Anisotropic dissipative fluid dynamics – theory and applications in heavy-ion physics

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thanks to: Etele Molnár, Harri Niemi

**based on: PRD 93 (2016) 11, 114025;
PRD 94 (2016) 12, 125003**

Walter, ICNFP, and Anisotropic Fluid Dynamics

Walter on the cruise to Balos lagoon, 3rd edition of ICNFP 2014

J. Phys. G: Nucl. Phys. **13** (1987) L181–L188. Printed in the UK

LETTER TO THE EDITOR

Generalised non-equilibrium equation of state for nuclear matter with momentum-dependent interactions

L W Neise, H Stöcker and W Greiner

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Microscopic foundations of dissipative fluid dynamics (I)

Boltzmann equation: $$k^\mu \partial_\mu f_{\rm k} = C[f]$$

where: k^μ particle 4-momentum

$f_{\rm k}$ single-particle distribution function

$\mathcal{C}[f]$ collision integral

$\Longrightarrow$ $0^{\rm th}$ and $1^{\rm st}$ moment of the Boltzmann equation:

$$\partial_\mu N^\mu = \mathcal{C}$$
$$\partial_\mu T^{\mu\nu} = \mathcal{C}^\nu$$

where: $N^\mu \equiv \int_k k^\mu f_{\rm k}$ particle no. 4-current,

$T^{\mu\nu} \equiv \int_k k^\mu k^\nu f_{\rm k}$ energy-momentum tensor,

$\int_k \equiv g \int \frac{d^3k}{(2\pi)^3 k_0}$, g: internal quantum no. degeneracy of momentum state

Note: $\mathcal{C} \equiv \int_k C[f] = 0$ and $\mathcal{C}^\nu \equiv \int_k k^\nu C[f] \equiv 0$ for binary elastic collisions (particle no. and 4-momenta are microscopic collisional invariants)

$\Longrightarrow$ macroscopic conservation of particle number, energy, and momentum!

$$\partial_\mu N^\mu = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$

Microscopic foundations of dissipative fluid dynamics (II)

general tensor decomposition with respect to u^μ in Landau frame:
(where u^μ is 4-velocity of energy flow)

$$N^\mu = n u^\mu + n^\mu$$
$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

where:

$n \equiv N^\mu u_\mu$	particle density (1)
$\epsilon \equiv T^{\mu\nu} u_\mu u_\nu$	energy density (1)
$p(\epsilon, n)$	pressure in a fictitious local-equilibrium state with given ϵ, n
$\Pi \equiv -\frac{1}{3} T^{\mu\nu}\Delta_{\mu\nu} - p$	bulk viscous pressure (1)
$n^\mu \equiv \Delta^{\mu\nu} N_\nu$	particle diffusion current (3)
$\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$	shear-stress tensor (5)

with: $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ 3-space projector onto direction orthogonal to u^μ

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$$

$\Longrightarrow$ **equations of motion not closed:**
5 equations, 14 unknowns

$$\dot{n} + n\,\theta + \partial \cdot n = 0$$
$$\dot{\epsilon} + (\epsilon + p + \Pi)\,\theta - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0$$
$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi \dot{u}^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda}$$

where:

$\dot{A} \equiv u^\mu \partial_\mu A$	comoving derivative
$\theta \equiv \partial_\mu u^\mu$	expansion scalar
$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$	3-space gradient orthogonal to u^μ

$\Longrightarrow$ **need 9 additional equations of motion for Π, n^μ, $\pi^{\mu\nu}$!**

Microscopic foundations of dissipative fluid dynamics (III)

Consider small deviations from local thermodynamical equilibrium:

$$f_{\mathrm{k}} = f_{0\mathrm{k}} + \delta f_{\mathrm{k}} \qquad |\delta f_{\mathrm{k}}| \ll |f_{0\mathrm{k}}|$$

where

$$f_{0\mathrm{k}} = \left[\exp\left(-\alpha + \beta E_{\mathrm{k}u}\right) + a\right]^{-1}$$

with: $\beta = 1/T$, T temperature, $\alpha = \beta\mu$, μ chemical potential,
$E_{\mathrm{k}u} = k^\mu u_\mu$ energy of particle in rest frame of fluid,
$a = \pm 1, 0$ for fermions/bosons, Boltzmann particles

$\Longrightarrow$ irreducible moments of δf_{k}:

$$\rho_r^{\mu_1 \cdots \mu_\ell} \equiv \int_k E_{\mathrm{k}u}^r \, k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \, \delta f_{\mathrm{k}}$$

where: $A^{\langle \mu_1 \cdots \mu_\ell \rangle} \equiv \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} A^{\nu_1 \cdots \nu_\ell}$,
$\Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell}$ projectors onto subspaces orthogonal to u^μ ,
formed from $\Delta^{\mu\nu}$, symmetric in μ_i, ν_j , traceless,

Note: $-\frac{m^2}{3}\rho_0 \equiv \Pi$, $\rho_0^\mu \equiv n^\mu$, $\rho_0^{\mu\nu} \equiv \pi^{\mu\nu}$
matching conditions in Landau frame: $n = n_0 \Longrightarrow \rho_1 = 0$
$\epsilon = \epsilon_0 \Longrightarrow \rho_2 = 0$
$q^\mu = 0 \Longrightarrow \rho_1^\mu = 0$

Microscopic foundations of dissipative fluid dynamics (IV)

$\Longrightarrow$ derive equations of motion for irreducible moments:

$$\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\, u^\alpha \partial_\alpha \int_k E^r_{\mathrm{k}u}\, k^{\langle\nu_1} \cdots k^{\nu_\ell\rangle} \delta f_{\mathrm{k}}$$

$\Longrightarrow$ use Boltzmann equation:

$$\delta \dot{f}_{\mathrm{k}} = - \dot{f}_{0\mathrm{k}} - \frac{1}{E_{\mathrm{k}u}} \left\{ k^\mu \nabla_\mu \left(f_{0\mathrm{k}} + \delta f_{\mathrm{k}} \right) - C[f] \right\}$$

$\Longrightarrow$ system of infinitely many coupled equations for irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$, completely equivalent to Boltzmann equation $\Longrightarrow$ truncation required!

systematic power counting:

$$\mathrm{Kn} \equiv \frac{\ell_{\mathrm{mfp}}}{L_{fluid}} \sim \ell_{\mathrm{mfp}}\, \partial_\mu \qquad \text{Knudsen number}$$

$$\mathrm{Re}^{-1} \equiv \frac{\Pi}{p} \sim \frac{n^\mu}{n} \sim \frac{\pi^{\mu\nu}}{p} \qquad \text{inverse Reynolds number}$$

for $\ell \geq 3 : \rho_r^{\mu_1\cdots\mu_\ell} \sim O(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{Re}^{-1})$ $\Longrightarrow$ will be neglected (work to O_2)

eigenmode analysis: consider slowest eigenmodes to be dynamical,
take all faster eigenmodes equal to their asymptotic values

$\Longrightarrow$ allows to express all irreducible moments $\rho_r\,,\ \rho_r^\mu\,,\ \rho_r^{\mu\nu}$ for $r > 0$ by $\rho_0 \equiv -\frac{3}{m^2}\,\Pi\,,\ \rho_0^\mu \equiv n^\mu\,,\ \rho_0^{\mu\nu} \equiv \pi^{\mu\nu}$ and gradients of $\alpha\,,\ \beta\,,\ u^\mu$

Microscopic foundations of dissipative fluid dynamics (V)

$\Longrightarrow$ **equations of motion for Π, n^μ, $\pi^{\mu\nu}$:**

$$\tau_\Pi \, \dot{\Pi} + \Pi = -\zeta_0 \theta + \mathcal{K} + \mathcal{J} + \mathcal{R}$$

$$\tau_n \, \dot{n}^{<\mu>} + n^\mu = \kappa_0 \nabla^\mu \alpha + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu$$

$$\tau_\pi \, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = 2\,\eta_0\,\sigma^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$$

Kn^2:

$$\mathcal{K} = \bar{\zeta}_1\,\omega_{\mu\nu}\,\omega^{\mu\nu} + \bar{\zeta}_2\,\sigma^{\mu\nu}\,\sigma_{\mu\nu} + \bar{\zeta}_3\,\theta^2 + \bar{\zeta}_4\,(\nabla\alpha)^2 + \bar{\zeta}_5\,(\nabla p)^2 + \bar{\zeta}_6\,\nabla_\mu\alpha\nabla^\mu p + \bar{\zeta}_7\,\nabla^2\alpha + \bar{\zeta}_8\,\nabla^2 p\,,$$

$$\mathcal{K}^\mu = \bar{\kappa}_1\,\sigma^{\mu\nu}\,\nabla_\nu\alpha + \bar{\kappa}_2\,\sigma^{\mu\nu}\,\nabla_\nu p + \bar{\kappa}_3\,\theta\,\nabla^\mu\alpha + \bar{\kappa}_4\,\theta\,\nabla^\mu p + \bar{\kappa}_5\,\omega^{\mu\nu}\,\nabla_\nu\alpha + \bar{\kappa}_6\,\Delta^{\mu\lambda}\partial^\nu\sigma_{\lambda\nu} + \bar{\kappa}_7\,\nabla^\mu\theta\,,$$

$$\mathcal{K}^{\mu\nu} = \bar{\eta}_1\,\omega_\lambda{}^{\langle\mu}\,\omega^{\nu\rangle\lambda} + \bar{\eta}_2\,\theta\,\sigma^{\mu\nu} + \bar{\eta}_3\,\sigma_\lambda{}^{\langle\mu}\,\sigma^{\nu\rangle\lambda} + \bar{\eta}_4\,\sigma_\lambda{}^{\langle\mu}\,\omega^{\nu\rangle\lambda} + \bar{\eta}_5\,\nabla^{\langle\mu}\alpha\,\nabla^{\nu\rangle}\alpha$$
$$+ \bar{\eta}_6\,\nabla^{\langle\mu}p\,\nabla^{\nu\rangle}p + \bar{\eta}_7\,\nabla^{\langle\mu}\alpha\,\nabla^{\nu\rangle}p + \bar{\eta}_8\,\nabla^{\langle\mu}\nabla^{\nu\rangle}\alpha + \bar{\eta}_9\,\nabla^{\langle\mu}\nabla^{\nu\rangle}p$$

Re^{-1}Kn:

$$\mathcal{J} = -\ell_{\Pi n}\,\nabla_\mu n^\mu - \tau_{\Pi n}\,n^\mu\nabla_\mu p - \delta_{\Pi\Pi}\,\theta\,\Pi - \lambda_{\Pi n}\,n^\mu\nabla_\mu\alpha + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu}$$

$$\mathcal{J}^\mu = \tau_n\,\omega^{\mu\nu}\,n_\nu - \delta_{nn}\,\theta\,n^\mu - \ell_{n\Pi}\,\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\,\nabla^\lambda\pi_{\nu\lambda} + \tau_{n\Pi}\,\Pi\,\nabla^\mu p - \tau_{n\pi}\,\pi^{\mu\nu}\,\nabla_\nu p - \lambda_{nn}\,\sigma^{\mu\nu}\,n_\nu$$
$$+ \lambda_{n\Pi}\,\Pi\,\nabla^\mu\alpha - \lambda_{n\pi}\,\pi^{\mu\nu}\,\nabla_\nu\alpha$$

$$\mathcal{J}^{\mu\nu} = 2\,\tau_\pi\,\pi_\lambda{}^{\langle\mu}\,\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\,\theta\,\pi^{\mu\nu} - \tau_{\pi\pi}\,\pi_\lambda{}^{\langle\mu}\,\sigma^{\nu\rangle\lambda} + \lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu} - \tau_{\pi n}\,n^{\langle\mu}\,\nabla^{\nu\rangle}p + \ell_{\pi n}\,\nabla^{\langle\mu}n^{\nu\rangle}$$
$$+ \lambda_{\pi n}\,n^{\langle\mu}\nabla^{\nu\rangle}\alpha \qquad \text{where } \omega^{\mu\nu} \equiv \left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)/2$$

Re^{-2}:

$$\mathcal{R} = \varphi_1\,\Pi^2 + \varphi_2\,n_\mu n^\mu + \varphi_3\,\pi^{\mu\nu}\pi_{\mu\nu}$$

$$\mathcal{R}^\mu = \varphi_4\,\pi^{\mu\nu}\,n_\nu + \varphi_5\,\Pi\,n^\mu$$

$$\mathcal{R}^{\mu\nu} = \varphi_6\,\Pi\,\pi^{\mu\nu} + \varphi_7\,\pi_\lambda{}^{\langle\mu}\,\pi^{\nu\rangle\lambda} + \varphi_8\,n^{\langle\mu}\,n^{\nu\rangle}$$

G.S. Denicol, H. Niemi, E. Molnar, DHR,
PRD 85 (2012) 114047,
Erratum PRD 91 (2015) 3, 039902

Anisotropic fluid dynamics

Initial gradients in heavy-ion collisions are large

$\Longrightarrow$ **deviations from local thermodynamical equilibrium are large!**

$\Longrightarrow$ **may invalidate dissipative fluid dynamics**

Idea: "resum" dissipative corrections into single-particle distribution function,

e.g.: W. Florkowski, PLB 668 (2008) 32; M. Martinez, M. Strickland, PRC 81 (2010) 024906

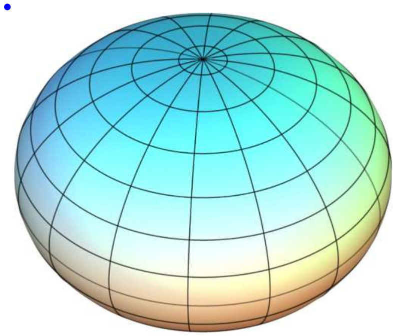
$$\hat{f}_{0\mathrm{k}} = \left[\exp\left(-\hat{\alpha} + \hat{\beta}_u \sqrt{E_{\mathrm{k}u}^2 + \xi\, E_{\mathrm{k}l}^2}\right) + a\right]^{-1}$$

where $E_{\mathrm{k}l} \equiv -l^\mu k_\mu$ **, with** l^μ **direction of anisotropy,** $l^\mu l_\mu = -1\,,\;\; l^\mu u_\mu = 0\,,$

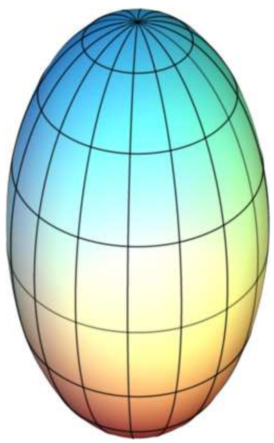
usually: $l^\mu = \gamma_z(v_z, 0, 0, 1)\,,\;\gamma_z = (1 - v_z^2)^{-1/2}\,,$

ξ **anisotropy parameter**

$\Longrightarrow$ **in LR frame of fluid:**



$\xi > 0$



$\xi < 0$

$\Longrightarrow$ **5 conservation equations determine** $\hat{\alpha}$, $\hat{\beta}_u$, u^μ **(3)**

$\Longrightarrow$ **need additional equation to determine** ξ**!**

Microscopic foundations of anisotropic dissipative fluid dynamics (I)

$$f_{\rm k} = f_{0\rm k} + \delta f_{\rm k} \equiv \hat{f}_{0\rm k} + \delta \hat{f}_{\rm k}$$

If $\delta f_{\rm k} \sim f_{0\rm k}$, choose $\hat{f}_{0\rm k}$ such that $|\delta \hat{f}_{\rm k}| \ll |\hat{f}_{0\rm k}|$

$\Longrightarrow$ improved convergence properties of expansion around $\hat{f}_{0\rm k}$!

D. Bazow, U.W. Heinz, M. Strickland, PRC 90 (2014) 5, 054910

E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

$\Longrightarrow$ irreducible moments of $\delta \hat{f}_{\rm k}$:

$$\hat{\rho}^{\mu_1\cdots\mu_\ell}_{rs} \equiv \int_k E^r_{{\rm k}u}\, E^s_{{\rm k}l}\, k^{\{\mu_1} \cdots k^{\mu_\ell\}}\, \delta \hat{f}_{\rm k}$$

where: $A^{\{\mu_1\cdots\mu_\ell\}} \equiv \Xi^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} A^{\nu_1\cdots\nu_\ell}$,

$\Xi^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$ projectors onto subspaces orthogonal to both u^μ and l^μ , formed from $\Xi^{\mu\nu}$,

symmetric in μ_i, ν_j , traceless,

$\Xi^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu$ 2-space projector onto direction orthogonal to both u^μ and l^μ

$\Longrightarrow$ derive equations of motion for irreducible moments:

$$\dot{\hat{\rho}}^{\{\mu_1\cdots\mu_\ell\}}_{rs} \equiv \Xi^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\, u^\alpha \partial_\alpha \int_k E^r_{{\rm k}u}\, E^s_{{\rm k}l}\, k^{\{\nu_1} \cdots k^{\nu_\ell\}} \delta \hat{f}_{\rm k}$$

$\Longrightarrow$ use Boltzmann equation:

$$\delta \dot{\hat{f}}_{\rm k} = -\dot{\hat{f}}_{0\rm k} - \frac{1}{E_{{\rm k}u}} \left\{ -E_{{\rm k}l} D_l \left(\hat{f}_{0\rm k} + \delta \hat{f}_{\rm k} \right) + k^\mu \tilde{\nabla}_\mu \left(\hat{f}_{0\rm k} + \delta \hat{f}_{\rm k} \right) - C[f] \right\}$$

where: $D_l \equiv -l^\mu \partial_\mu$, $\tilde{\nabla}^\mu \equiv \Xi^{\mu\nu} \partial_\nu$

Microscopic foundations of anisotropic dissipative fluid dynamics (II)

Truncation: so far, no eigenmode analysis, only 14-moment approximation

Define $$\hat{I}_{nrq}(\hat{\alpha}, \hat{\beta}_u, \xi) \equiv \frac{1}{(2q)!!} \int_k E_{\mathrm{k}u}^n \, E_{\mathrm{k}l}^r \, (-\Xi^{\alpha\beta} k_\alpha k_\beta)^q \, \hat{f}_{0\mathrm{k}}$$

$\Longrightarrow$ the 14 moments are:

particle density	$n \equiv \hat{n} = \hat{I}_{100} \Longleftrightarrow \hat{\rho}_{10} = 0$ (1st Landau matching cond.)
particle diffusion in l^μ-direction	$n_l \equiv \hat{n}_l + \hat{\rho}_{01} = \hat{I}_{110} + \hat{\rho}_{01}$
energy density	$\epsilon \equiv \hat{\epsilon} = \hat{I}_{200} \Longleftrightarrow \hat{\rho}_{20} = 0$ (2nd Landau matching cond.)
heat flow in l^μ-direction	$M \equiv \hat{M} + \hat{\rho}_{11} = \hat{I}_{210} + \hat{\rho}_{11}$
pressure in l^μ-direction	$P_l \equiv \hat{P}_l = \hat{I}_{220} \Longleftrightarrow \hat{\rho}_{02} = 0$ (3rd Landau matching cond.)
transverse pressure	$P_\perp \equiv \hat{P}_\perp + \frac{3}{2}\Pi = \hat{I}_{201} - \frac{m^2}{2}\hat{\rho}_{00}$
particle diffusion in transverse direction	$V_\perp^\mu \equiv \hat{\rho}_{00}^\mu$
heat flow in transverse direction	$W_{\perp u}^\mu \equiv \hat{\rho}_{10}^\mu$
shear-stress current in l^μ-direction	$W_{\perp l}^\mu \equiv \hat{\rho}_{01}^\mu$
shear-stress tensor in transverse direction	$\pi_\perp^{\mu\nu} \equiv \hat{\rho}_{00}^{\mu\nu}$

$\Longrightarrow$ Landau frame: $M = W_{\perp u}^\mu = 0 \quad \Longleftrightarrow \quad \hat{\rho}_{11} = -\hat{M}\,,\ \hat{\rho}_{10}^\mu = 0$

$\Longrightarrow$ eliminate all other moments by linear relation:

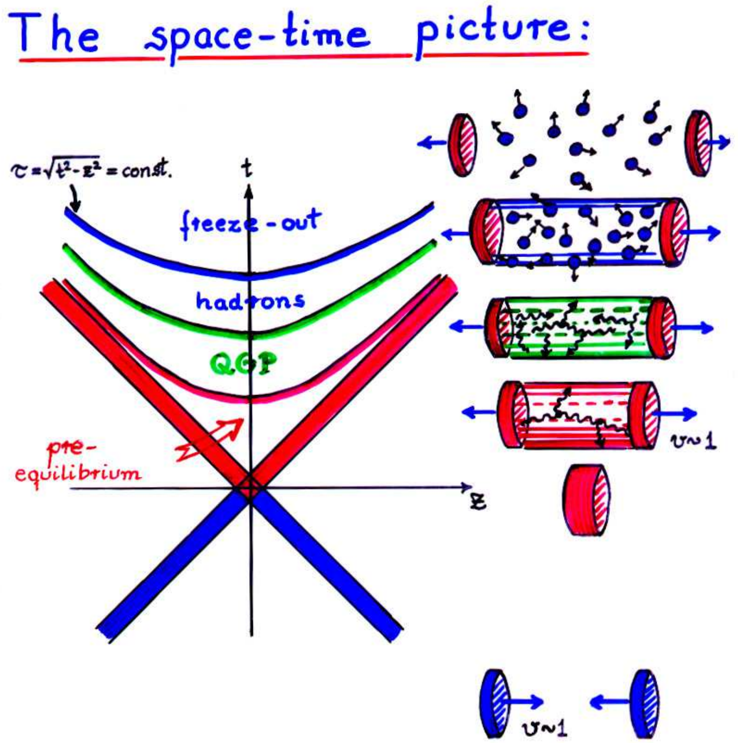
$$\hat{\rho}_{ij}^{\mu_1\cdots\mu_\ell} = (-1)^\ell \ell! \sum_{n=0}^{N_\ell} \sum_{m=0}^{N_\ell - n} \hat{\rho}_{nm}^{\mu_1\cdots\mu_\ell} \gamma_{injm}^{(\ell)}$$ where $\gamma_{injm}^{(\ell)}$ function of $\hat{\alpha}$, $\hat{\beta}_u$, ξ

for details, see E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

Application to heavy-ion collisions (I)

Bjorken flow:

J.D. Bjorken, PRD 27 (1983) 140



"Pure" anisotropic fluid dynamics

($\delta \hat{f}_\mathrm{k} \equiv 0 \iff$ all $\hat{\rho}_{rs}^{\mu_1 \cdots \mu_\ell} \equiv 0$)

$\Longrightarrow$ eqs. of motion for irreducible moments become eqs. of motion for moments $\hat{I}_{nrq}$:

$$\partial_\tau \hat{I}_{i+j,j,0} + \frac{(j+1)\hat{I}_{i+j,j,0} + (i-1)\hat{I}_{i+j,j+2,0}}{\tau} = \hat{\mathcal{C}}_{i-1,j}$$

$\Longrightarrow$ conservation equations:

$$i = 1, j = 0: \quad \partial_\tau \hat{n} + \frac{\hat{n}}{\tau} = 0$$

$$i = 2, j = 0: \quad \partial_\tau \hat{\epsilon} + \frac{\hat{\epsilon} + \hat{P}_l}{\tau} = 0$$

$\Longrightarrow$ 2 equations, 3 unknowns: $\hat{\alpha}$, $\hat{\beta}_u$, ξ

$\Longrightarrow$ need additional equation to close equations of motion!

$\Longrightarrow$ in principle, equation of motion for any moment $\hat{I}_{i+j,j,0}$ suffices, but which one is the best choice?

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Application to heavy-ion collisions (II)

assume relaxation-time approximation for collision term: $\hat{\mathcal{C}}_{i-1,j} \equiv -\dfrac{\hat{I}_{i+j,j,0} - I_{i+j,j,0}}{\tau_{\rm eq}}$

where $I_{i+j,j,0} = \lim_{\xi \to 0} \hat{I}_{i+j,j,0}$

$\Longrightarrow$ study the following choices:

(1) $i=0,\ j=2: \quad \partial_\tau \hat{P}_l + \dfrac{3\hat{P}_l - \hat{I}_{240}}{\tau} = -\dfrac{\hat{P}_l - I_{220}}{\tau_{\rm eq}}$

(2) $i=3,\ j=0: \quad \partial_\tau \hat{I}_{300} + \dfrac{\hat{I}_{300} - 2\hat{I}_{320}}{\tau} = -\dfrac{\hat{I}_{300} - I_{300}}{\tau_{\rm eq}}$

(3) $i=1,\ j=2: \quad \partial_\tau \hat{I}_{320} + \dfrac{3\hat{I}_{320}}{\tau} = -\dfrac{\hat{I}_{320} - I_{320}}{\tau_{\rm eq}}$

(4) $i=0,\ j=0: \quad \partial_\tau \hat{I}_{000} + \dfrac{\hat{I}_{000} - \hat{I}_{020}}{\tau} = -\dfrac{\hat{I}_{000} - I_{000}}{\tau_{\rm eq}}$

(5) $i=0,\ j=4: \quad \partial_\tau \hat{I}_{440} + \dfrac{5\hat{I}_{440} - \hat{I}_{460}}{\tau} = -\dfrac{\hat{I}_{440} - I_{440}}{\tau_{\rm eq}}$

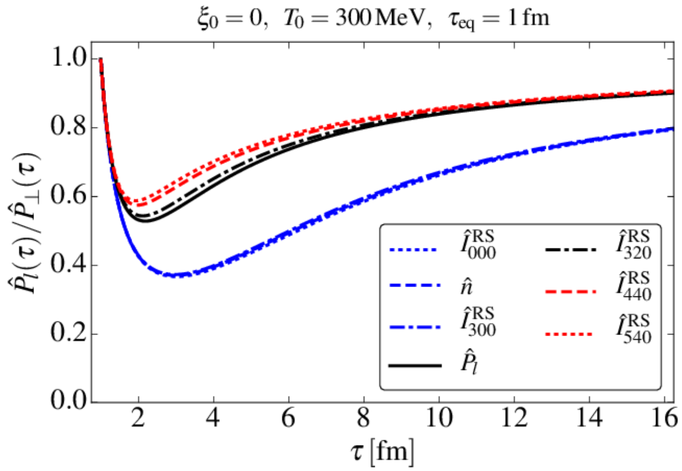
(6) $i=1,\ j=4: \quad \partial_\tau \hat{I}_{540} + \dfrac{5\hat{I}_{540}}{\tau} = -\dfrac{\hat{I}_{540} - I_{540}}{\tau_{\rm eq}}$

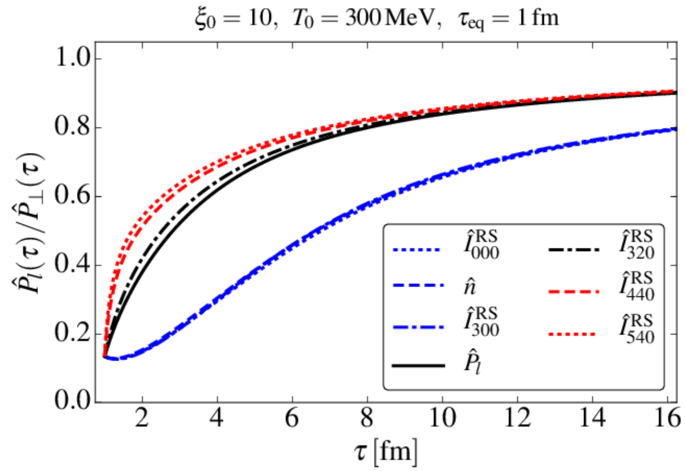
(7) in case particle number is not conserved: $i=1,\ j=0: \quad \partial_\tau \hat{n} + \dfrac{\hat{n}}{\tau} = -\dfrac{\hat{n} - I_{100}}{\tau_{\rm eq}}$

Note: different moments probe $\hat{f}_{0\rm k}$ in different regions of momentum space!

Application to heavy-ion collisions (III)

no particle number conservation:

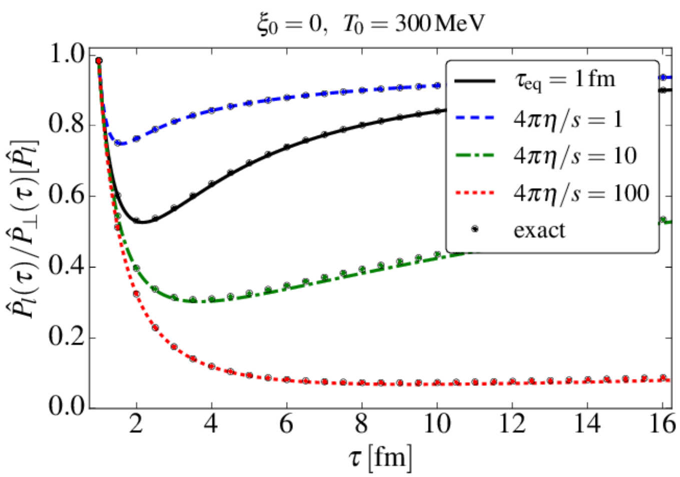


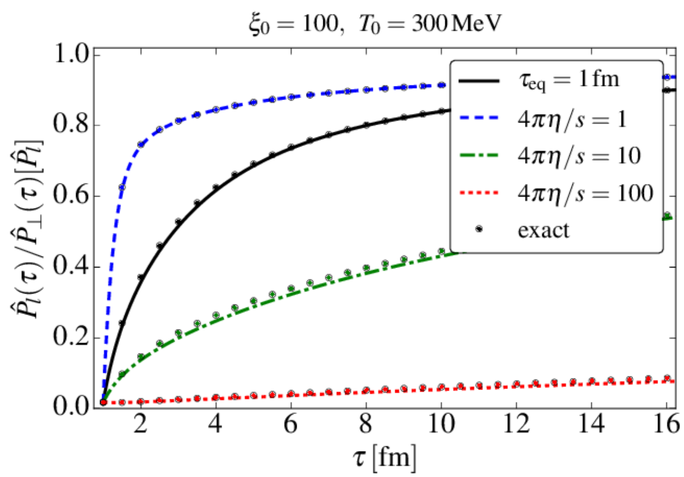


⟹ all cases (1) – (7) give different results! ⟹ which one is the best?

⟹ comparison of case (1) to solution of Boltzmann equation

W. Florkowski, R. Ryblewski, M. Strickland, PRC 88 (2013) 024903

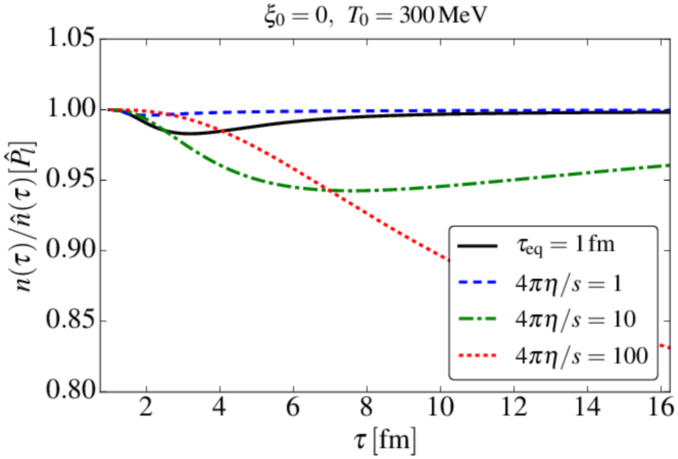


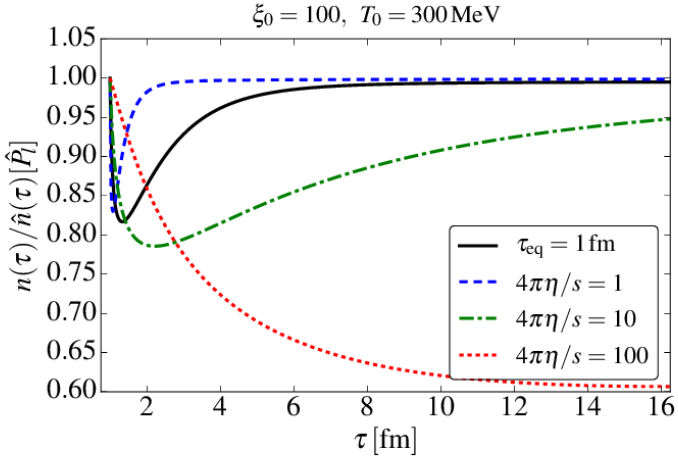


⟹ relaxation equation for $\hat{P}_l$ gives best match to solution of Boltzmann eq.!

Application to heavy-ion collisions (V)

However: other moments not necessarily also agree well with Boltzmann equation





Conclusions and Outlook

1. **Derivation of equations of motion of anisotropic dissipative fluid dynamics from Boltzmann equation**

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   $\Longrightarrow$ **still need to do eigenmode analysis!**

2. **Closure of equations of motion of "pure" anisotropic fluid dynamics**

   $\Longrightarrow$ **best agreement to solution of Boltzmann equation provided by $\hat{P}_l$**

   but: not all moments agree with solution of Boltzmann equation

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   $\Longrightarrow$ **need to improve $\hat{f}_{0\mathbf{k}}$?!**