text{QED}} = \bar{\psi} \left( i \gamma^\mu D^\mu - m \right) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$ 

- $\gamma^\mu$ are Dirac matrices;
- $\psi$ a bispinor field of spin-1/2 particles (e.g. electron-positron field);
- $\bar{\psi} \equiv \psi^\dagger \gamma^0$, called ”psi-bar”, also referred to as the Dirac adjoint;
- $D^\mu \equiv \partial^\mu + i e A^\mu + i e B^\mu$ is the gauge covariant derivative;
- $e$ is the coupling constant, i.e., the electric charge of the bispinor field;
- $m$ is the mass of the electron or positron;
- $A^\mu$ is the covariant four-potential of the electromagnetic field generated by the electron itself;
- $B^\mu$ is the external field imposed by external source;
- $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor.
High energy physics - Interacting theory

- Construct vertex from the interaction coupled-term $\mathcal{L}_I \sim e \bar{\psi} \gamma^\mu \psi A_\mu$;
- Coupling strength $\lambda \sim e$;
- Energy, momentum, charge must be conserved in physical processes;
Semiclassical theory

Low energy physics of light and matter

In the low energy physics regime we can safely ignore details of light-matter interaction contained in $\mathcal{L}_{\text{QED}}$. We can employ a semiclassical theory.
Semiclassical theory

Low energy physics of light and matter

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Classical and quantum fields (the latter with $[\hat{a}_{k,s}, \hat{a}^\dagger_{k',s'}] = \delta^3(k - k') \delta_{ss'}$)

\[
A(x, t) = \sum_s \int d^3 k \ e_{k,s} \left[ A_{k,s} e^{i(k \cdot x - \omega_k t)} + A_{k,s}^* e^{-i(k \cdot x - \omega_k t)} \right],
\]

\[
\hat{A}(x, t) = \sum_s \int d^3 k \ e_{k,s} \left[ \hat{a}_{k,s} e^{i(k \cdot x - \omega_k t)} + \hat{a}_{k,s}^\dagger e^{-i(k \cdot x - \omega_k t)} \right].
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\]

Classical field

Classical Hamiltonian of e.m. field:

\[
H_0 = \sum_s \int d^3x \omega(k) A^*_{k,s} A_{k,s}.
\]

Quantized field

Quantum Hamiltonian of e.m. field

\[
\hat{H}_0 = \sum_s \int d^3k \omega(k) \hat{a}^\dagger_{k,s} \hat{a}_{k,s} + E_0.
\]
Moving to Quantum Optics

Quantized electric and magnetic fields

\[ \hat{E}(x, t) = i \sum_s \int d^3 k \, \tilde{e}_{k,s} \left[ \hat{a}_{k,s} e^{i(k \cdot x - \omega_k t)} + \hat{a}_{k,s}^\dagger e^{-i(k \cdot x - \omega_k t)} \right] \]

\[ \hat{B}(x, t) = \frac{i}{c} \sum_s \int d^3 k \left( \frac{k}{|k|} \times \tilde{e}_{k,s} \right) \left[ \hat{a}_{k,s} e^{i(k \cdot x - \omega_k t)} + \hat{a}_{k,s}^\dagger e^{-i(k \cdot x - \omega_k t)} \right]. \]

We now make important considerations:

i) Most quantum optical situations, coupling of field to matter is through electric field interacting with a dipole moment or through some nonlinear type of interaction involving powers of the electric field;

ii) Focus on the electric field \( \hat{E}(x, t) \);

iii) Magnetic field is “weaker” than the electric field by a factor of \( \frac{1}{c} \);

iv) Field couples to the spin magnetic moment of the electrons;

v) This interaction is negligible for essentially all the aspects of Quantum Optics that we are concerned with;

vi) Negligible spatial variation of field over dimensions of atomic system.
From the electromagnetic field to modes of light

Dipole approximation:

\[ \hat{E}(x, t) \sim \tilde{e}_x \left[ \hat{a} e^{-i \omega_k t} + \hat{a}^\dagger e^{i \omega_k t} \right], \quad [\hat{a}, \hat{a}^\dagger] = 1. \]

For our purposes: replace field operator \( \hat{E}(x, t) \) by single bosonic mode \( \hat{a} \).
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For our purposes: replace field operator \( \hat{E}(x, t) \) by single bosonic mode \( \hat{a} \).

### Significant states of one-mode \( \hat{a} \)

**Vacuum state** \( |0\rangle \): \( \hat{a} |0\rangle = 0 \).

**Number state** \( |n\rangle \): \( \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle \),

\[ |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle. \]

**Coherent state** \( |\alpha\rangle \):

\[ |\alpha\rangle = \exp[\alpha \hat{a} - \alpha^* \hat{a}^\dagger] |0\rangle. \]

**Single-mode squeezed state** \( |s\rangle \):

\[ |s\rangle = \exp[s \hat{a}^\dagger^2 - s^* \hat{a}^2] |0\rangle. \]

### Significant states of two-modes \( \hat{a}, \hat{c} \)

**Vacuum state** \( |0\rangle \): \( \hat{a} |0\rangle = \hat{c} |0\rangle = 0 \).

**Number state** \( |n, m\rangle \):

\[ [\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}] |n, m\rangle = (n + m) |n, m\rangle , \]

\[ |n, m\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \frac{(\hat{b}^\dagger)^m}{\sqrt{m!}} |0\rangle. \]

**Two-mode squeezed state** \( |r\rangle \):

\[ |r\rangle = \exp[r \hat{a}^\dagger \hat{b}^\dagger - r^* \hat{a} \hat{b}] |0\rangle. \]

We have \([\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0\).
Phase space representation of interesting states

\[ |\alpha\rangle = \exp[\alpha \hat{a} - \alpha^* \hat{a}^\dagger] |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \]

\[ \rho(T) = \frac{1}{\cosh^2 r} \sum_n \tanh^n r |n\rangle\langle n|, \quad \tanh r = e^{-\frac{\hbar \omega}{k_B T}}. \]

\[ |s\rangle = \exp[s \hat{a}^\dagger^2 - s^* \hat{a}^2] |0\rangle = \frac{1}{\cosh s} \sum_n \tanh^n s |2n\rangle. \]
Operations in Quantum Optics

Time evolution / transformation of states $\rho$: $\rho(\lambda) = U^\dagger(\lambda) \rho(0) U(\lambda),$
Operations in Quantum Optics

Time evolution / transformation of states $\rho$: 

$$
\rho(\lambda) = U^\dagger(\lambda) \rho(0) U(\lambda),
$$

Linear unitary operations: effectively reduced to

- Free evolution/phase shifting: 
  $$
  U(t) = \exp[-i \omega_a t (= \phi) \hat{a} \dagger \hat{a}].
  $$
- Beam splitting: 
  $$
  U(\theta) = \exp[-i \theta (\hat{a} \dagger \hat{b} + \hat{a} \dagger \hat{b} \dagger)].
  $$
- Single mode squeezing: 
  $$
  U(s) = \exp[(s \hat{a} \dagger \dagger - s^* \hat{a}^2)].
  $$
- Two mode squeezing: 
  $$
  U(r) = \exp[(r \hat{a} \dagger \hat{b} \dagger - r^* \hat{a} \hat{b})].
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\]

Single mode squeezing:
\[
U(s) = \exp[(s \hat{a}^{\dagger}, 2 - s^* \hat{a}^2)].
\]

Two mode squeezing:
\[
U(r) = \exp[(r \hat{a}^{\dagger} \hat{b}^{\dagger} - r^* \hat{a} \hat{b})].
\]

Non-linear unitary operations: effectively reduced to

PDC:
\[
U(\xi) = \exp[\xi \hat{a}^{\dagger} \hat{b}^{\dagger} \hat{c} - \xi^* \hat{a} \hat{b} \hat{c}^{\dagger}].
\]
Quantum Optics laboratory

Now use BCH: \[ e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+...} \].

Figure: Figure from phys.org
Gaussian states of light

Picking experimentally realisable states of light

i) Very few states can be realised in optics laboratories (i.e., not $|\psi_1\rangle$);
ii) Employable states are prepared using linear optics;
iii) These state can be manipulated with linear optics.
Gaussian states of light

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Gaussian states

i) Have Gaussian Wigner function;  
$W(\xi) = \frac{1}{\pi^2} \int_{\mathbb{R}^2N} d^2N \chi_s(\kappa) e^{iX \cdot \Omega \xi}$  
i) Wigner function is positive;  
iii) Are defined by finite d.o.f.;  
iv) Produced in every Q.O. lab.
Covariance matrix formalism

Gaussian states

- N modes of light $\hat{a}_1, \ldots, \hat{a}_N$;
Covariance matrix formalism

Gaussian states

- \( N \) modes of light \( \hat{a}_1, ..., \hat{a}_N \);
- Introduce \( \hat{X} := (\hat{a}_1, ..., \hat{a}_N, \hat{a}_1^\dagger, ..., \hat{a}_N^\dagger)^T \);
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- Commutation relations: $[\hat{a}_n, \hat{a}_m^\dagger] = i \Omega_{nm}$. 
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- Symplectic form $\Omega$: $\Omega_{nm} = \text{diag}(-i, -i, ..., i, i, ...)$.
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- First moments \( d \): \( d := \langle \hat{X} \rangle_\rho \).
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Linear transformations

- Quadratic in $\hat{a}_n, \hat{a}_m^\dagger$;
- $N$-mode linear represented by $2^N \times 2^N$ symplectic matrix $S$.
- Symplectic: $S^\dagger \Omega S = \Omega$.
- Unitary $U = \exp[-i H t]$: represented by $S = \exp[-i F(t) \Omega H]$. 

Here $H = \frac{1}{2} \mathbf{X}^T H \mathbf{X}$. 
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**Linear transformations**

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  Here $H = \frac{1}{2} \hat{X}^T \hat{H} \hat{X}$.
Covariance matrix formalism

Gaussian states

Gaussian states are defined by only **first** and **second moments**: they are defined **univocally** by the covariance matrix $\sigma$ and the first moments $d$. 

$$\sigma = S^{\dagger} \nu \otimes S$$

Here $\nu \otimes = \text{diag}(\nu_1, \nu_2, ..., \nu_1, \nu_2, ...) \text{ is the Williamson form of } \sigma \text{ and } \nu_k \geq 1 \text{ are the symplectic eigenvalues of } \sigma.$$

Purity

N.B. The state $\sigma$ is pure iff $\nu_k = 1$ for all $k$. The symplectic eigenvalues are $\nu_k = \coth(\hbar \omega_k K b T_k)$. 


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Quantum optics with covariance matrix formalism

Gaussian state evolution/transformations

\[ \sigma(\lambda) = S^\dagger(\lambda) \sigma(0) S(\lambda) \].
Quantum optics with covariance matrix formalism

### Gaussian state evolution/transformations

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### Gaussian vs. non-Gaussian states

<table>
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<td>ii) Infinite d.o.f.;</td>
<td>ii) Finite d.o.f.;</td>
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<td>vi) Entanglement: see next.</td>
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Entanglement of Gaussian states

Separability and entanglement

The state $\rho_{AB}$ separable if exists $\rho_{AB} = \rho_A \otimes \rho_B$. If not, it is entangled. EPR state: $|\Psi\rangle = \frac{1}{\sqrt{2}} [|01\rangle + |10\rangle]$. 

Symplectic eigenvalues $\nu_k \geq 1$: spectrum of $i\Omega\sigma$. Symplectic spectrum of the partial transpose

Compute spectrum $\tilde{\nu}_k$ of $i\Omega P^\dagger\sigma P$. Here $P$ implements partial transposition of one mode.
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Symplectic spectrum of the partial transpose

Compute spectrum $\tilde{\nu}_k$ of $i \Omega P^\dagger \sigma P$. Here $P$ implements partial transposition of one mode.

Of these eigenvalues $\tilde{\nu}_k$ take the smallest (in absolute value), called $\tilde{\nu}_-$.
An example: using Hilbert space formalism

Start from the vacuum state $|0\rangle$. Use linear optics and two-mode squeeze the vacuum with $U(r) := \exp[-r (\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})]$. (Lots of) algebra:

$$ |\Psi(r)\rangle = U(r) |0\rangle = \sum_n \frac{\tanh^n r}{\cosh r} |n, n\rangle. $$

The state $\rho(r)$ is:

$$ \rho(r) = |\Psi(r)\rangle\langle\Psi(r)| = \sum_{n,m} \frac{\tanh^{n+m} r}{\cosh^2 r} |n, n\rangle\langle m, m|. $$

Partial transpose mode $\hat{b}$ is

$$ \tilde{\rho}(r) = |\Psi(r)\rangle\langle\Psi(r)| = \sum_{n,m} \frac{\tanh^{n+m} r}{\cosh^2 r} |n, m\rangle\langle m, n|. $$

Choose measure: Negativity $\mathcal{N} := \frac{\text{Tr}(\sqrt{\tilde{\rho}^\dagger \tilde{\rho}}) - 1}{2}$

More algebra: $\mathcal{N} = \frac{e^{2r} - 1}{2}$. 
An example: using **Covariance Matrix** formalism

Start from the vacuum state $\sigma_0$. Use linear optics and two-mode squeeze the vacuum with $S(r)$. (Little) algebra:

\[
\sigma_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad S(r) = \begin{pmatrix}
cosh r & 0 & 0 & \sinh r \\
0 & \cosh r & \sinh r & 0 \\
0 & \sinh r & \cosh r & 0 \\
\sinh r & 0 & 0 & \cosh r \\
\end{pmatrix}.
\]

The state $\sigma(r) := S^\dagger(r) \sigma_0 S(r)$ is:

\[
\sigma(r) = \begin{pmatrix}
cosh 2r & 0 & 0 & \sinh 2r \\
0 & \cosh 2r & \sinh 2r & 0 \\
0 & \sinh 2r & \cosh 2r & 0 \\
\sinh 2r & 0 & 0 & \cosh 2r \\
\end{pmatrix}.
\]
An example: using Covariance Matrix formalism

Partial transpose mode $\hat{b}$ is implemented by $P$ and obtain $P^\dagger \sigma P$

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad P^\dagger \sigma P = \begin{pmatrix}
\cosh 2r & \sinh 2r & 0 & 0 \\
\sinh 2r & \cosh 2r & 0 & 0 \\
0 & 0 & \cosh 2r & \sinh 2r \\
0 & 0 & \sinh 2r & \cosh 2r \\
\end{pmatrix}.
\]

Compute the spectrum of $i \Omega P^\dagger \sigma P$, which is

\[
\{ e^{2r}, \tilde{\nu}_- = e^{-2r}, -e^{2r}, -e^{-2r}, \}.
\]

Choose measure: Negativity $\mathcal{N} := \frac{1 - \tilde{\nu}_-}{2 \tilde{\nu}_-}$.

Simply obtain: $\mathcal{N} = \frac{e^{2r} - 1}{2}$.
Quantum Information

Information theory aims at understanding how to...

- Store
- Transmit
- Decode
- Quantify
- Employ
- Secure

...information
Quantum Information

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...information

<table>
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Discrete Variables

i) Generate EPR. One qubit to A, other to B;

ii) Bell measurement at A of EPR pair qubit and the qubit (|φ⟩) to be teleported. Yield one of four measurement outcomes, encoded in two classical bits;

iii) Using the classical channel, the two bits are sent from A to B. (Speed less than c);

iv) Result of measurement at A, EPR pair qubit at B in one of four possible states. Of these, one identical to the original quantum state |φ⟩, other three are closely related. Which of these four possibilities actually obtains is encoded in the two classical bits. Knowing this, the qubit at location B is modified to result in a qubit identical to |φ⟩.

Figure: Figure from Wikipedia
Continuous Variables

i) Generate EPR-like state: two-mode squeezed state;

ii) CV version of Bell measurement. Beam split modes $\hat{a}$ and “in”. Homodyne detect quadratures $\hat{x}_-\text{ and } \hat{p}_+$;

iii) Classical channel, same as before;

iv) Similar as before. Use classical information to perform extra displacement. Obtain initial state $\rho_{in}$.

Figure: Protocol and figure based on Section IV.B in Laser Physics 16, 1418 (2006)
Relativistic Quantum Teleportation

i) Generate usual squeezed state;

ii.i) Homodyne measurement at A of local mode and the mode to be teleported;

ii.ii) Rob moves! Field inside Rob’s cavity is affected by motion. Mode from squeezed state is mixed to other modes;

iii) Using the classical channel, the two bits are sent from A to R. (Speed less than $c$);

iv) Result like before but depends also on motion.

Figure: PRL 110, 113602 (2013)
Conclusions

Quantum Information

Information Theory
Thermodynamics

Quantum Mechanics

Relativity

Black Hole Thermodynamics

QFT in curved spacetime

Relativistic Quantum Information
Acknowledgments

Thank you