Gauge Theories

and

Non-Commutative Geometry

A Review

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Motivation

- Short distance singularities.

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- The construction of gauge theories using the techniques of non-commutative geometry.

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  $\partial^\mu x_\nu = \delta^\mu_\nu \qquad [x_\mu, f(x)] = i\theta_{\mu\nu}\partial^\nu f(x)$
- Define a * product
  $f * g = e^{\frac{i}{2}\frac{\partial}{x_\mu}\theta_{\mu\nu}\frac{\partial}{y_\nu}} f(x)g(y)|_{x=y}$

All computations can be viewed as expansions in θ
expansions in the external field

More efficient ways?

Quantum field theory in a space with non-commutative geometry?
BRS Symmetry?

Large N field theories

- $\phi^i(x)\ i = 1, ..., N\ ;\ N \to \infty$

  $\phi^i(x) \to \phi(\sigma, x)\ 0 \leq \sigma \leq 2\pi$

  $\sum_{i=1}^{\infty} \phi^i(x)\phi^i(x) \to \int_0^{2\pi} d\sigma(\phi(\sigma, x))^2$

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- For a Yang-Mills theory, the resulting expression is local

Gauge theories on surfaces

E.G. Floratos and J.I.

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- there exists a reformulation in $d{+}2$ dimensions

$$A_\mu(x) \to \mathcal{A}_\mu(x, z_1, z_2) \qquad F_{\mu\nu}(x) \to \mathcal{F}_{\mu\nu}(x, z_1, z_2)$$

with $\qquad [z_1, z_2] = \frac{2i}{N}$

$$[A_\mu(x), A_\nu(x)] \rightarrow \{\mathcal{A}_\mu(x, z_1, z_2), \mathcal{A}_\nu(x, z_1, z_2)\}_{Moyal}$$

$$[A_\mu(x), \Omega(x)] \rightarrow \{\mathcal{A}_\mu(x, z_1, z_2), \Omega(x, z_1, z_2)\}_{Moyal}$$

$$\int d^4x \ Tr\left(F_{\mu\nu}(x)F^{\mu\nu}(x)\right) \quad \rightarrow \quad \int d^4x dz_1 dz_2 \ \mathcal{F}_{\mu\nu}(x, z_1, z_2) * \mathcal{F}^{\mu\nu}(x, z_1, z_2)$$

These expressions are defined for *all* N!

Not necessarily integer ???

I. Large N

-A simple algebraic result:

At large N

The $SU(N)$ algebra $\rightarrow$ The algebra of the area preserving diffeomorphisms of a closed surface. (sphere or torus).

-The structure constants of $[SDiff(S^2)]$ are the limits for large N of those of $SU(N)$.

-Alternatively: For the sphere

$$x_1 = cos\phi\ sin\theta, \quad x_2 = sin\phi\ sin\theta, \quad x_3 = cos\theta$$

$$Y_{l,m}(\theta,\phi) = \sum_{\substack{i_k=1,2,3\\ k=1,\ldots,l}} \alpha^{(m)}_{i_1\ldots i_l}\ x_{i_1}\ldots x_{i_l}$$

where $\alpha^{(m)}_{i_1\ldots i_l}$ is a symmetric and traceless tensor.
For fixed l there are $2l+1$ linearly independent tensors $\alpha^{(m)}_{i_1\ldots i_l}$, $m = -l, \ldots, l$.

Choose, inside $SU(N)$, an $SU(2)$ subgroup.

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

A basis for $SU(N)$:

$$S^{(N)}_{l,m} = \sum_{\substack{i_k=1,2,3 \\ k=1,\ldots,l}} \alpha^{(m)}_{i_1\ldots i_l} \; S_{i_1}\ldots S_{i_l}$$

$$[S^{(N)}_{l,m},\; S^{(N)}_{l',m'}] = i f^{(N)l'',m''}_{l,m;\; l',m'} \; S^{(N)}_{l'',m''}$$

The three $SU(2)$ generators S_i, rescaled by a factor proportional to $1/N$, will have well-defined limits as N goes to infinity.

$$S_i \to T_i = \frac{2}{N} S_i$$
$$[T_i, T_j] = \frac{2i}{N} \epsilon_{ijk} T_k$$
$$T^2 = T_1^2 + T_2^2 + T_3^2 = 1 - \frac{1}{N^2}$$

In other words: under the norm $\|x\|^2 = Trx^2$, the limits as N goes to infinity of the generators T_i are three objects x_i which commute and are constrained by

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$\frac{N}{2i}\,[f,g] \to \quad \epsilon_{ijk}\; x_i\; \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}$$

$$\frac{N}{2i}\,[T^{(N)}_{l,m}, T^{(N)}_{l',m'}] \to \;\{Y_{l,m}, Y_{l',m'}\}$$

$$N[A_\mu, A_\nu] \to \quad \{A_\mu(x,\theta,\phi), A_\nu(x,\theta,\phi)\}$$

$\Rightarrow$ The d-dim. $SU(N)$ Yang-Mills theory for $N \to \infty$

$\equiv$

A classical theory on a $d+2$-dim space.

The quantum theory??

II. To all orders

We can parametrise the T_i's in terms of two operators, z_1 and z_2.

$$T_+ = T_1 + iT_2 = e^{\frac{iz_1}{2}}\,(1 - z_2^2)^{\frac{1}{2}}\,e^{\frac{iz_1}{2}}$$
$$T_- = T_1 - iT_2 = e^{-\frac{iz_1}{2}}\,(1 - z_2^2)^{\frac{1}{2}}\,e^{-\frac{iz_1}{2}}$$
$$T_3 = z_2$$

If we assume that z_1 and z_2 satisfy:

$[z_1, z_2] = \frac{2i}{N}$

The T_i's satisfy the $SU(2)$ algebra.

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For the torus

Choose, inside $SU(N)$, a quantum $U(1) \times U(1)$

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix} \quad ; \quad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(N odd), $\omega = e^{4\pi i/N}$

$$g^N = h^N = 1 \quad ; \quad hg = \omega gh$$

For the torus

We can use the integer mod$N$ powers of these matrices to express the $SU(N)$ generators:

$$S_{m_1,m_2} = \omega^{m_1 m_2/2} g^{m_1} h^{m_2} \quad ; \quad S^{\dagger}_{m_1,m_2} = S_{-m_1,-m_2}$$

$$[S_{\mathbf{m}}, S_{\mathbf{n}}] = 2\mathrm{i} \sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n}\right) S_{\mathbf{m}+\mathbf{n}}$$

$\mathbf{n} = (n_1, n_2)$ and $\mathbf{n}\times\mathbf{m} = n_1 m_2 - m_1 n_2$

$$SU(N)|_{N\to\infty} = \mathrm{SDiff}(T^2)$$

z_1 , z_2 the two angular variables:

$$h = \mathrm{e}^{\mathrm{i}z_1} \quad g = \mathrm{e}^{-2\mathrm{i}\pi z_2} \Rightarrow [z_1, z_2] = \frac{2\mathrm{i}}{N} \to hg = \omega gh$$

For the torus

The generators of the Heisenberg algebra z_1 and z_2,
as well as the group elements $h = \mathrm{e}^{\mathrm{i}z_1}$ and $g = \mathrm{e}^{-2\mathrm{i}\pi z_2}$

are infinite dimensional operators

but we can represent the $SU(N)$ algebra by the finite dimensional matrices g , h and S_{m_1,m_2}

They form a discrete subgroup of the Heisenberg group
$\Rightarrow$
quantum mechanics on a discrete phase space

We can define two new operators
$\hat{q}$ ("position" in the discrete space) and $\hat{p}$ (its FFT):
They are represented by finite matrices but, obviously, they do not satisfy the Heisenberg algebra.

The techniques of non-com. geometry

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- Gauge transformations are:
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- Question: Is there a space on which Internal symmetry transformations act as Diffeomorphisms?
- Answer: Yes, but it is a space with non-commutative geometry. A space defined by an algebra of matrix-valued functions

Some useful references:

- Ali H. Chamseddine and Alain Connes ;

Why the Standard Model, arXiv:0706.3688 [hep-th]

- Ali H. Chamseddine, Alain Connes, and Viatcheslav Mukhanov ;

Geometry and the Quantum: Basics, arXiv:1411.0977 [hep-th]

- Ali H. Chamseddine, Alain Connes, and Viatcheslav Mukhanov ;

Quanta of Geometry: Noncommutative Aspects, arXiv:1409.2471 [hep-th]

The construction involves A fundamental spectral triplet :

Given a spin manifold M, the triplet consists of:

1. A Hilbert space $\mathcal{H}$
2. An algebra of functions $\mathcal{A}$ which are $C^{\infty}(M)$
3. The Dirac operator $\mathcal{D}$

(If we ignore gravity, $\mathcal{D}$ can be replaced by the chirality operator)

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- *New predictions for the Standard Model parameters?*

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- Answer: Compute the corresponding β-function.

$$16\pi^2\beta_{g_1} = g_1^3\frac{1}{10}$$

$$16\pi^2\beta_{g_2} = -g_2^3\frac{43}{6}$$

$$16\pi^2\beta_{\lambda} = 12\lambda^2 - \frac{9}{5}g_1^2\lambda - 9g_2^2\lambda + \frac{27}{100}g_1^4 + \frac{9}{10}g_1^2g_2^2 + \frac{9}{4}g_2^4$$

$$\beta_z = \beta_{\eta_1} + \beta_{\eta_2} =$$
$$= \frac{-\lambda w}{16\pi^2 \rho z} \left[\left(\frac{27}{100}\rho^2 + \frac{9}{10}\rho + \frac{9}{4} \right) z^2 - \left(2\rho^2 + \frac{54}{5}\rho - \frac{16}{3} \right) z \right.$$
$$\left. + 12(\rho + 1)^2 \right]$$

$$\eta_1 = \frac{g_1^2}{\lambda} \quad ; \quad \eta_2 = \frac{g_2^2}{\lambda} \quad ; \quad z = \eta_1 + \eta_2 \quad ; \quad \rho = \frac{\eta_1}{\eta_2} \quad ; \quad w = \eta_1 \eta_2$$

- β_z has no zeroes! $\Rightarrow$ The Standard Model is irreducible.

Related question: Is there a B.R.S. symmetry for the model on non-com. geometry?

The spectacular accuracy reached by experiments, as well as theoretical calculations, made particle physics a precision science

Example: $m_W = 80.385 \pm 0.015\text{GeV}$
$\Rightarrow$ "Approximate" theories are no more sufficient!

A discrepancy by a few percent implies that we do not have the right theory!

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- But, for the moment, we see no corner!

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- Non-Commutative Geometry has come to stay!
- Whether it will turn out to be convenient for us to use, is still questionable.
- It will depend on our ability to simplify the mathematics sufficiently, or to master them deeply, in order to get new insights