Strong coupling and quark masses from lattice QCD

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α_s current determination

PDG world average [PDG 2016]

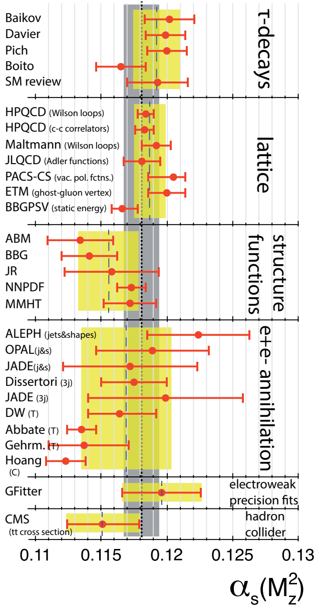
$$\boxed{\alpha(M_Z) = 0.1181 \pm 0.0013 \ (1\%)}$$

4σ tension between determinations

$$\text{lattice}: \ \alpha(M_Z) = 0.1183 \pm 0.0007$$
$$e^+e^-\text{annihilation}: \ \alpha(M_Z) = 0.1123 \pm 0.0015$$

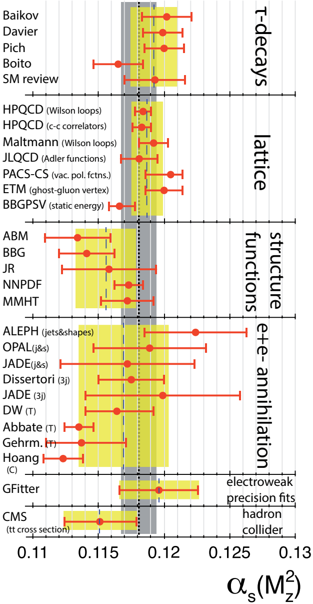
FLAG average [arXiv:1607.00299]

$$\alpha(M_Z) = 0.1182 \pm 0.0012 \ (1\%)$$
$$\Lambda^{(5)}_{\overline{\text{MS}}} = 211 \pm 14 \text{ MeV } (7\%)$$



lattice determination

- aim for subpercent precision
- only $n_f + 1$ masses from the hadron spectrum are needed $(m_{ud}, m_s, \ldots, \alpha_s)$
- challenge: connecting low scale hadronic quantities to perturbative definition of α_s
- what are the limiting factors?



strong coupling constant

$\alpha_S(\mu) = g_S(\mu)^2/(4\pi)$ is scale (μ) and scheme (S) dependent

$$\boxed{\mathcal{Q}(\mu) = c_1\alpha_S(\mu) + c_2\alpha_S(\mu)^2 + \ldots + O(\alpha_S^{n+1}) + O\left(\exp\left(-\gamma/\alpha_S\right)\right)}$$

lattice approach:

- ✓ design short-distance Euclidean observables $\mathcal{Q}$, UV/IR safe
- ✓ compute them by Monte Carlo simulations

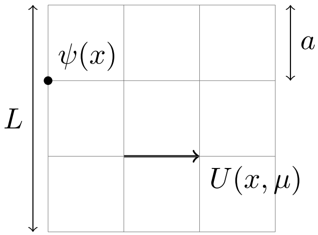
- ✗ truncation of perturbative expansion
- ✗ systematic errors in the lattice computations
- ✗ number of quarks

a problem of scales

- $\mathcal{Q}(\mu) = \lim_{a\to 0} \mathcal{Q}(\mu, a)$, deal with lattice artefacts
- match perturbative expansion: large μ or small $\alpha_{\text{eff}} = \mathcal{Q}/c_1$
- hadron mass/intermediate scales ($\sqrt{t_0}, w_0, r_0, r_1$)
- $a\mu$, r_0/a (am_h) need to be measured on the same lattices
- $r_0 \ll L, \quad \mu \ll a^{-1} \implies \boxed{\mu \ll (L/a) \times r_0^{-1} \simeq 5 - 20 \text{ GeV}}$

lattice path integrals

$$\boxed{Z = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi}\; e^{-S[\psi,\bar{\psi},U]}}$$



$$S = \sum_{x,y \in \Lambda} \bar{\psi}(x)\, D(x,y;U)\, \psi(y) + S_g[U]$$

only inputs: am_f, g

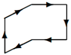
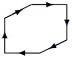
$$r_0 \ll L, \quad \mu \ll a^{-1}$$

three examples

- Wilson lines [HPQCD10]
- current correlators [HPQCD14A]
- step scaling & gradient flow [ALPHA16]

wilson lines

[HPQCD10] simulations with $n_f = 2+1$

BR =  CC = 

$$\boxed{Y_i = \log\langle \mathrm{Re\ Tr\ P} \exp -i \oint_C dx^\mu A_\mu \rangle = Y\left(a_i, (am_q)_i, \alpha_0, y_m^{(1)}, c_n, d\right)}$$

$$Y\left(a_i, (am_q)_i, \alpha_0, y_m^{(1)}, c_n, d\right) = \\ = \left(1 + y_m^{(1)}(2am_l + am_s)_i\right) \sum_{n=1,2,3,\ldots}^{10} c_n\, \alpha_V(d/a_i)^n$$

$$a_i = \frac{r_1}{(r_1/a)_i}\left(1 + r_{1a}^{(2)}(a/r_1)_i^2 + r_{1m}^{(1)}(2r_1 m_l)_i\right)$$

$$\alpha_0 = \alpha_{\overline{\mathrm{MS}}}(\mu = 5\ \mathrm{GeV}, n_f = 3)$$

bayesian fitting

$$\chi^2 = \sum_i \frac{\left[Y_i - Y\left(a_i, (am_q)_i, \alpha_0, y_m^{(1)}, c_n, d\right)\right]^2}{\sigma_{Y_i}^2} + \sum_\xi \delta\chi_\xi^2$$

prior for the fit parameters

$$\delta\chi_\xi^2 = \sum_\xi \frac{(\xi - \bar{\xi})^2}{\sigma_\xi^2}$$

conservative choices, but difficult to judge the impact of the priors

error budget and results

Collaboration	Ref.	N_f	publication status	renormalization scale	perturbative behaviour	lattice spacings	scale	$\Lambda_{\overline{\rm MS}}$[MeV]	$r_0\Lambda_{\overline{\rm MS}}$
HPQCD 10[a §]	[9]	2+1	A	○	★	★	$r_1 = 0.3133(23)$ fm	340(9)	0.812(22)
HPQCD 08A[a]	[613]	2+1	A	○	★	★	$r_1 = 0.321(5)$ fm[††]	338(12)[⋆]	0.809(29)
Maltman 08[a]	[63]	2+1	A	○	○	★	$r_1 = 0.318$ fm	352(17)[†]	0.841(40)
HPQCD 05A[a]	[612]	2+1	A	○	○	○	r_1[††]	319(17)[⋆⋆]	0.763(42)

	$\log W_{11}$	$\log W_{12}$	$\log W_{22}$	$\log W_{11}W_{22}/W_{12}^2$	$\log W_{12}/u_0^6$	$\log W_{22}/u_0^8$	$\alpha_{\rm lat}/W_{11}$
$c_1 \ldots c_3$	0.1%	0.1%	0.1%	0.3%	0.1%	0.1%	0.1%
c_n for $n \geq 4$	0.2	0.3	0.3	0.4	0.3	0.4	0.3
am_q, $r_1 m_q$ extrapolation	0.1	0.1	0.0	0.1	0.1	0.1	0.0
$(a/r_1)^2$ extrapolation	0.2	0.3	0.4	0.3	0.2	0.2	0.0
$(r_1/a)_i$ errors	0.4	0.4	0.4	0.3	0.3	0.3	0.3
r_1 errors	0.3	0.3	0.3	0.3	0.3	0.3	0.3
gluon condensate	0.1	0.1	0.1	0.2	0.1	0.1	0.1
statistical errors	0.0	0.0	0.0	0.1	0.0	0.0	0.0
$V \to \overline{\rm MS} \to M_Z$	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Total	0.6%	0.6%	0.7%	0.7%	0.6%	0.6%	0.5%

lattice artefacts / poor convergence of pert th

$$\boxed{\alpha_s^{(5)}(M_Z) = 0.1184(6)}$$

moments of correlators

[HPQCD14A] simulations with $n_f = 2+1+1$

$$G_n = \sum_t (t/a)^n \, G(t)$$

$$G(t) = a^3 \sum_{\mathbf{x}} (am_{0h})^2 \, \langle j_5(\mathbf{x},t) j_5(0) \rangle \, , \qquad j_5 = \bar{\psi}_h \gamma_5 \psi_h$$

reduced moments

$$\tilde{R}_n = \begin{cases} G_4/G_4^{(0)} \, , & n=4 \\ \frac{1}{m_{0c}} \left(G_n/G_n^{(0)} \right)^{1/(n-4)} \, , & n \geq 6 \end{cases}$$

for $n \geq 4$, $\tilde{R}_n$ are UV finite $\longrightarrow \lim_{a\to 0} \tilde{R}_n$

perturbative expression

$$\tilde{R}_n \to \begin{cases} r_4\left(\alpha_{\overline{\mathrm{MS}}}, \mu\right) \times \left[1 + d_4\left(\alpha_{\overline{\mathrm{MS}}}, \mu\right) \frac{\langle \alpha_s G^2/\pi \rangle}{(2m_h)^4} + \ldots\right], & n = 4 \\ \frac{r_n\left(\alpha_{\overline{\mathrm{MS}}}, \mu\right)}{m_c(\mu)} \times \left[1 + d_n\left(\alpha_{\overline{\mathrm{MS}}}, \mu\right) \frac{\langle \alpha_s G^2/\pi \rangle}{(2m_h)^4} + \ldots\right], & n \geq 6 \end{cases}$$

r_n, d_n known in pert th to NNLO, LO respectively

$$r_n\left(\alpha_{\overline{\mathrm{MS}}}, \mu\right) = 1 + \sum_j r_{nj}(\mu) \alpha_{\overline{\mathrm{MS}}}(\mu)^j$$

$\mu = 3m_h(\mu)$

fitting form

$$\tilde{R}_n = \left\{ \begin{array}{rr} 1 & n = 4 \\ 1/\xi_m m_c(\xi_\alpha \mu) & n \geq 6 \end{array} \right\} \times r_n \left(\alpha_{\overline{\mathrm{MS}}}(\xi_\alpha \mu), \mu \right) \times$$

$$\left(1 + d_n \frac{\langle \alpha_s G^2/\pi \rangle}{(2m_h)^4} \right) \times \left(1 + d_n^{h,c} \frac{m_{0h}^2 - m_{0c}^2}{m_{0h}^2} \right) +$$

$$\left(\frac{am_\eta}{2.26} \right)^2 \sum_i c_i(m_\eta, n) \left(\frac{am_\eta}{2.26} \right)^{2i}$$

$m_c(\mu), \alpha_{\overline{\mathrm{MS}}}(\mu)$ obtained using RG evolution as functions of

$$\begin{cases} \alpha_{\overline{\mathrm{MS}}}(5\ \mathrm{GeV}, n_f = 4) = \alpha_0 \\ \\ m_c(5\ \mathrm{GeV}, n_f = 4) = m_0 \end{cases}$$

error budget and results

Collaboration	Ref.	N_f	publication status	renormalization scale	perturbative behaviour	continuum extrapolation	scale	$\Lambda_{\overline{MS}}$[MeV]	$r_0\Lambda_{\overline{MS}}$
HPQCD 14A	[5]	2+1+1	A	○	★	○	$w_0 = 0.1715(9)$ fm^a	$294(11)^{bc}$	0.703(26)
HPQCD 10	[9]	2+1	A	○	★	○	$r_1 = 0.3133(23)$ fm†	$338(10)^\star$	0.809(25)
HPQCD 08B	[152]	2+1	A	■	■	■	$r_1 = 0.321(5)$ fm†	$325(18)^+$	0.777(42)

	$m_c(3)$	$\alpha_{\overline{MS}}(M_Z)$	m_c/m_s	m_b/m_c
Perturbation theory	0.3	0.5	0.0	0.0
Statistical errors	0.2	0.2	0.3	0.3
$a^2 \to 0$	0.3	0.3	0.0	1.0
$\delta m_{uds}^{\text{sea}} \to 0$	0.2	0.1	0.0	0.0
$\delta m_c^{\text{sea}} \to 0$	0.3	0.1	0.0	0.0
$m_h \neq m_c$ (Eq. (15))	0.1	0.1	0.0	0.0
Uncertainty in $w_0, w_0/a$	0.2	0.0	0.1	0.4
α_0 prior	0.0	0.1	0.0	0.0
Uncertainty in m_{η_s}	0.0	0.0	0.4	0.0
$m_h/m_c \to m_b/m_c$	0.0	0.0	0.0	0.4
δm_{η_c}: electromag., annih.	0.1	0.0	0.1	0.1
δm_{η_b}: electromag., annih.	0.0	0.0	0.0	0.1
Total:	0.64%	0.63%	0.55%	1.20%

Bayesian fit up to $O(a^{20})$

$$\Delta\alpha = |c_4/c_1|\alpha_{\text{eff}}^4$$

$$\boxed{\alpha_s^{(5)}(M_Z) = 0.11822(74)}$$

schrödinger functional

1. Finite volume scheme $\mu = 1/L, \quad \mu a \ll 1 \Longrightarrow L/a \gg 1$

$$\boxed{\sigma(u) = g(2L)^2\Big|_{g(L)^2=u}}$$

   step scaling: $g(L_{\rm had})^2 \longrightarrow g(\mu)^2, \quad \mu = 2^k/L_{\rm had}$

2. compute in a large volume:

$$L_{\rm had}/a\,, \quad r_0/a \;\Longrightarrow\; \left(\frac{L_{\rm had}}{r_0}\right)$$

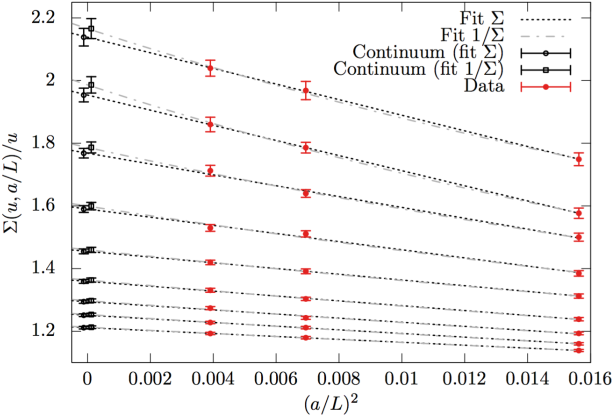
   so that

$$\mu = 2^k \left[\left(\frac{L_{\rm had}}{r_0}\right) r_0\right]^{-1}$$

   r_0 is the physical input that determines α_s

lattice step scaling

$$\Sigma(u, a/L) = \bar{g}_{\rm GF}(2L)^2\big|_{m=0, u=\bar{g}_{\rm GF}(L)^2}$$

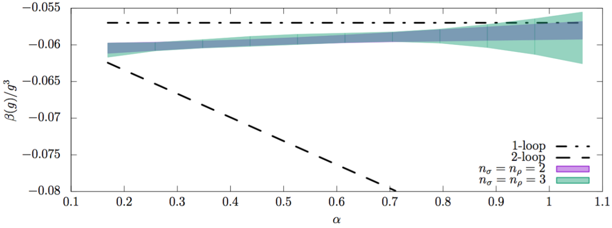


beta function

$$\beta(g) = -\frac{g^3}{P(g)}\,, \quad P(g) = p_0 + p_1 g^2 + p_2 g^4 + \ldots$$

$$\log 2 = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{dx}{\beta(x)}$$

$$= -\frac{p_0}{2}\left[\frac{1}{\sigma(u)} - \frac{1}{u}\right] + \frac{p_1}{2}\log\left[\frac{\sigma(u)}{u}\right] + \sum_{1}^{n_{\max}} \frac{p_{n+1}}{2n}\left[\sigma(u)^n - u^n\right]$$



error budget and results

[1604.06193]

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} L_0 = 0.0791(21)\ (2.7\%)$$

[1607.06423]

$$L_{\mathrm{had}}/L_0 = 21.86(42)\ (1.9\%)$$

[next step]

computation of L_{had} in physical units (e.g. L_{had}/r_0)

$\hookrightarrow$ approx. 4% accuracy on Λ $\longrightarrow$ aim for percent accuracy

three independent methods all with percent accuracy

light quark masses

- ETM14, $n_f = 2+1+1$

$$\hat{M}^2_{K^0} - \hat{M}^2_{K^+} = (m_d - m_u) \left(\frac{\partial M_K^2}{\partial m} \right)\Bigg|_{m=m_{ud}}$$

- QCDSF/UKQCD15: $n_f = 3$ + QED

$$M^2(a\bar{b}) = M_0^2 + \ldots + \beta_0(e_u^2 + e_d^2 + e_s^2) + \beta_1(e_a^2 + e_b^2) + \beta_2(e_a - e_b)^2 + \ldots$$

- RM123 11, MILC09: $n_f = 2, 2+1$

$$\left(\Delta^\gamma_{K^+} - \Delta^\gamma_{K^0}\right) - \left(\Delta^\gamma_{\pi^+} - \Delta^\gamma_{\pi^0}\right) = \epsilon \left(M^2_{\pi^+} - M^2_{\pi^0}\right)$$

- PACS-CS12: $n_f = 2+1$ reweighting

light quark masses

- BMW16: $n_f = 2 + 1$ full QCD + quenched QED computation

$$\epsilon = 0.73(2)(5)(17)$$

[1604.07112]

light quark summary

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	m_u	m_d	m_u/m_d
MILC 14	[113]	C	★	★	★	–	–			$0.4482(48)(^{+\ 21}_{-115})(1)(165)$
ETM 14	[4]	A	★	★	★	★	b	2.36(24)	5.03(26)	0.470(56)
QCDSF/UKQCD 15[⊖]	[165]	P	○	■	○	–	–			0.52(5)
PACS-CS 12[⋆]	[143]	A	★	■	■	★	a	2.57(26)(7)	3.68(29)(10)	0.698(51)
Laiho 11	[44]	C	○	★	★	○	–	1.90(8)(21)(10)	4.73(9)(27)(24)	0.401(13)(45)
HPQCD 10[‡]	[9]	A	○	★	★	★	–	2.01(14)	4.77(15)	
BMW 10A, 10B[+]	[7, 8]	A	★	★	★	★	b	2.15(03)(10)	4.79(07)(12)	0.448(06)(29)
Blum 10[†]	[103]	A	○	■	○	★	–	2.24(10)(34)	4.65(15)(32)	0.4818(96)(860)
MILC 09A	[6]	C	○	★	★	○	–	1.96(0)(6)(10)(12)	4.53(1)(8)(23)(12)	0.432(1)(9)(0)(39)
MILC 09	[89]	A	○	★	★	○	–	1.9(0)(1)(1)(1)	4.6(0)(2)(2)(1)	0.42(0)(1)(0)(4)
MILC 04, HPQCD/ MILC/UKQCD 04	[107] [148]	A	○	○	○	■	–	1.7(0)(1)(2)(2)	3.9(0)(1)(4)(2)	0.43(0)(1)(0)(8)
RM123 13	[16]	A	○	★	○	★	c	2.40(15)(17)	4.80 (15)(17)	0.50(2)(3)
RM123 11[⊕]	[166]	A	○	★	○	★	c	*2.43(11)(23)*	*4.78(11)(23)*	*0.51(2)(4)*
Dürr 11[*]	[132]	A	○	★	○	–	–	2.18(6)(11)	4.87(14)(16)	
RBC 07[†]	[105]	A	■	■	★	★	–	3.02(27)(19)	5.49(20)(34)	0.550(31)

$$\text{BMW}: \quad m_u = 2.27(6)(5)(4)\ \text{MeV}\,, \quad m_d = 4.67(6)(5)(4)\ \text{MeV}\,,$$

charm quark

Two complementary methods:

- mass of hadrons vs quark mass

$$M_{\mathrm{had}}(am_c) = M_{\mathrm{had}}^{\mathrm{exp}} \quad \Longrightarrow \quad am_c^{\mathrm{phys}} \quad \Longrightarrow \quad m_c(m_c)$$

- moments method described above

- choice of the heavy quark action

charm summary

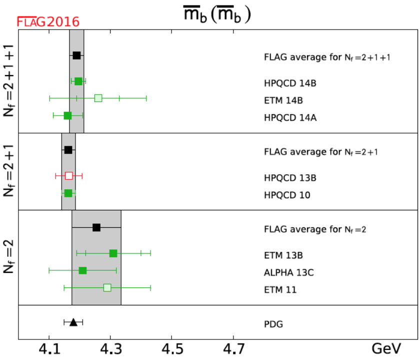
Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	$\overline{m}_c(\overline{m}_c)$	$\overline{m}_c(3\text{ GeV})$
HPQCD 14A	[5]	2+1+1	A	★	★	★	–	1.2715(95)	0.9851(63)
ETM 14A	[175]	2+1+1	A	○	★	○	★	1.3478(27)(195)	1.0557(22)(153)
ETM 14	[4]	2+1+1	A	○	★	○	★	1.348(46)	1.058(35)
JLQCD 15B	[173]	2+1	C	○	★	★	–	1.2769(21)(89)	0.9948(16)(69)
χQCD 14	[17]	2+1	A	○	○	○	★	1.304(5)(20)	1.006(5)(22)
HPQCD 10	[9]	2+1	A	○	★	○	–	1.273(6)	0.986(6)
HPQCD 08B	[152]	2+1	A	○	★	○	–	1.268(9)	0.986(10)
ALPHA 13B	[176]	2	C	★	○	★	★	1.274(36)	0.976(28)
ETM 11F	[174]	2	C	○	★	○	–	1.279(12)/1.296(18)*	0.979(09)/0.998(14)*
ETM 10B	[11]	2	A	○	★	○	★	1.28(4)	1.03(4)
PDG	[151]							1.275(25)	

b quark

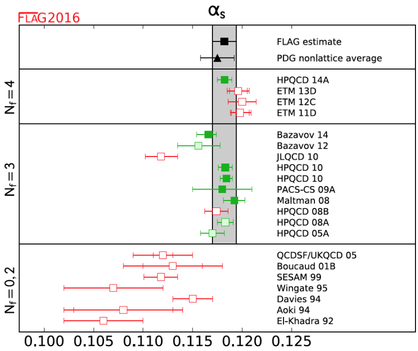
problem fitting the different scales on current lattices, $am_b \simeq 1$

- static action
- HISC relativistic action
- NRQCD effective description
- relativistic heavy quark action
- lattice HQET

b quark summary



outlook



- subpercent accuracy on α_s - ILC EW fit
- impact on N3LO calculations
- taming of lattice systematic errors
- m_c, m_b at the percent level
- more results will become available: better understanding of syst err
- aggressive/conservative analyses