Temperature dependence of bulk and shear viscosities from lattice $SU(3)$–gluodynamics

N. Astrakhantsev, V. Braguta, A. Kotov

& new results

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Outline

- Introduction
- Details of the calculation
- Shear viscosity
  - Fitting of the data
  - Backus-Gilbert method
- Bulk viscosity
  - Middle point method
  - Backus-Gilbert method
- Conclusion
Relativistic Hydrodynamics

- \( T^{\mu\nu} = (e + p)u^\mu u^\nu + p g^{\mu\nu} + (\eta \nabla^{\langle \mu} u^{\nu \rangle} + \zeta \Delta^{\mu\nu} \nabla_\alpha u^\alpha) + \ldots \)
- \( \nabla^\alpha = \Delta^{\alpha\nu} \partial_\nu, \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \)
- \( \nabla^{\langle \mu} u^{\nu \rangle} = \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla_\alpha u^\alpha \)

- EOM \( \partial_\mu T^{\mu\nu} = 0 \)

- Non-relativistic limit \( (u^\mu = (1, \vec{v})) \)
  - Continuity equation: \( \partial_t \rho + \rho (\vec{\partial} \vec{v}) + \vec{v} \vec{\partial} \rho = 0 \)
  - Navier–Stokes equation: \( \frac{\partial v_i}{\partial t} + v^k \frac{\partial v_i}{\partial x^k} = -\frac{1}{\rho} \frac{\partial p}{\partial x^i} - \frac{1}{\rho} \frac{\partial \Pi_{ki}}{\partial x^i} \)
  - Viscous stress tensor: \( \Pi^{ik} = -\eta \left( \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} - \frac{2}{3} \delta^{ik} \frac{\partial v^l}{\partial x^l} \right) - \zeta \delta^{ik} \frac{\partial v^l}{\partial x^l} \)

- \( \eta \) — shear viscosity, \( \zeta \) — bulk viscosity
Elliptic flow from STAR experiment (Nucl. Phys. A 757, 102 (2005))

\[ \frac{dN}{d\phi} \sim (1 + 2v_1 \cos(\phi) + 2v_2 \cos^2(\phi)), \phi - \text{scattering angle} \]

Quark-gluon plasma is close to ideal liquid \( (\frac{\eta}{s} = (1 - 3) \frac{1}{4\pi}) \)

QM2017 (talk by S. Bass)

Temperature Dependence of Shear & Bulk Viscosities

**Temperature dependent shear viscosity:**
- Analysis favors small value and shallow rise
- Results do not fully constrain temperature dependence:
  - Inverse correlation between \((\eta/s)_{\text{slope}}\) slope and intercept \((\eta/s)_{\text{min}}\)
  - Insufficient data to obtain sharply peaked likelihood distributions for \((\eta/s)_{\text{slope}}\) and curvature \(\beta\) independently
- Current analysis most sensitive to \(T < 0.23\) GeV
  - RHIC data may disambiguate further

\[
\eta/s(T) = (\eta/s)_{\text{min}} + (\eta/s)_{\text{slope}} \times (T-T_C) \times (T/T_C)^\beta
\]

**Temperature dependent bulk viscosity:**
- Setup of analysis allows for vanishing value of bulk viscosity
- Significant non-zero value at \(T_C\) favored, confirming the presence / need for bulk viscosity
- Either high sharp peak or broad & shallow temperature dependence
caveat of current analysis.
- Bulk-viscous corrections are implemented using relaxation-time approximation & regulated to prevent negative particle densities
Shear viscosity in two limits

\[ \eta \propto \frac{f_\pi^4}{T^4} \]

\[ \alpha_s < 0.1 \]

\[ 0.1 < \alpha_s < 0.2 \]

\[ \text{pion gas} \]

\[ \alpha_s^2 \ln \alpha_s^{-1} \]

(Prakash, Prakash, Venugopalan, Welke; Dobado, Llanes-Estrada; Csernai, Kapusta, McLerran; JWC, Nakano)

(Arnold, Moore, Yaffe)
Our goal

First-principle determination of shear and bulk viscosities!
Lattice calculation of shear & bulk viscosity

The first step:
Measurement of the correlation functions:

\[ C_{sh}(t) = \int d^3 \vec{x} \langle T_{12}(t, \vec{x}) T_{12}(0) \rangle \]

\[ C_b(t) = \int d^3 \vec{x} \langle T_{\mu \mu}(t, \vec{x}) T_{\nu \nu}(0) \rangle \]
Lattice calculation of shear & bulk viscosity

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The second step (analytical continuation):
Calculation of the spectral function \( \rho(\omega) \):

\[ C(t) = \int_0^\infty d\omega \rho(\omega) \frac{\cosh(\frac{\omega}{2T} - \omega t)}{\sinh(\frac{\omega}{2T})} \]

\[ \eta = \pi \lim_{\omega \to 0} \frac{\rho_{sh}(\omega)}{\omega} \]

\[ \zeta = \frac{\pi}{9} \lim_{\omega \to 0} \frac{\rho_b(\omega)}{\omega} \]
Details of the calculation

- $SU(3)$–gluodynamics
- Two-level algorithm (only for gluodynamics)
- Lattice size $32^3 \times 16$
- Temperatures $T/T_c = 0.9, 0.925, 0.95, 1.0, 1.2, 1.35, 1.5$
- Accuracy $\sim 2 - 3\%$ at $t = \frac{1}{2T}$
- For $\langle T_{12}(x)T_{12}(y) \rangle \sim (\langle T_{11}(x)T_{11}(y) \rangle - \langle T_{11}(x)T_{22}(y) \rangle)$
- Clover discretization for the $\hat{F}_{\mu\nu}$
Shear viscosity
Correlation functions (shear viscosity)

\[ C(\tau) / T^5 \]

\[ T/T_c = 0.9 \]
\[ T/T_c = 1.0 \]
\[ T/T_c = 1.2 \]
\[ T/T_c = 1.5 \]
Spectral function

\[ C_{sh}(t) = \int_0^\infty d\omega \rho_{sh}(\omega) \frac{\cosh\left(\frac{\omega}{2T} - \omega t\right)}{\sinh\left(\frac{\omega}{2T}\right)} \]

Properties of the spectral function:

▶ \( \rho(\omega) \geq 0, \rho(-\omega) = -\rho(\omega) \)

▶ Asymptotic freedom: \( \rho(\omega)\big|_{\omega \to \infty} \sim \frac{1}{10} \frac{d_A}{(4\pi)^2} \omega^4 \left(1 - \frac{5N_c \alpha_s}{9\pi}\right) \)

\( \sim 90\% \) of the total contribution \( t = \frac{1}{2T} \)

▶ Hydrodynamics: \( \rho(\omega)\big|_{\omega \to 0} = \frac{\eta}{\pi} \omega \)
Spectral function

\[ C_{sh}(t) = \int_0^\infty d\omega \rho_{sh}(\omega) \frac{\cosh\left(\frac{\omega}{2T} - \omega t\right)}{\sinh\left(\frac{\omega}{2T}\right)} \]

Properties of the spectral function:

▶ \( \rho(\omega) \geq 0, \rho(-\omega) = -\rho(\omega) \)

▶ Asymptotic freedom: \( \rho(\omega)\big|_{\omega \to \infty}^{NLO} = \frac{1}{10} \frac{dA}{(4\pi)^2} \omega^4 \left(1 - \frac{5N_c \alpha_s}{9\pi}\right) \)

\( \sim 90\% \) of the total contribution \( t = 1/(2T) \)

▶ Hydrodynamics: \( \rho(\omega)\big|_{\omega \to 0} = \frac{\eta}{\pi} \omega \)

Ansatz for the spectral function (QCD sum rules motivation)

\[ \rho(\omega) = \frac{\eta}{\pi} \omega \theta(\omega_0 - \omega) + A\rho_{lat}(\omega) \theta(\omega - \omega_0) \]
Lattice spectral function
Properties of the spectral function

- Hydrodynamical approximation works well up to \( \omega < \pi T \sim 1 \text{ GeV} \) (H.B. Meyer, arXiv:0809.5202)
Properties of the spectral function

- Hydrodynamical approximation works well up to $\omega < \pi T \sim 1$ GeV (H.B. Meyer, arXiv:0809.5202)

- Asymptotic freedom works well from $\omega > 3$ GeV
Properties of the spectral function

- Hydrodynamical approximation works well up to $\omega < \pi T \sim 1 \text{ GeV}$ (H.B. Meyer, arXiv:0809.5202)

- Asymptotic freedom works well from $\omega > 3 \text{ GeV}$

- Poor knowledge of the spectral function in the region $\omega \in (1, 3) \text{ GeV}$
  $\Rightarrow$ Main source of uncertainty in the fitting procedure
Backus-Gilbert method for the spectral function

- Problem: find $\rho(\omega)$ from the integral equation

$$C(x_i) = \int_0^\infty d\omega \rho(\omega) K(x_i, \omega), \quad K(x_i, \omega) = \frac{\cosh\left(\frac{\omega}{2T} - \omega x_i\right)}{\sinh\left(\frac{\omega}{2T}\right)}$$

- Define an estimator $\tilde{\rho}(\bar{\omega})$ ($\delta(\bar{\omega}, \omega)$ – resolution function):

$$\tilde{\rho}(\bar{\omega}) = \int_0^\infty d\omega \hat{\delta}(\bar{\omega}, \omega) \rho(\omega)$$

- Let us expand $\delta(\bar{\omega}, \omega)$ as

$$\delta(\bar{\omega}, \omega) = \sum_i b_i(\bar{\omega}) K(x_i, \omega) \quad \tilde{\rho}(\bar{\omega}) = \sum_i b_i(\bar{\omega}) C(x_i)$$

- Goal: minimize the width of the resolution function

$$b_i(\bar{\omega}) = \frac{\sum_j W_{ij}^{-1} R_j}{\sum_{ij} R_i W_{ij}^{-1} R_j},$$

$$W_{ij} = \int d\omega K(x_i, \omega)(\omega - \bar{\omega})^2 K(x_j, \omega), \quad R_i = \int d\omega K(x_i, \omega)$$

- Regularization by the covariance matrix $S_{ij}$:

$$W_{ij} \to \lambda W_{ij} + (1 - \lambda) S_{ij}, \quad 0 < \lambda < 1$$
Resolution function $\delta(0, \omega) \ (T/T_c = 1.35)$

- Width of the resolution function $\omega/T \sim 4$
- Hydrodynamical approximation works up to $\omega/T < \pi$
- Problem: large contribution from ultraviolet tail ($\sim 50\%$)
- Solution: UV contribution can be subtracted as we know UV part from the fitting procedure quite well
Results
Bulk viscosity

Preliminary results
Bulk viscosity in two limits

\[ \zeta = \frac{1}{9\omega_0} \left( T^5 \frac{\partial}{\partial T} \frac{e^{-3p}}{T^4} + 16\epsilon_v \right) \]

D. Kharzeev, K. Tuchin, JHEP 0809 (2008) 093,
Previous lattice works ($SU(3)$–gluodynamics)

Correlation functions (bulk viscosity)

\[ \frac{C(\tau)}{T^5} = \begin{cases} 
0.9 & \text{for } T/T_c = 0.9 \\
1.1 & \text{for } T/T_c = 1.1 \\
1.35 & \text{for } T/T_c = 1.35 \\
1.5 & \text{for } T/T_c = 1.5 
\end{cases} \]
Correlation functions (shear & bulk viscosity)

Shear

Bulk
Spectral function

\[ C_b(t) = \int_0^\infty d\omega \rho_b(\omega) \frac{\cosh(\frac{\omega}{2T} - \omega t)}{\sinh(\frac{\omega}{2T})} \]

Properties of the spectral function:

- \( \rho(\omega) \geq 0, \rho(-\omega) = -\rho(\omega) \)

- Asymptotic freedom: \( \rho(\omega)|_{\omega \to \infty}^{\text{NLO}} = d_A \left( \frac{11 \alpha_s}{(4\pi)^2} \right)^2 \omega^4 \)
  compare with shear channel \( \sim d_A \frac{1}{10(4\pi)^2} \omega^4 \)

- Hydrodynamics: \( \rho(\omega)|_{\omega \to 0} = \frac{9}{\pi} \zeta \omega \)
Perturbative part vs hydrodynamical part

- In the region $\tau/a \sim \frac{\beta}{2}$, hydrodynamics is dominant.
- In the region $\tau/a \sim \text{few}$, perturbative contribution is dominant.
Hydrodynamical approximation

\[
C_h(\tau) = \int_0^\infty d\omega \rho_h(\omega) \frac{\cosh\left(\frac{\omega}{2T} - \omega\tau\right)}{\sinh\left(\frac{\omega}{2T}\right)}, \quad \rho_h(\omega) = \frac{9}{\pi} \zeta \omega \theta(\omega_0 - \omega)
\]
Middle point estimation of bulk viscosity

\[ C(\tau)/C_h(\tau) \]

- In the vicinity of the phase transition hydrodynamics works very well!
- \[ C_h(\beta/2) = \frac{9}{\pi} \zeta \int_0^{\omega_0} d\omega \frac{\omega}{\sinh(\frac{\omega}{2T})}. \]
Backus-Gilbert method

Resolution function $\delta(0, \omega) \ (T/T_c = 1.5, \ \lambda = 0.1)$

▶ Width of the resolution function $\omega/T \sim 5$
Bulk viscosity $\zeta/T^4$ vs $T$ (preliminary!)
Bulk viscosity $\zeta/s$ vs $T$ (preliminary!)
Comparison with other approaches

- Agreement with other lattice studies
- Large deviation from perturbative results
Is QGP weakly or strongly coupled?

- Weakly coupled system $\zeta/\eta \sim (1 - 3v_s^2)^2 \ (\chi^2/dof \sim 4)$
- Strongly coupled system $\zeta/\eta \sim (1 - 3v_s^2) \ (\chi^2/dof \sim 1)$
- $\zeta/\eta \geq \frac{2}{3}(1 - 3v_s^2)$ \footnote{A. Buchel, Physics Letters B663, 286 (2008)}
Results and Conclusions

We calculated $\eta/s$ & $\zeta/s$ for set of temperatures $T/T_c \in (0.9, 1.5)$

Agreement with previous lattice results

Large deviation from perturbative calculation

QGP reveals the properties of strongly coupled system