Critical nucleus charge in a superstrong magnetic field

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Outline

$\hbar = c = 1$
$e^2 = \alpha = 1/137.0..$

- Critical nucleus charge in the absence of external fields
- Critical nucleus charge in a magnetic field
- Superstrong magnetic field: modification of the Coulomb potential
- Critical nucleus charge in a superstrong magnetic field
- Positron scattering on a supercritical nucleus

Critical charge in the absence of a magnetic field

Pointlike Coulomb center with charge $Z > 137$ can not exist.

But if the finite size of a nucleus is taken into account then Dirac equation can be solved for $Z > 137$. At $Z = Z_c \sim 170$ ground energy level reaches lower continuum ($\varepsilon = -m_e$) and spontaneous electron-positron pair production occurs. Electrons occupy ground level and one can observe two free positrons.

Since the value of the critical charge is too large, it seems impossible to produce such a nucleus for a sufficiently long period of time. There were searches for this effect in heavy ion collisions, but *no evidence so far*. (However, there are some hints from graphene physics).

Critical charge in a magnetic field

V.N. Oraevskii, A.I. Rex, V.B. Semikoz, JETP, Vol. 45, No 3, p. 428 (March 1977)

Analytical results for a strong magnetic fields: $B \gg m^2 e^3 Z^2$; $B \gg \frac{m^2}{e(Ze^2)^2}$

Critical field:

$$\frac{B}{B_0} = 2(Z_{cr}e^2)^2 \exp\left(-\gamma + \frac{\pi - 2\arg\Gamma(1 + 2iZ_{cr}e^2)}{Z_{cr}e^2}\right).$$

Here $B_0 = m^2/e \approx 4.4 \cdot 10^{13}$ Gauss.

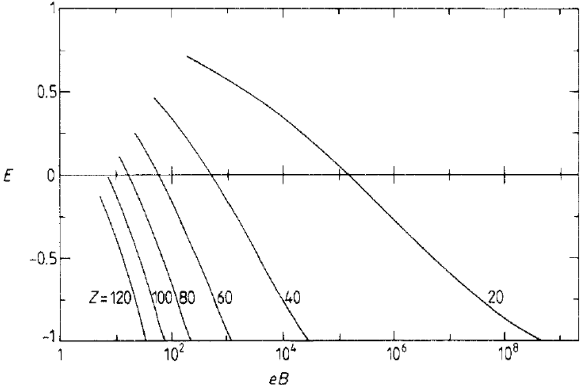
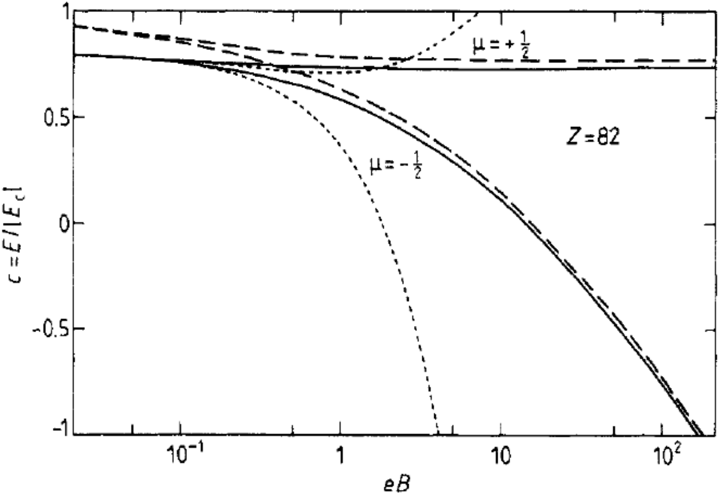


Fig. 1 from paper by *P. Schlüter, G. Soff, K.-H. Wietschorke, W. Greiner, J. Phys. B: At. Mol. Phys.* 18 ***(1985)***

Critical charge in a magnetic field

P. Schlüter, G. Soff, K.-H. Wietschorke, W. Greiner, J. Phys. B: At. Mol. Phys. 18 *(1985)*

Exact numerical results for a wide range of magnetic fields



Influence of magnetic field on the Coulomb potential

- $a_B \sim a_H$: $\quad B_a = m^2 e^3 \approx 2.4 \times 10^9$ G

$$U(r) = -\frac{e^2}{r} + \ldots$$

- $\hbar\omega_B \sim mc^2$: $\quad B_0 = \frac{m^2}{e} \approx 4.4 \times 10^{13}$ G

$$U(r) = -\frac{e^2}{r} + \ldots$$

- $B = \frac{m^2}{e^3} \approx 6 \times 10^{15}$ G

  $U(r)$ changes

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- $\hbar\omega_B \sim mc^2$: $B_0 = \frac{m^2}{e} \approx 4.4 \times 10^{13}$ G $= B_0$

  Schwinger field

  $$U(r) = -\frac{e^2}{r} + \ldots$$

- $B = \frac{m^2}{e^3} \approx 6 \times 10^{15}$ G – superstrong magnetic field

  $U(r)$ changes $U(r) = ?$

Coulomb potential is modified due to the enchancement of the vacuum polarization at one loop:

$$\Phi\left(\rho, z\right) = 4\pi e \int \frac{d^2 k_\perp dk_\parallel}{\left(2\pi\right)^3} \frac{e^{-i\vec{k}_\perp \vec{\rho}} e^{-ik_\parallel z}}{k_\parallel^2 + k_\perp^2 - \Pi^{(2)}\left(k_\perp, k_\parallel\right)}$$

Polarization operator:

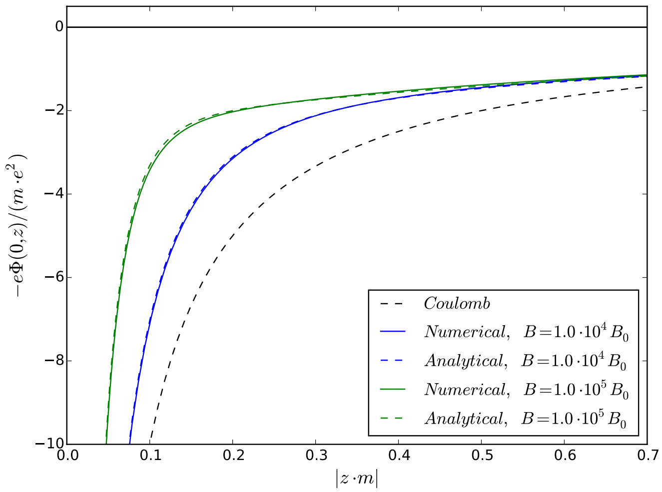
$$\Pi^{(2)}\left(k_\perp, k_\parallel\right) = -\frac{2e^3 B}{\pi} \exp\left(-\frac{k_\perp^2}{2eB}\right) T\left(t\right),$$

$$T(t) = 1 - \frac{1}{\sqrt{t(1+t)}} \log\left(\sqrt{1+t} + \sqrt{t}\right), \quad t \equiv k_\parallel^2 / 4m^2.$$

- One–loop calculation
- Lowest Landau Level

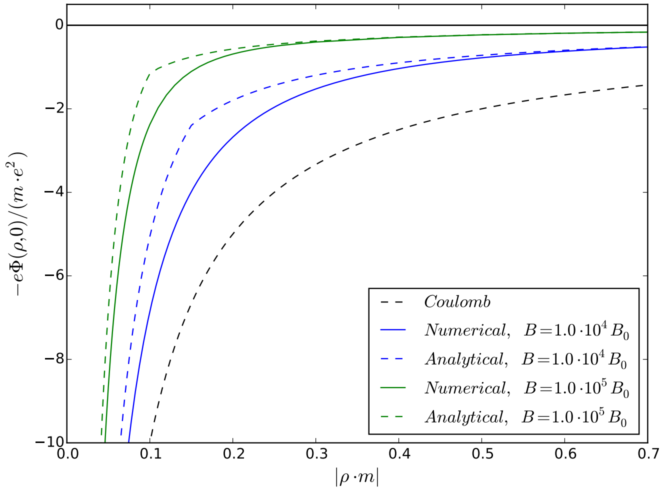
Potential for $(0, z)$

$$\Phi(0,z) = \frac{e}{|z|}\left(1 - e^{-|z|\sqrt{6m^2}} + e^{-|z|\sqrt{(2/\pi)e^3 B + 6m^2}}\right)$$

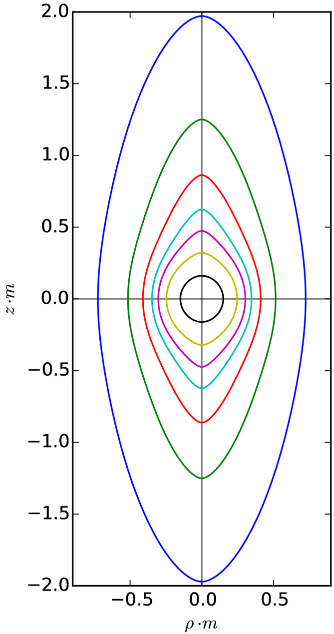


$$\Phi(\rho,0) = \begin{cases} \dfrac{e}{\rho} \cdot \exp\left(-\rho\sqrt{(2/\pi)e^3 B}\right), & \rho \ll l_0, \\ \dfrac{e}{\rho} \cdot \sqrt{\dfrac{3\pi m^2}{e^3 B}}, & \rho \gg l_0, \end{cases}$$

where $l_0 \equiv \sqrt{\dfrac{\pi}{2e^3 B}} \ln \sqrt{\dfrac{e^3 B}{3\pi m^2}}$



Potential structure for $B = 10^4 B_0$



Details in JETP Lett. 105, 147 (2017) (in collaboration with S.I. Glazyrin)

Analytic:

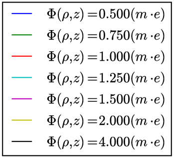
- Small distances: Yukawa screening

$$\Phi(\rho, z) = \frac{e}{r} \cdot e^{-r\sqrt{2e^3 B/\pi}},$$

$$r = \sqrt{\rho^2 + z^2}$$

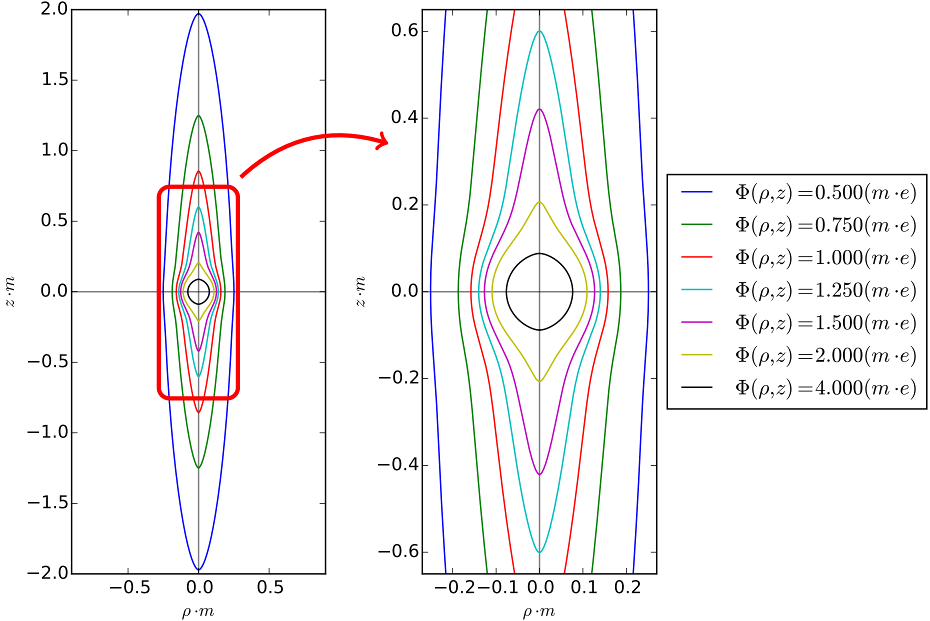
- Large distances: ellipses

$$\Phi\left(\rho, z\right) = \frac{e}{\sqrt{z^2 + \rho^2\left(1 + \frac{e^3 B}{3\pi m^2}\right)}}$$



Shabad, Usov (2007), (2008)
Vysotsky (2010)
Machet, Vysotsky (2011)
Godunov, Machet, Vysotsky (2012)

Internal structure for $B = 10^5 B_0$

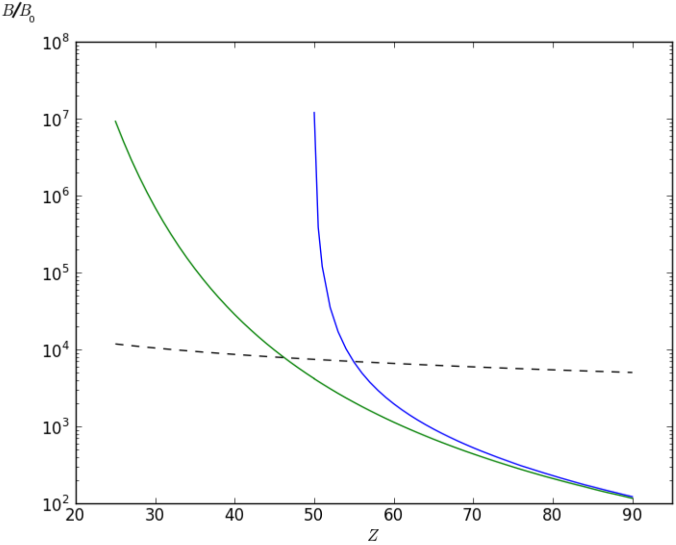


Results for $Z = 40$

$\frac{B}{B_0}$	ε/m (ORS equation)	ε/m (Numerical results)	ε/m (Numerical results with the account of screening)
10^0	0.819	0.850	0.850
10^1	0.653	0.667	0.667
10^2	0.336	0.339	0.346
10^3	-0.158	-0.159	-0.0765
10^4	-0.758	-0.759	-0.376
$2 \cdot 10^4$	-0.926	-0.927	-0.423
$\ldots$	at $B/B_0 \approx 2.85 \cdot 10^4,\ \varepsilon = -m$		$\ldots$
10^5	—	—	-0.4887
10^6	—	—	-0.5241
10^7	—	—	-0.5351
10^8	—	—	-0.5386
10^9	—	—	-0.5397

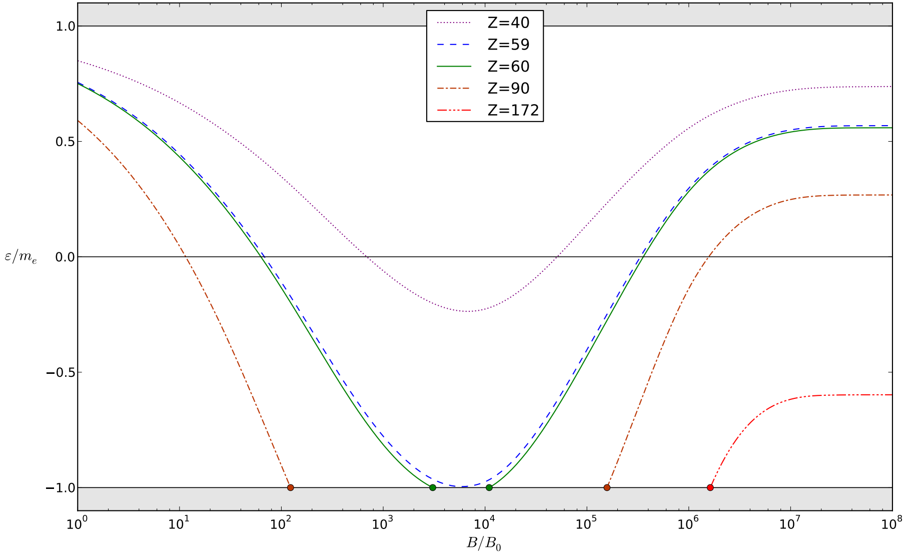
Critical charge with the account of screening

Details in Phys.Rev.D 85, 044058 (2012)



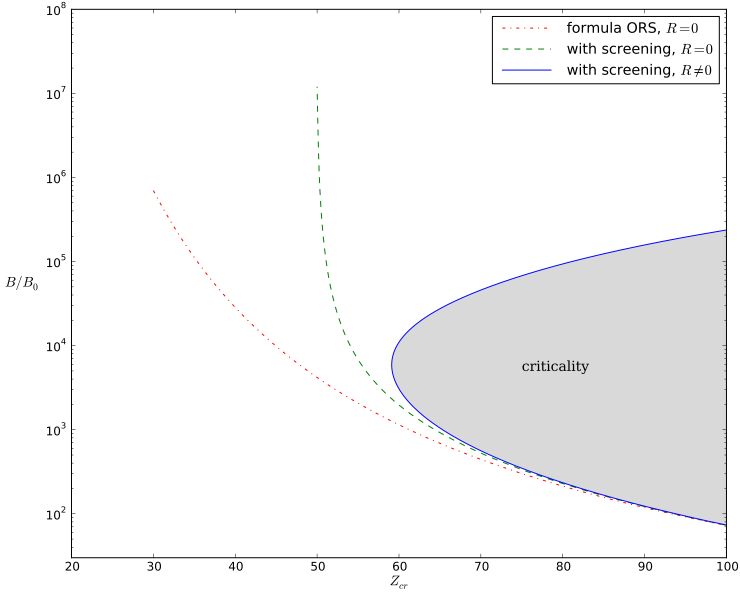
Ground energy level for various Z

[Details in Phys.Rev.D 87, 124035 (2013)]



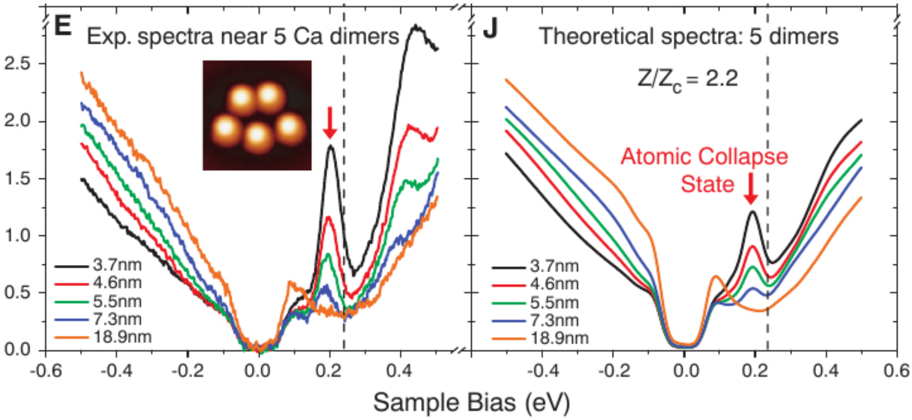
Critical nucleus charge with the account of finite nucleus radius

Details in Phys.Rev.D 87, 124035 (2013)



Critical charge in graphene

Details in Y. Wang *et al*, Science 10, 734 (2013)



The radial functions of the Dirac equation $F(r) \equiv rf(r)$ and $G(r) \equiv rg(r)$ are determined by the following differential equations:

$$\begin{cases} \dfrac{dF}{dr} + \dfrac{\varkappa}{r}F - (\varepsilon + m - V(r))\, G = 0, \\ \dfrac{dG}{dr} - \dfrac{\varkappa}{r}G + (\varepsilon - m - V(r))\, F = 0, \end{cases}$$

where $\varkappa = -(j + 1/2) = -1, -2, \ldots$ for $j = l + 1/2$ and $\varkappa = (j + 1/2) = 1, 2, 3 \ldots$ for $j = l - 1/2$ and the ground state corresponds to $\varkappa = -1$.

In order to deal with the case $Z\alpha > 1$ the Coulomb potential should be regularised at $r = 0$:

$$V(r) = \begin{cases} -\dfrac{Z\alpha}{R}, & r < R, \\ -\dfrac{Z\alpha}{r}, & r > R. \end{cases}$$

Solution of the Dirac equation

Solution at $r < R$:

$$\begin{pmatrix} F \\ G \end{pmatrix} = \text{const} \cdot \sqrt{\beta r} \cdot \begin{pmatrix} \mp J_{\mp(1/2+\varkappa)}(\beta r) \\ J_{\pm(1/2-\varkappa)}(\beta r) \frac{\beta}{\varepsilon + m + \frac{Z\alpha}{R}} \end{pmatrix},$$

where $\beta = \sqrt{(\varepsilon + Z\alpha/R)^2 - m^2}$. Upper (lower) signs correspond to $\varkappa < 0$ ($\varkappa > 0$).

Solution at $r > R$:

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} \sqrt{m+\varepsilon} \\ -\sqrt{m-\varepsilon} \end{pmatrix} \exp(-\rho/2)\rho^{i\tau} \begin{pmatrix} Q_1 + Q_2 \\ Q_1 - Q_2 \end{pmatrix}, \quad (2)$$

where $\tau = \sqrt{(Z\alpha)^2 - \varkappa^2}$, $\rho = 2\lambda r = -2ikr$, and

$$\begin{cases} Q_1 = C \cdot \dfrac{-\frac{iZ\alpha m}{k} + \varkappa}{-i\tau + \frac{iZ\alpha\varepsilon}{k}} \cdot {}_1F_1\left(i\tau - \dfrac{iZ\alpha\varepsilon}{k}, 2i\tau + 1, \rho\right) + \\ \qquad + D \cdot \dfrac{-\frac{iZ\alpha m}{k} + \varkappa}{i\tau + \frac{iZ\alpha\varepsilon}{k}} \rho^{-2i\tau} {}_1F_1\left(-i\tau - \dfrac{iZ\alpha\varepsilon}{k}, -2i\tau + 1, \rho\right), \\ Q_2 = C \cdot {}_1F_1\left(1 + i\tau - \dfrac{iZ\alpha\varepsilon}{k}, 2i\tau + 1, \rho\right) + D\rho^{-2i\tau} {}_1F_1\left(1 - i\tau - \dfrac{iZ\alpha\varepsilon}{k}, -2i\tau + 1, \rho\right), \end{cases}$$

where C and D are arbitrary coefficients. The ratio C/D should be determined by matching with the solution at $r < R$.

The scattering phase $\delta_\varkappa(\varepsilon, Z)$ is determined by investigating the behavior of the wave function at large r:

$$e^{2i\delta_\varkappa} = -\frac{1}{\varkappa + \frac{iZ\alpha m}{k}} \cdot \frac{\frac{C}{D} \cdot \frac{\Gamma(2i\tau)}{\Gamma\left(i\tau + \frac{iZ\alpha\varepsilon}{k}\right)} \rho^{i\tau} \left(-\rho\right)^{-i\tau} - \frac{\Gamma(-2i\tau)}{\Gamma\left(-i\tau + \frac{iZ\alpha\varepsilon}{k}\right)} \rho^{-i\tau} \left(-\rho\right)^{i\tau}}{\frac{C}{D} \cdot \frac{\Gamma(2i\tau)}{\Gamma\left(1 + i\tau - \frac{iZ\alpha\varepsilon}{k}\right)} - \frac{\Gamma(-2i\tau)}{\Gamma\left(1 - i\tau - \frac{iZ\alpha\varepsilon}{k}\right)}},$$

The resonance of the scattering amplitude corresponds to the pole of the S-matrix element $S \equiv e^{2i\delta}$, so we immediately get an equation for the position of this pole in the ε-plane:

$$\frac{C}{D} \cdot \frac{\Gamma\left(2i\tau\right)}{\Gamma\left(1 + i\tau - \frac{iZ\alpha\varepsilon}{k}\right)} - \frac{\Gamma\left(-2i\tau\right)}{\Gamma\left(1 - i\tau - \frac{iZ\alpha\varepsilon}{k}\right)} = 0.$$

Approximate results: V. M. Kuleshov *et al*, Phys. Usp. 58, 785 (2015), [Usp. Fiz. Nauk 185, 845 (2015)].

Exact results: SG, B. Machet, M.I. Vysotsky, arXiv:1707.07497

$$\frac{C}{D} = -\rho_0^{-2i\tau} \cdot \frac{F_g^- - MF_f^-}{F_g^+ - MF_f^+},$$

where

$$M = \pm\frac{\sqrt{m+\varepsilon}}{\sqrt{m-\varepsilon}} \cdot \frac{J_{\pm(1/2-\varkappa)}(\beta R)}{J_{\mp(1/2+\varkappa)}(\beta R)} \cdot \frac{\beta}{\varepsilon + m + \frac{Z\alpha}{R}},$$

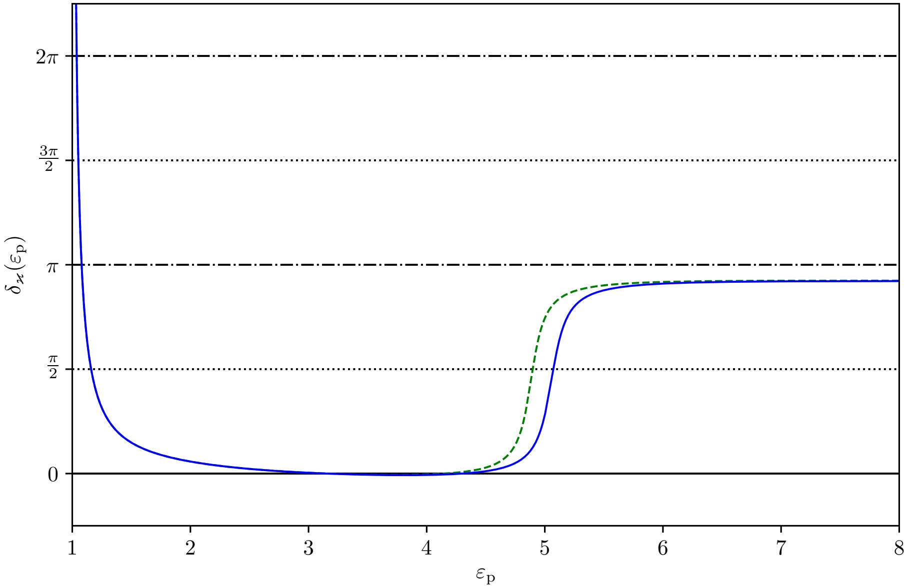
$$F_f^\pm = {}_1F_1\left(\alpha_1^\pm, \gamma^\pm, \rho_0\right) \frac{\frac{iZ\alpha m}{k} - \varkappa}{\alpha_1^\pm} + {}_1F_1\left(\alpha_2^\pm, \gamma^\pm, \rho_0\right),$$

$$F_g^\pm = {}_1F_1\left(\alpha_1^\pm, \gamma^\pm, \rho_0\right) \frac{\frac{iZ\alpha m}{k} - \varkappa}{\alpha_1^\pm} - {}_1F_1\left(\alpha_2^\pm, \gamma^\pm, \rho_0\right),$$

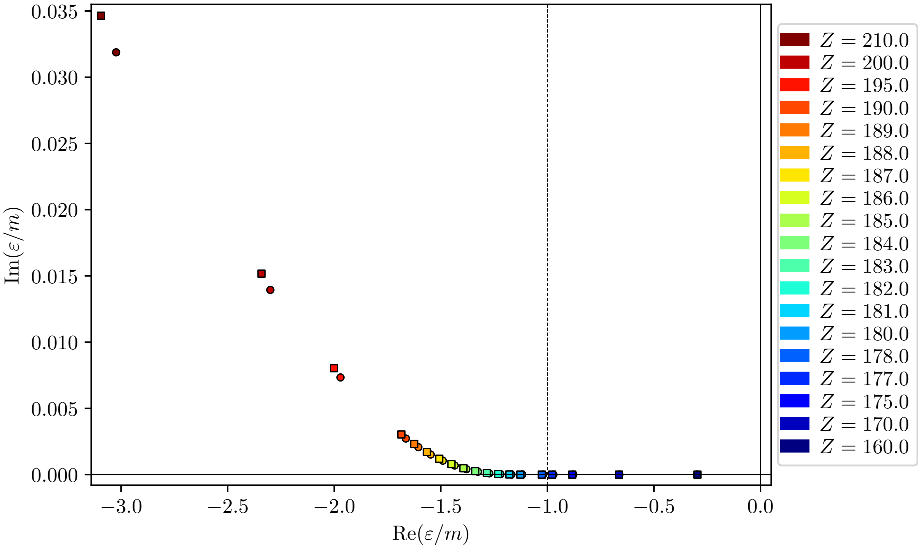
and

$$\alpha_1^\pm = \pm i\tau - \frac{iZ\alpha\varepsilon}{k}, \alpha_2^\pm = 1 \pm i\tau - \frac{iZ\alpha\varepsilon}{k}, \gamma^\pm = \pm 2i\tau + 1, \rho_0 = -2ikR.$$

Scattering phase for $Z = 232$



Position of the poles



Position of the poles

Z	$\mathrm{Re}\,(\varepsilon_{\mathrm{appr}})$	$\mathrm{Im}\,(\varepsilon_{\mathrm{appr}})$	$\mathrm{Re}\,(\varepsilon)$	$\mathrm{Im}\,(\varepsilon)$
160	-0.297	0	-0.296	0
170	-0.662	0	-0.664	0
175	-0.879	0	-0.883	0
177	-0.972	0	-0.978	0
178	-1.020	0	-1.026	0
180	-1.118	5.375e-07	-1.127	9.229e-07
181	-1.169	6.644e-06	-1.178	9.475e-06
182	-1.220	3.198e-05	-1.231	4.168e-05
183	-1.273	9.562e-05	-1.284	1.183e-04
184	-1.326	2.164e-04	-1.338	2.591e-04
185	-1.380	4.097e-04	-1.394	4.794e-04
186	-1.435	6.863e-04	-1.450	7.903e-04
187	-1.491	1.053e-03	-1.507	1.198e-03
188	-1.548	1.515e-03	-1.565	1.707e-03
189	-1.605	2.071e-03	-1.625	2.318e-03
190	-1.664	2.723e-03	-1.685	3.030e-03
195	-1.971	7.335e-03	-2.001	8.031e-03
200	-2.300	1.394e-02	-2.341	1.517e-02
210	-3.023	3.188e-02	-3.094	3.464e-02
232	-4.885	8.773e-02	-5.057	9.638e-02

Conclusions

- Superstrong magnetic field $B > m^2/e^3$ modifies the Coulomb potential (it becomes screened)
- We numerically calculated the modified potential in all space with high precision
- The screening of the Coulomb potential leads to the freezing of atomic energy levels. As a result, ions with $Z < 50$ cannot reach criticality regardless of the magnetic field strength.
- Finite nucleus radius leads to nontrivial dependence of the critical nucleus charge on magnetic field: ions are critical only in the finite range of magnetic fields. Only ions with $Z > 210$ remain critical regardless of the value of magnetic field.
- The scattering of positrons on a supercritical nucleus was considered. The exact expressions for the scattering phase and the position of the pole were obtained.

References

1. M.I. Vysotsky, JETP Lett. 92, 15 (2010).
2. B. Machet, M.I. Vysotsky, Phys. Rev. D 83, 025022 (2011).
3. S.I. Godunov, B. Machet, M.I. Vysotsky, Phys.Rev.D 85, 044058 (2012).
4. S.I. Godunov, M.I. Vysotsky, Phys.Rev.D 87, 124035 (2013).
5. S.I. Godunov, Phys. Atom. Nucl. 76, 901 (2013).
6. S.I. Glazyrin, S.I. Godunov, JETP Lett. 105, 147 (2017).
7. S.I. Godunov, B. Machet, M.I. Vysotsky, arXiv:1707.07497.