Higher order calculation for W mass

Two-loop master integrals for the mixed QCD$\times$EW corrections to Drell-Yan processes

Stefano Di Vita

based on work with Roberto Bonciani, Pierpaolo Mastrolia and Ulrich Schubert, JHEP 1609 (2016) 091 [arXiv:1604.08581]

DESY (Hamburg)

Precision theory for precise measurements at LHC and future colliders
Sep 28, 2016

I barely have 1 "phenomenological" slide . . . hold on, the coffee break is close!

Outline

1. Drell-Yan processes: a very (very!) compact introduction
2. Two-loop mixed QCD$\times$EW corrections: what to compute
3. Two-loop mixed QCD$\times$EW corrections: how we computed

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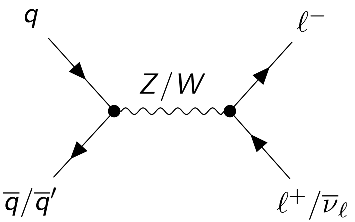
What this is about [Drell, Yan 70; . . . ; Alioli et al. 16]

see F. Piccinini's talk for pheno details
& V. Croft's talk for experimental status

"my phenomenological slide" ☹

- $\ell\bar{\ell}'$ production @ hadron colliders
- LO: s-channel Z/W exchange
- useful process:
  - constrain PDFs
  - direct determination of m_W

    template fit of $\ell\nu_\ell$ transverse mass distribution
  - background to BSM



all diagrams drawn with tikz-feynman [Ellis 16]
axodraw [Vermaseren 94]

- recall: SM relates m_W to m_Z and EW fit is a factor 2 more precise than direct determination (PDG 80.385 ± 0.015 GeV)
- direct measurement limited by stat. (PDFs uncert. ~ 10 MeV)

Higher order corrections to Drell-Yan cross section

Fixed order expansion

power-counting: $\alpha_s^2 \sim \alpha$

$$\sigma_{DY} = \sigma_0 \qquad \text{LO}$$

$$+ \alpha_s \sigma_{\alpha_s} + \alpha_s^2 \sigma_{\alpha_s^2} + \alpha_s^3 \sigma_{\alpha_s^3} + \dots \qquad \text{QCD}$$

$$+ \alpha \sigma_{\alpha} + \alpha^2 \sigma_{\alpha^2} + \alpha^3 \sigma_{\alpha^3} + \dots \qquad \text{EW}$$

$$+ \alpha \alpha_s \sigma_{\alpha\alpha_s} + \alpha \alpha_s^2 \sigma_{\alpha\alpha_s^2} + \dots \qquad \text{EW} \times \text{QCD}$$

NLO   NNLO   N3LO

- ✓ QCD NLO, QCD NNLO, EW NLO — fully differential, matched to PS
- ☺ QCD N3LO — hopefully soon, it's almost for free from $gg \to H$
- ☹ EW×QCD NNLO — full result not yet available

see G. Zanderighi's talk for NNLO and N3LO QCD

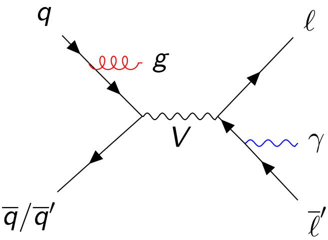
History of QCD corrections I apologize for any omission

- $pp' \to V$ $\left[d^2\sigma/dQ^2dy\right]$
  - NLO [Altarelli, Ellis, Martinelli 79; + Greco 84]
  - NNLO [Matsuura, van der Marck, van Neerven 89; Hamberg, van Neerven, Matsuura 91]
- $pp' \to V \to \ell\bar{\ell}'$, fully differential
  - Full NLO (MCFM) [Giele, Glover, Kosower 93; Campbell, Ellis, Rainwater 03]
  - Full NNLO (FEWZ, DYNNLO) [Melnikov, Petriello 06; + Anastasiou, Dixon 04; Catani, Grazzini 07; + Cieri, Ferrera, de Florian 09]
- $pp' \to V(\to \ell\bar{\ell}') + X$
  - NLO, $V + j$ $\left[d^3\sigma/dQ^2dp_Tdy\right]$ [Ellis, Martinelli, Petronzio 83; Arnold, Reno 89; Gonsalves, Pawlowski, Wai 89; Brandt, Kramer, Nyeo 91]
  - NLO, $\ell\bar{\ell}' + 1, 2\,j$ [Giele, Glover, Kosower 93; Campbell, Ellis 02; + Rainwater 03]
  - NLO, $\ell\bar{\ell}' + \gamma$ [Dixon, Kunszt, Signer 98]
  - NNLO, $V + j$ [Boughezal et al. 15,16; Gehrmann-De Ridder et al. 15;]
- Resummation and matching to PS
  - Soft g through N^{3}LL, p_T^V/M_V through NLL, NLO and NNLO matching (MC@NLO, POWHEG, DYNNLOPS), ... [Sterman 87; Catani, Trentadue 89; 91; Moch, Vogt 05; Balazs, Yuan 97; Bozzi, Catani, De Florian, Ferrera, Grazzini 10; Alioli, Nason, Oleari, Re 08; Karlberg, Re, Zanderighi 14; Hoeche, Li, Prestel 14; Alioli, Bauer, Berggren, Tackmann, Walsh 15; ...]

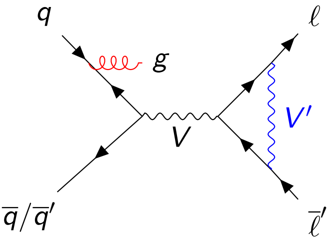
History of EW corrections I apologize for any omission

- W production at NLO EW
  - Pole approx [Wackeroth, Hollik 97; Baur, Keller, Wackeroth 99]
  - Full [Zykunov et al. 01; Dittmaier, Krämer 02,05; Baur, Wackeroth 04 (WZGRAD); Arbuzov et al. 06 (SANC); Carloni Calame et al. 06 (HORACE); Hollik, Kasprzik, Kniehl 08; Bardin et al. 08 WINHAC]
- Z production at NLO EW
  - QED [Barberio, van Eijk, Was 91,94; Baur, Keller, Sakumoto 98; Golonka, Was 06 (PHOTOS); Placzek, Jadach 03; + Krasny 13 (WHINAC)]
  - Full [Baur, Wackeroth 04; + Brein, Hollik, Schappacher 02 (WZGRAD); Zykunov et al. 07; Carloni Calame et al. 07 (HORACE); Dittmaier, Huber 12; Arbuzov et al. 07 (SANC)]
- $V + j$ production at NLO EW
  - large p_T^W [Kühn, Kulesza, Pozzorini, Schulze 04]
  - EW [Denner, Dittmaier, Kasprzik, Muck 09,11,12; Kallweit, Lindert, Maierhöfer, Pozzorini, Schönherr 14, 15]
  - also 2-loop $V + \gamma$ [Gehrmann, Tancredi 11]
- NNLO QCD, NLO EW (FEWZ) [Melnikov, Petriello 06; Li, Petriello 12; + Gavin, Quackenbush 12]
- NLO+PS (POWHEG) [Barze, Montagna, Nason, Nicrosini, Piccinini 12; +Vicini 13; Bernaciak, Wackeroth 12]

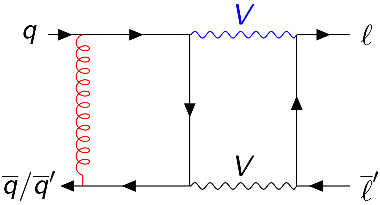
QCD×EW corrections: not yet fully available



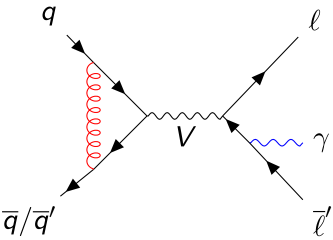
double real



real-virtual



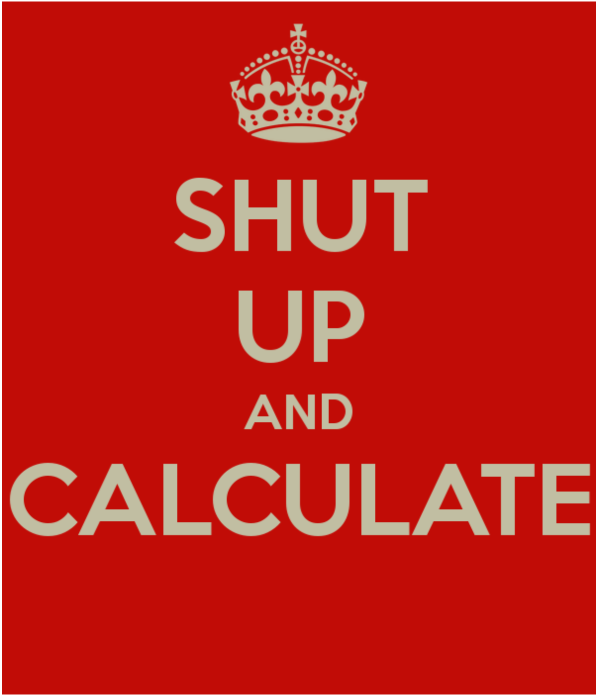
double virtual



real-virtual

QCD×EW corrections: not yet fully available

- What is available?
  - Two-loop W/Z form factors [Czarnecki, Kühn 96; Kotikov, Kühn, Veretin 08; Kara 13]
  - Virtual QCD×QED [Kilgore, Sturm 11]
  - Expansion around pole (in the resonant region) [Dittmaier, Huss, Schwinn 14,16]
  - Monte Carlo estimates through NLO QCD × NLO EW (with higher orders) see F. Piccinini's talk
- Why bother?
  - Bulk of corrections to inclusive obs comes from resonant region ...
  - ... but for accurate differential distributions in regions different from resonance (and to check the pole expansion), the full calculation is needed
  - Interesting problem from the math perspective
- What to do?
  - Tree-level $2 \to 4$ is by now a solved problem
  - $\mathcal{O}(\alpha)$ corrections to $V + j$ are known
  - $\mathcal{O}(\alpha_s)$ corrections to $V + \gamma$ are known
  - Let's tackle the two-loop contribution!


SHUT
UP
AND
CALCULATE

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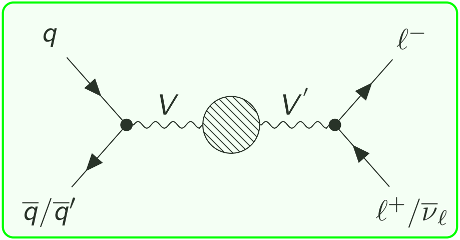
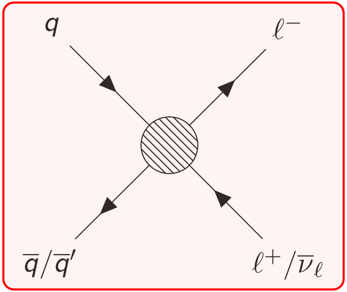
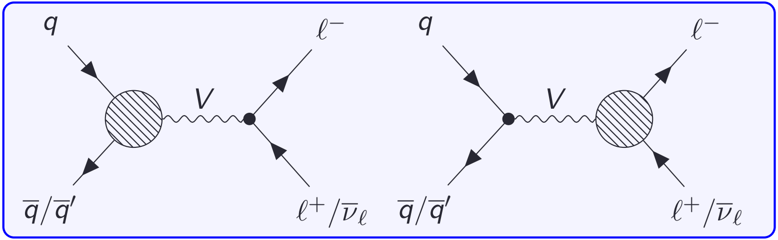
Drell-Yan dilepton production: virtual corrections

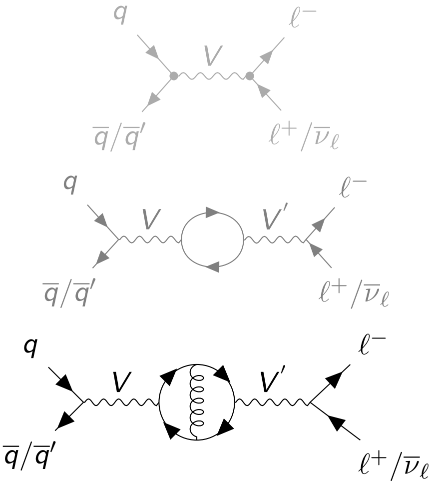
$m_{q,\ell} = 0$





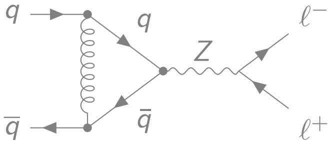


Propagator NNLO QCD×EW corrections: e.g.

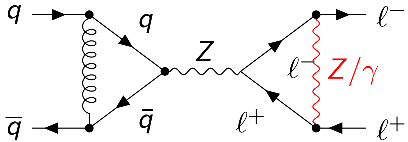


- gauge bosons couple to quarks, and quarks to gluons
- general two-loop self-energies are in principle solved, at least numerically
  - TSIL [Martin and Robertson 04]
  - S2LSE [Bauberger]
- essential building block of SM renormalization at two loops

Vertex NNLO QCD×EW corrections: e.g.

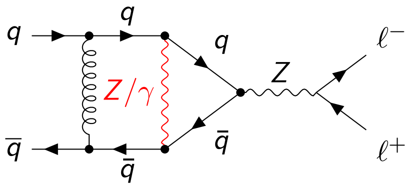


NLO QCD



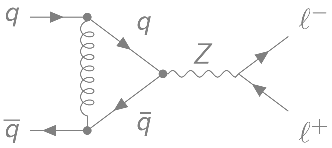
NNLO QCD×EW, factorizable, (1-loop)^2

- quarks in the initial state
- leptons in the final state
  - no QCD corrections there at 1- and 2-loops
  - no gluon exchange with initial state at 1- and 2-loops

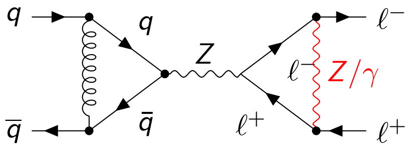


NNLO QCD×EW, factorizable, 1PI

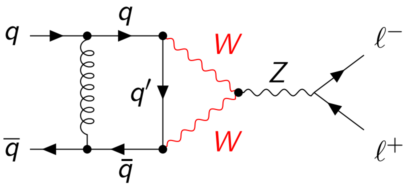
Vertex NNLO QCD×EW corrections: e.g.



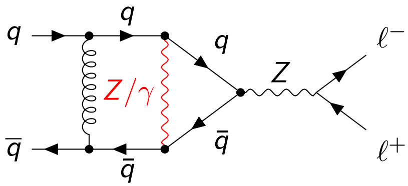
NLO QCD



NNLO QCD×EW, factorizable, (1-loop)^2

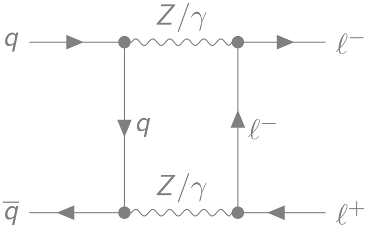


[Kotikov, Kühn, Veretin 08]



NNLO QCD×EW, factorizable, 1PI

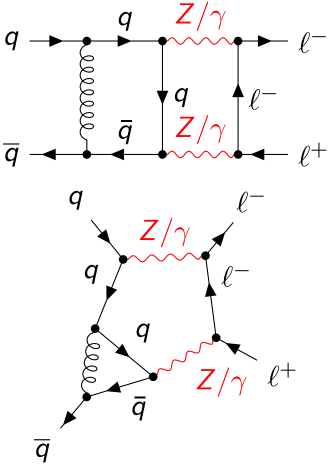
Box NNLO QCD×EW corrections: e.g.



NLO EW, non-factorizable

leptons in the final state

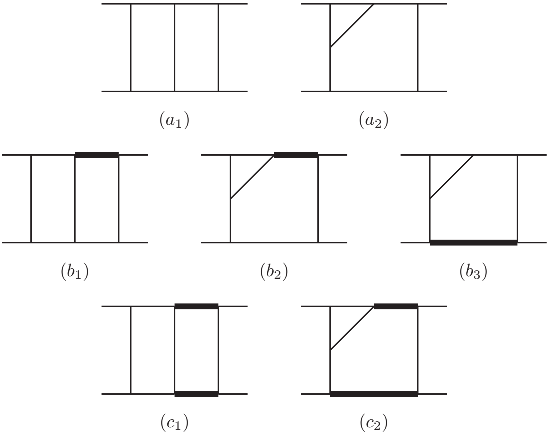
- no QCD corrections at 1-loop
- no gluon exchange with initial state
- can get boxes only by dressing the non-factorizable NLO EW



NNLO QCD×EW, non-factorizable

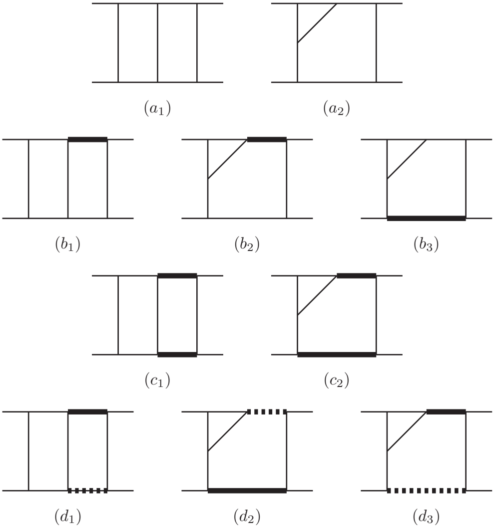
Two-loop mixed QCD×EW corrections: $q\bar{q} \to \ell^+\ell^-$

- Do it carefully (FeynArts [Hahn 01])
- One can map all the Feynman diagrams onto 3 families
- The corrections to the neutral current DY process never involve W and Z at the same time
- Topology A well known [Smirnov 99; Gehrmann, Remiddi 99]
- Topologies B-C unknown so far



Two-loop mixed QCD×EW corrections: $q\bar{q}' \to \ell^- \overline{\nu}_\ell$

- Do it carefully (FeynArts [Hahn 01])
- One can map all the Feynman diagrams onto 4 families
- The corrections to the charged current DY process also involve W and Z at the same time
- Topology A well known [Smirnov 99; Gehrmann, Remiddi 99]
- Topologies B-C-D unknown so far



Outline

Let's make life a bit simpler

- Families with 1 or 2 degenerate massive propagators $\Rightarrow (s, t, m^2_{W,Z})$
- Family with 2 different massive propagators $\Rightarrow (s, t, m^2_W, m^2_Z)$
- We exploit $\Delta m^2 \equiv m^2_Z - m^2_W \ll m^2_Z$
- Expanding for instance the Z propagators around m_W

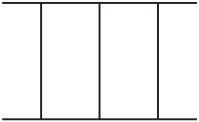
$$\frac{1}{p^2 - m^2_Z} = \frac{1}{p^2 - m^2_W - \Delta m^2} \approx \frac{1}{p^2 - m^2_W} + \frac{m^2_Z}{(p^2 - m^2_W)^2}\,\xi + \ldots$$

where

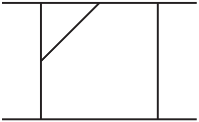
$$\xi = \frac{\Delta m^2}{m^2_Z} = \frac{m^2_Z - m^2_W}{m^2_Z} \sim \frac{1}{4}$$

- The coefficients of the series in ξ are Feynman diagrams with 3 scales
- The expanded denominators will appear raised to powers $> 1 \Rightarrow$ IBP

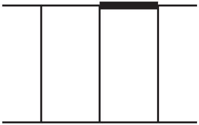
We computed these “master integrals” Bonciani, Mastrolia, Schubert, **DV** 16



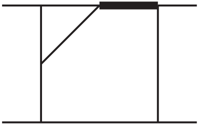
(a_1)



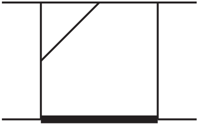
(a_2)



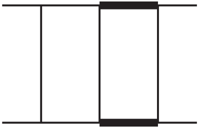
(b_1)



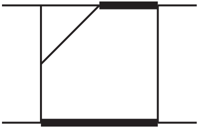
(b_2)



(b_3)

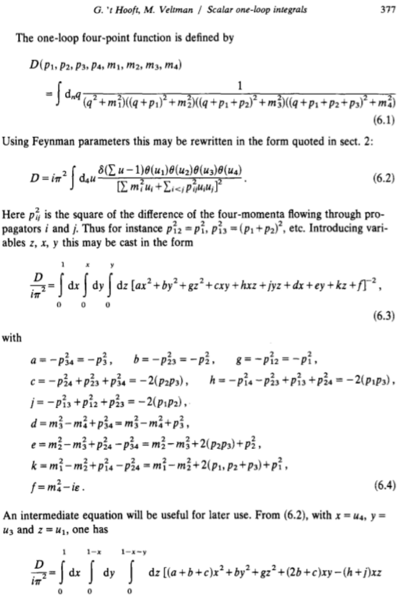


(c_1)



(c_2)

The art of computing Feynman integrals



G. 't Hooft, M. Veltman / Scalar one-loop integrals 377

The one-loop four-point function is defined by

$$D(p_1,p_2,p_3,p_4,m_1,m_2,m_3,m_4)$$
$$= \int d_nq \frac{1}{(q^2+m_1^2)((q+p_1)^2+m_2^2)((q+p_1+p_2)^2+m_3^2)((q+p_1+p_2+p_3)^2+m_4^2)} \quad (6.1)$$

Using Feynman parameters this may be rewritten in the form quoted in sect. 2:

$$D = i\pi^2 \int d_4u \frac{\delta(\sum u-1)\theta(u_1)\theta(u_2)\theta(u_3)\theta(u_4)}{[\sum m_i^2 u_i + \sum_{i<j} p_{ij}^2 u_i u_j]^2}. \quad (6.2)$$

Here p_{ij}^2 is the square of the difference of the four-momenta flowing through propagators i and j. Thus for instance $p_{12}^2 = p_1^2$, $p_{13}^2=(p_1+p_2)^2$, etc. Introducing variables z, x, y this may be cast in the form

$$\frac{D}{i\pi^2} = \int_0^1 dx \int_0^x dy \int_0^y dz\,[ax^2+by^2+gz^2+cxy+hxz+jyz+dx+ey+kz+f]^{-2}, \quad (6.3)$$

with

$$a=-p_{34}^2=-p_3^2, \quad b=-p_{23}^2=-p_2^2, \quad g=-p_{12}^2=-p_1^2,$$
$$c=-p_{24}^2+p_{23}^2+p_{34}^2=-2(p_2p_3), \quad h=-p_{14}^2-p_{23}^2+p_{13}^2+p_{24}^2=-2(p_1p_3),$$
$$j=-p_{13}^2+p_{12}^2+p_{23}^2=-2(p_1p_2),$$
$$d=m_3^2-m_4^2+p_{34}^2=m_3^2-m_4^2+p_3^2,$$
$$e=m_2^2-m_3^2+p_{24}^2-p_{34}^2=m_2^2-m_3^2+2(p_2p_3)+p_2^2,$$
$$k=m_1^2-m_2^2+p_{14}^2-p_{24}^2=m_1^2-m_2^2+2(p_1,p_2+p_3)+p_1^2,$$
$$f=m_4^2-i\varepsilon. \quad (6.4)$$

An intermediate equation will be useful for later use. From (6.2), with $x=u_4$, $y=u_3$ and $z=u_1$, one has

$$\frac{D}{i\pi^2} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz\,[(a+b+c)x^2+by^2+gz^2+(2b+c)xy-(h+j)xz$$

- Box scalar 1-loop integral from classic 't Hooft and Veltman paper
- Much has changed since the old days . . .
- Automation (you know better than me)
- New methods (Mellin Barnes, unitarity-based, differential equations, sector decomposition)
- *Divide and conquer* approach: exploit "algebraic redundancies" and reduce the number of integrals to be computed

Differential equations for Master Integrals [Kotikov 91; Remiddi 97]

Integration by parts identities

Loop integrals in d dimensions satisfy linear identities (IBPs + other). E.g.

$$\int \frac{d^d k}{(k^2-m^2)^2[(k-p)^2-m^2]} \equiv \int \frac{d^d k}{D_1^2\, D_2}$$

$$= \frac{d-3}{(p^2-4m^2)} \int \frac{d^d k}{D_1\, D_2} - \frac{d-2}{2m^2(p^2-4m^2)} \int \frac{d^d k}{D_1}$$

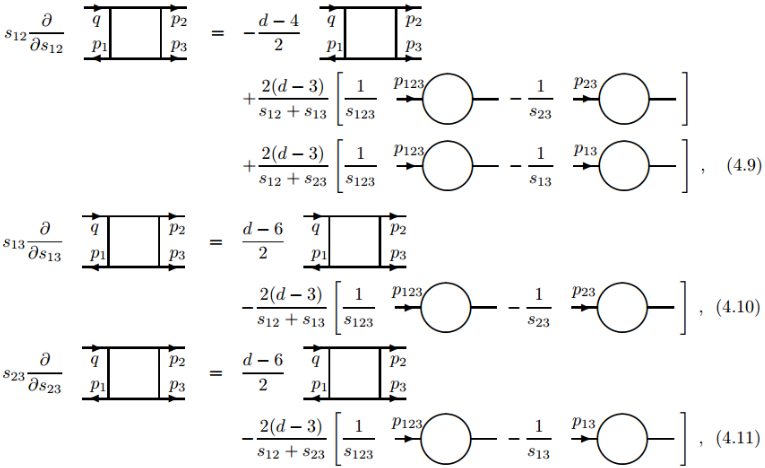
Only a finite number of them are independent (hence MIs)! ☺

- Public codes for IBP generation and solution: `AIR` [Anastasiou, Lazopoulos 04], `FIRE` [Smirnov 08], `REDUZE` [Studerus 10; + von Manteuffel 12], `LiteRed` [Lee 12]
- Take derivatives wrt external p_{ij}^2's and m_i^2's → use IBPs → obtain system of linear differential equations for the MIs (ODEs or PDEs)

$\mathbf{F} \equiv$ vector of MIs
$\mathbb{K} \equiv$ coeff. matrix

$$d\mathbf{F}(\vec{x},\epsilon) = \mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon) \qquad \epsilon = (4-d)/2$$

$s_{23} \equiv s$, $s_{13} \equiv t$, $s_{12} \equiv u$,
$s_{123} \equiv s + t + u$



$+$ other equations for the bubbles, *not* involving the boxes

$\Rightarrow$ hierarchical structure

Canonical DEs systems and iterated integrals

A smart change of the MIs basis can bring to big simplifications [Henn 13]

old basis $\leftarrow$ $\mathbf{F}(\vec{x}, \epsilon) = \mathbb{B}(\vec{x}, \epsilon)\, \mathbf{I}(\vec{x}, \epsilon)$ $\rightarrow$ new basis

bad basis ☹

$$d\mathbf{F}(\vec{x}, \epsilon) = \mathbb{K}(\vec{x}, \epsilon)\, \mathbf{F}(\vec{x}, \epsilon)$$

good basis ☺

$$d\mathbf{I}(\vec{x}, \epsilon) = \epsilon\, d\mathbb{A}(\vec{x})\, \mathbf{I}(\vec{x}, \epsilon)$$

Solution order by order in ϵ remember Dyson's series, $i\, dU(t, t_0) = \epsilon V(t) U(t, t_0) dt$?

$$\mathbf{I}(\epsilon, \vec{x}) = \mathcal{P} \exp\left\{\epsilon \int_\gamma d\mathbb{A}\right\} \mathbf{I}(\epsilon, \vec{x}_0)$$

$\mathbf{I}(\epsilon, \vec{x}_0) \equiv$ boundary constants e.g. value at $x_0 = 0$ etc

$$\mathcal{P} \exp\left\{\epsilon \int_\gamma d\mathbb{A}\right\} = \mathbb{1} + \epsilon \int_\gamma d\mathbb{A} + \epsilon^2 \int_\gamma d\mathbb{A}\, d\mathbb{A} + \epsilon^3 \int_\gamma d\mathbb{A}\, d\mathbb{A}\, d\mathbb{A} + \ldots$$

Canonical DEs systems and iterated integrals

A smart change of the MIs basis can bring to big simplifications [Henn 13]

$$\text{old basis} \leftarrow \boxed{\mathbf{F}(\vec{x},\epsilon) = \mathbb{B}(\vec{x},\epsilon)\,\mathbf{I}(\vec{x},\epsilon)} \rightarrow \text{new basis}$$

bad basis ☹

$$d\mathbf{F}(\vec{x},\epsilon) = \mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon)$$

good basis ☺

$$d\mathbf{I}(\vec{x},\epsilon) = \epsilon\, d\mathbb{A}(\vec{x})\,\mathbf{I}(\vec{x},\epsilon)$$

Solution order by order in ϵ remember Dyson's series, $i\,dU(t,t_0) = \epsilon V(t)U(t,t_0)dt$?

$$\mathbf{I}(\epsilon,\vec{x}) = \mathcal{P}\exp\left\{\epsilon\int_\gamma d\mathbb{A}\right\}\mathbf{I}(\epsilon,\vec{x}_0) \qquad \mathbf{I}(\epsilon,\vec{x}_0) \equiv \begin{array}{l}\text{boundary constants}\\ \text{e.g. value at } s=0 \text{ etc}\end{array}$$

γ is *any* path from $\vec{x}_0$ to $\vec{x}$ (that does not cross branch cuts and singularities of the integrand). $\mathcal{P}$ is like $\mathcal{T}$-ordering, but in more dimensions!

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It follows from Chen's theorem . . .

. . . that the matrices

$$\int_\gamma \underbrace{d\mathbb{A}\ldots d\mathbb{A}}_{\text{k times}}$$

are invariant under smooth deformations of the path γ (provided branch cuts and singularities are avoided)! A lot of freedom ☺

Canonical DEs systems and iterated integrals

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$$d\mathbf{F}(\vec{x},\epsilon) = \mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon)$$

good basis ☺

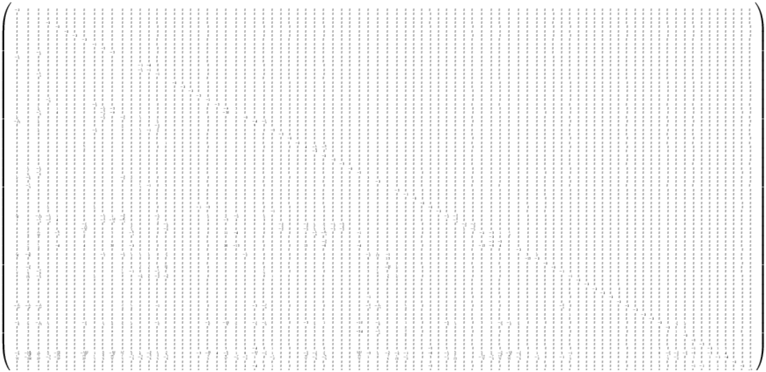
$$d\mathbf{I}(\vec{x},\epsilon) = \epsilon\, d\mathbb{A}(\vec{x})\,\mathbf{I}(\vec{x},\epsilon)$$

Achieving a "canonical" basis

No general algorithm devised yet, mathematical status of a "conjecture". Some ideas and special cases (constant leading singularity, ϵ-linear DEs, triangular DEs for $\epsilon \to 0$, Moser algorithm, ...) [Henn 13; Argeri et al. 14; Bern et al. 14; Lee 14; Höschele et al. 14; Gehrmann et al. 14; Tancredi 15]

How it looks like e.g. Higgs + 1Jet 3-loop ladder [Mastrolia, Schubert, Yundin, **DV** 14]

$1/x$



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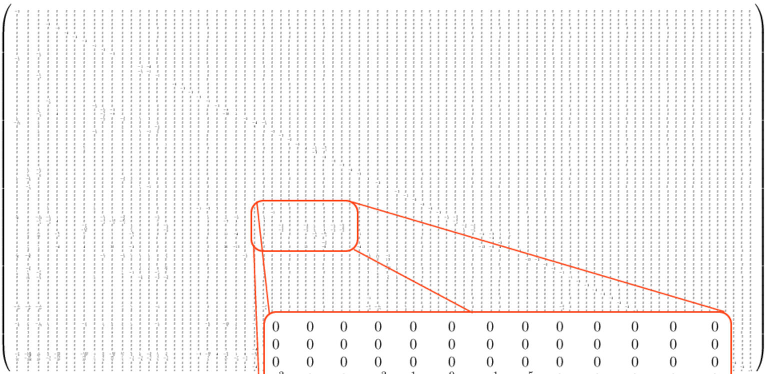
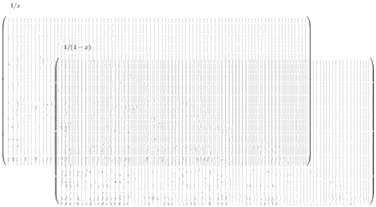


$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & 0 & 0 & -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} & -\frac{1}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & 0 & 0 & -\frac{7}{2} & \frac{1}{3} & 2 & -\frac{1}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -\frac{1}{6} & -\frac{11}{4} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12 & 0 & -6 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{12} & -1
\end{matrix}
$$

How it looks like e.g. Higgs + 1Jet 3-loop ladder [Mastrolia, Schubert, Yundin, **DV** 14]



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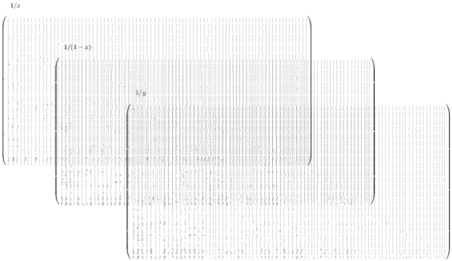

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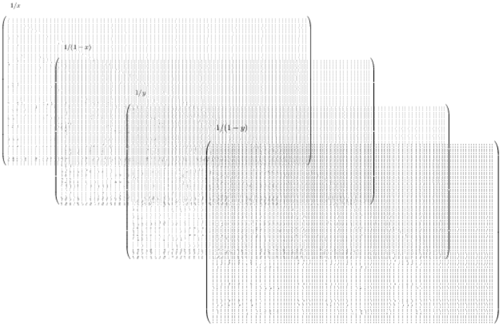

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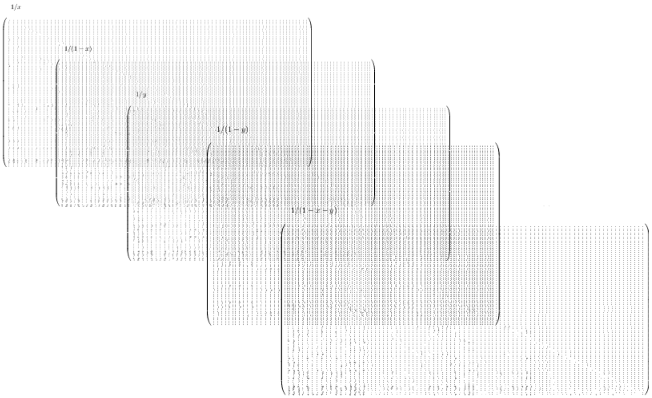

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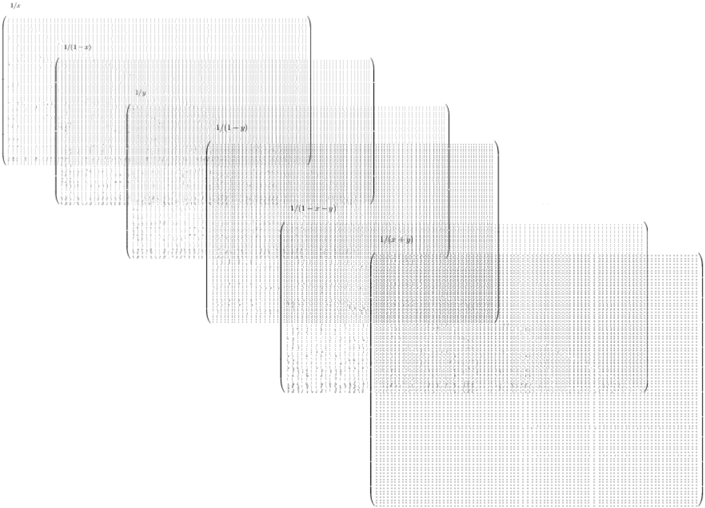

How it looks like e.g. Higgs + 1Jet 3-loop ladder [Mastrolia, Schubert, Yundin, **DV** 14]

Chen's iterated integrals [Chen 77]

For DY, the "canonical" coefficient matrix is a $d\log$ form

$$d\mathbb{A} = \sum_{i=1}^{n} \mathbb{M}_i \; d\log \eta_i(\vec{x}) \qquad \text{where} \begin{cases} \text{the } \mathbb{M}_i \text{ are matrices of numbers} \\ \text{the "letters" } \eta_i \text{ are functions of } \vec{x} \end{cases}$$

Therefore the entries of

$$\int_\gamma \underbrace{d\mathbb{A} \ldots d\mathbb{A}}_{\text{k times}}$$

are linear combinations of Chen's iterated integrals of the form

$$\underbrace{\int_\gamma d\log \eta_{i_k} \ldots d\log \eta_{i_1}}_{\equiv \mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1}} \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} g^\gamma_{i_k}(t_k) \ldots g^\gamma_{i_1}(t_1)\, dt_1 \ldots dt_k$$

where, given a parametrization $\gamma(t)$, $t \in [0,1]$, $g^\gamma_i(t) = \frac{d}{dt} \log \eta_i(\gamma(t))$

Chen's iterated integrals [Chen 77]

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are linear combinations of Chen's iterated integrals of the form

Recall GPLs

$$G_{i_k,\ldots,i_1}(1) \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} \frac{1}{t_k - i_k} \ldots \frac{1}{t_1 - i_1} \, dt_1 \ldots dt_k$$

where, given a parametrization $\gamma(t)$, $t \in [0,1]$, $g_i^\gamma(t) = \frac{d}{dt} \log \eta_i(\gamma(t))$

Possibly more familiar ...

Path integral representation for complex functions

$$Log(z) \equiv \int_\gamma \frac{d\zeta}{\zeta}$$

$$Li_2(z) \equiv -\int_\gamma \frac{\log(1-\zeta)}{\zeta} d\zeta$$

where γ is a path in the complex plane that starts at some z_0 and ends at z and does not cross

- the point $\zeta = 0$ for the first integral
- the point $\zeta = 0$ and the branch cut for $\zeta > 1$ for the second integral

Chen integrals generalize GPLs, which in turn generalize the classical polylogarithms. Public codes are available for GPL evaluation, including their analytic continuation, e.g. GiNaC.

Chen's integral are more general, automation and optimization is harder.

Chen's iterated integrals: properties basically same as for GPLs

- Invariance under path reparametrization
- Reverse path formula: $\mathcal{C}^{[\gamma^{-1}]}_{i_k,\ldots,i_1} = (-1)^k \mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1}$
- Recursive structure: ($\gamma^s(t) \equiv \gamma(s\,t)$, with $s \in [0,1]$)

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} = \int_0^1 g^{\gamma}_{i_k}(s)\, \mathcal{C}^{[\gamma_s]}_{i_{k-1},\ldots,i_1} ds \qquad \frac{d}{ds} \mathcal{C}^{[\gamma_s]}_{i_k,\ldots,i_1} = g^{\gamma}_{i_k}(s)\, \mathcal{C}^{[\gamma_s]}_{i_{k-1},\ldots,i_1}$$

- Shuffle algebra:

$$\mathcal{C}^{[\gamma]}_{\vec{m}}\, \mathcal{C}^{[\gamma]}_{\vec{n}} = \sum_{\text{shuffles } \sigma} \mathcal{C}^{[\gamma]}_{\sigma(m_M),\ldots,\sigma(m_1),\sigma(n_N),\ldots,\sigma(n_1)}$$

- Path composition formula: if $\gamma \equiv \alpha\beta$, i.e. first α, then β

$$\mathcal{C}^{[\alpha\beta]}_{i_k,\ldots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\beta]}_{i_k,\ldots,i_{p+1}}\, \mathcal{C}^{[\alpha]}_{i_p,\ldots,i_1}$$

- Integration-by-parts formula: get rid of outermost integration

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} = \log \eta_{i_k}(\vec{x})\, \mathcal{C}^{[\gamma]}_{i_{k-1},\ldots,i_1} - \int_0^1 \log \eta_{i_k}(\vec{x}(t))\, g_{i_{k-1}}(t)\, \mathcal{C}^{[\gamma_t]}_{i_{k-2},\ldots,i_1} dt$$

Connection with GPLs in special cases

A representation in terms of GPLs can be obtained if the η_i's are multilinear in $\vec{x}$. E.g. single letter $\eta = 1 + xy$. Choose $\gamma = \alpha\beta$ with

$$\alpha(t) = (x_0 + t(x_1 - x_0),\, y_0)\,,$$
$$\beta(t) = (x_1,\, y_0 + t(y_1 - y_0))\,,$$

and $t \in [0, 1]$. Then

$$\int_{\alpha\beta} d\log(1+xy) = \int_\alpha d\log(1+xy) + \int_\beta d\log(1+xy)$$
$$= G\left(\tfrac{1+x_0y_0}{y_0(x_0-x_1)};1\right) + G\left(\tfrac{1+x_0y_0}{x_0(y_0-y_1)};1\right)$$

$$\int_{\alpha\beta} d\log(1+xy)\ d\log(1+xy) = \int_\alpha d\log(1+xy)\ d\log(1+xy) + \int_\alpha d\log(1+xy)\times$$
$$\times \int_\beta d\log(1+xy) + \int_\beta d\log(1+xy)\, d\log(1+xy)$$
$$= G\left(\tfrac{1+x_0y_0}{y_0(x_0-x_1)}, \tfrac{1+x_0y_0}{y_0(x_0-x_1)};1\right) + G\left(\tfrac{1+x_0y_0}{x_0(y_0-y_1)}, \tfrac{1+x_0y_0}{y_0(x_0-x_1)};1\right)$$
$$+ G\left(\tfrac{1+x_0y_0}{x_0(y_0-y_1)}, \tfrac{1+x_0y_0}{x_0(y_0-y_1)};1\right)$$

Integrating ϵ-linear DE's [Argeri, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, **DV** 14]

1. start with DE linear in ϵ (may need a bit of trial and error + expertise)

$$\partial_x \mathbf{F}(\epsilon, x) = A(\epsilon, x)\mathbf{F}(\epsilon, x)\,, \quad A(\epsilon, x) = A_0(x) + \epsilon\, A_1(x)$$

2. basis change with Magnus's exponential: $\mathbf{F}(\epsilon, x) = B_0(x)\ \mathbf{I}(\epsilon, x)$

$$B_0(x) \equiv e^{\Omega[A_0](x,x_0)} \quad \leftrightarrow \quad \partial_x B_0(x) = A_0(x) B_0(x)$$

3. obtain a canonical system for the $\mathbf{I}$'s

$$\partial_x \mathbf{I}(\epsilon, x) = \epsilon\, \hat{A}_1(x)\mathbf{I}(\epsilon, x)\,, \quad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)$$

4. obtain the solution with Magnus (or Dyson)

$$\mathbf{I}(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon\, \hat{A}_1](x,x_0)}$$

5. ϵ-expansion of g's will have uniform weight ("transcendentality") (if $\mathbf{I}(0)$'s are chosen wisely)

In two (or more!) dimensions [Mastrolia, Schubert, Yundin, **DV** 14]

- the $\mathbf{F}$'s obey an ϵ-linear DE system ($x = \frac{s}{m^2}$, $y = \frac{t}{m^2}$)

$$\partial_x \mathbf{F}(x,y,\epsilon) = \left(A_{1,0}(x,y) + \epsilon\, A_{1,1}(x,y)\right) \mathbf{F}(x,y,\epsilon)$$
$$\partial_y \mathbf{F}(x,y,\epsilon) = \left(A_{2,0}(x,y) + \epsilon\, A_{2,1}(x,y)\right) \mathbf{F}(x,y,\epsilon)$$

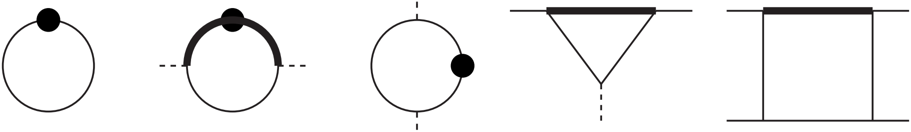
- After getting rid of $A_{i,0}$'s with Magnus (one variable at the time), the g's obey a canonical DE

$$\partial_x \mathbf{I}(x,y,\epsilon) = \epsilon\, \hat{A}_x(x,y) \mathbf{I}(x,y,\epsilon)$$
$$\partial_y \mathbf{I}(x,y,\epsilon) = \epsilon\, \hat{A}_y(x,y) \mathbf{I}(x,y,\epsilon)$$

- which can be cast in *dlog* form

$$d\mathbf{I}(x,y,\epsilon) = \epsilon\, d\mathbb{A}(x,y)\, \mathbf{I}(x,y,\epsilon)$$

- with some *alphabet* $\{\eta_1, \ldots, \eta_n\}$ $\Rightarrow$ Path-ordered exponential



$(\mathcal{T}_1)$ $(\mathcal{T}_2)$ $(\mathcal{T}_3)$ $(\mathcal{T}_4)$ $(\mathcal{T}_5)$

$$F_1 = \epsilon\, \mathcal{T}_1\,, \qquad F_2 = \epsilon\, \mathcal{T}_2\,, \qquad F_3 = \epsilon\, \mathcal{T}_3\,,$$
$$F_4 = \epsilon^2\, \mathcal{T}_4\,, \qquad F_5 = \epsilon^2\, \mathcal{T}_5$$

The vector **F** obeys an ϵ-linear DE: we obtain the canonical MIs with the Magnus procedure

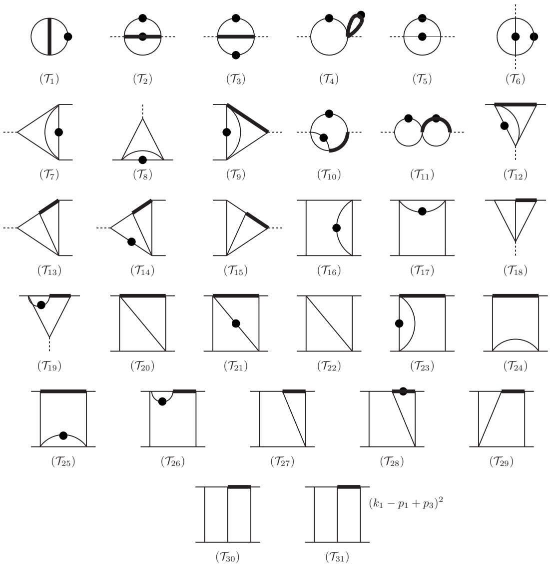
$$I_1 = F_1\,, \qquad I_2 = -s\, F_2\,, \qquad I_3 = -t\, F_3\,,$$
$$I_4 = -t\, F_4\,, \qquad I_5 = (s - m^2)\, t\, F_5$$

The alphabet of the corresponding $d\log$-form is $(x \equiv -s/m^2,\ y \equiv -s/m^2)$

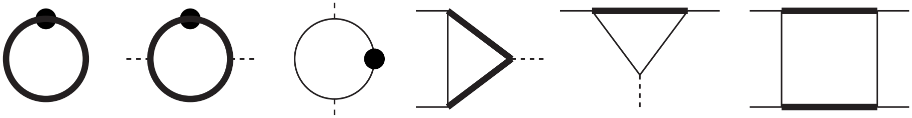
$$\eta_1 = x\,, \qquad \eta_2 = 1 + x\,, \qquad \eta_3 = y\,, \qquad \eta_4 = 1 - y\,, \qquad \eta_5 = x + y$$

One-mass DY MIs: 2-loop

- 1 extra letter
  $\eta_6 = x + y + xy$
- alphabet multilinear in $x, y \Rightarrow$ GPLs
- boundary conditions
  - regularity at pseudo-thresholds
  - zero momentum limits
  - direct integration
- analytic continuation straightforward $\Rightarrow$ complex (s, t, m^2)
- Checked against `SecDec` (Euclidean and in the physical regions)



Two-mass DY MIs: 1-loop



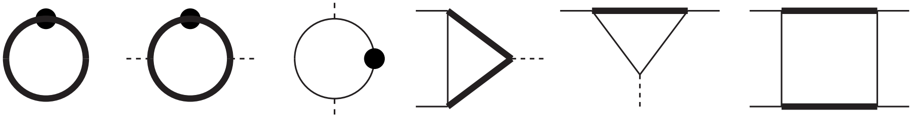
$(\mathcal{T}_1)$ $(\mathcal{T}_2)$ $(\mathcal{T}_3)$ $(\mathcal{T}_4)$ $(\mathcal{T}_5)$ $(\mathcal{T}_6)$

$$\mathrm{F}_1 = \epsilon\,\mathcal{T}_1\,, \qquad \mathrm{F}_2 = \epsilon\,\mathcal{T}_2\,, \qquad \mathrm{F}_3 = \epsilon\,\mathcal{T}_3\,,$$
$$\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, \qquad \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, \qquad \mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6$$

Canonical basis

$$\mathrm{I}_1 = \mathrm{F}_1\,, \qquad \mathrm{I}_2 = -s\sqrt{1-\frac{4m^2}{s}}\,\mathrm{F}_2\,, \quad \mathrm{I}_3 = -t\,\mathrm{F}_3\,,$$
$$\mathrm{I}_4 = -s\,\mathrm{F}_4\,, \quad \mathrm{I}_5 = -t\,\mathrm{F}_5\,, \qquad \mathrm{I}_6 = s\,t\sqrt{1-4\frac{m^2}{s}\left(1+\frac{m^2}{t}\right)}\,\mathrm{F}_6$$

Two-mass DY MIs: 1-loop



$(\mathcal{T}_1)$ $(\mathcal{T}_2)$ $(\mathcal{T}_3)$ $(\mathcal{T}_4)$ $(\mathcal{T}_5)$ $(\mathcal{T}_6)$

Four square roots appear

$$\sqrt{-s},\ \sqrt{4m^2 - s},\ \sqrt{-t},\ \text{and}\ \sqrt{1 - \frac{4\,m^2}{s}\left(1 + \frac{m^2}{t}\right)}$$

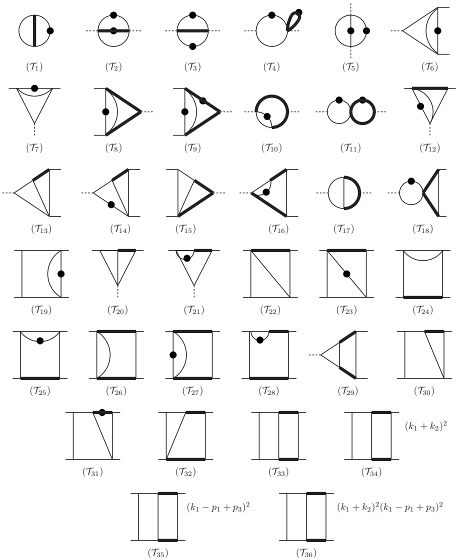
A change of variables gets rid of them

$$-\frac{s}{m^2} = \frac{(1-w)^2}{w}\,, \qquad -\frac{t}{m^2} = \frac{w}{z}\frac{(1+z)^2}{(1+w)^2}\,.$$

$$\eta_1 = z\,, \quad \eta_2 = 1+z\,, \quad \eta_3 = 1-z\,, \quad \eta_4 = w\,,$$
$$\eta_5 = 1+w\,, \quad \eta_6 = 1-w\,, \quad \eta_7 = z-w\,, \quad \eta_8 = z+w^2\,,$$

Two-mass DY MIs: 2-loop

- one extra sqrt $\sqrt{1+\frac{m^4}{t^2}-\frac{2\,m^2}{s}\left(1-\frac{u}{t}\right)}$
  - in DE for I_{32} at weight 3,4
  - in DEs for $I_{33,\dots,36}$ at weight 4
  - all the rest $\to$ GPLs
- boundary conditions
  - regularity at pseudo-thresholds
  - zero momentum limits
  - direct integration
- analytic continuation
  - straightforward for $I_{1,\dots,31}$
  - requires care for $I_{32,\dots,36}$
- checks against SecDec
  - $I_{1,\dots,31}$ (Eucl./phys.)
  - $I_{32,\dots,36}$ (Eucl.)



Mixed Chen-Goncharov representation for DY

Exploiting the recursive structure, the weight k coefficient of the MIs is

$$\mathbf{I}^{(k)}(\vec{x}) = \mathbf{I}^{(k)}(\vec{x}_0) + \int_0^1 \left[\frac{d\mathbb{A}(t)}{dt} \mathbf{I}^{(k-1)}(\vec{x}_t) \right] dt \,,$$

where $\vec{x}_t$ is the point $(x(t), y(t))$ along the curve identified by γ.

- Need weight-$(k-1)$ coefficient, which is independent of the path
- ☺ Rational alphabet → factorize over $\mathbb{C}$ → GPLs GiNaC
- ☹ In our case we have also square roots → path integration over GPLs
- ▶ Exploit IBP to perform always only 1 numerical path integration

$$\mathcal{C}^{[\gamma]}_{a|\vec{m}|\vec{n}} \equiv \int_0^1 g_a^\gamma(t)\, G^\gamma_{\vec{m}}(x)\, G^\gamma_{\vec{n}}(y)\, dt \,,$$

$$\mathcal{C}^{[\gamma]}_{a|\vec{m}|\varnothing} \equiv \int_0^1 g_a^\gamma(t)\, G^\gamma_{\vec{m}}(x)\, dt \,,$$

$$\mathcal{C}^{[\gamma]}_{a|\varnothing|\vec{n}} \equiv \int_0^1 g_a^\gamma(t)\, G^\gamma_{\vec{n}}(y)\, dt \,,$$

$$\mathcal{C}^{[\gamma]}_{a,\vec{b}|\vec{m}|\vec{n}} \equiv \int_0^1 g_a^\gamma(t)\, \mathcal{C}^{[\gamma t]}_{\vec{b}|\vec{m}|\vec{n}}\, dt \,,$$

where $G^\gamma_{\vec{m}}(x)$ and $G^\gamma_{\vec{n}}(y)$ stand for the GPLs $G_{\vec{m}}(x)$ and $G_{\vec{n}}(y)$ evaluated at $(x, y) = (\gamma^1(t), \gamma^2(t))$.

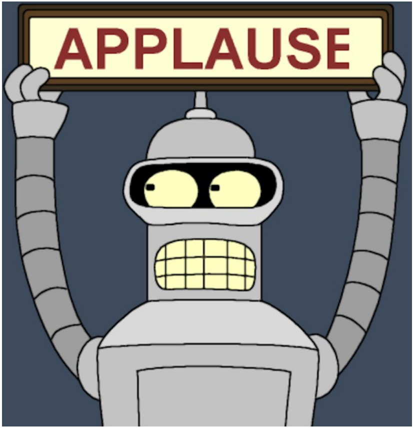
Summary and perspectives

- We computed the MIs for the virtual QCD$\times$EW two-loop corrections to the Drell-Yan scattering processes (for massless external particles)

$$q + \bar{q} \to l^- + l^+ \ , \qquad q + \bar{q}' \to l^- + \overline{\nu}$$

- We exploited $\Delta m^2 \equiv m_Z^2 - m_W^2 \ll m_Z^2$ to reduce the number of scales to 3
- We identified 49 canonical MIs (8 fully massless, 24 one-mass, 17 two-mass) with the help of the Magnus exponential
- The result is given as a Taylor series around $d = 4$ space-time dimensions in terms of iterated integrals up to weight four
- We adopted a mixed representation in terms of Chen-Goncharov iterated integrals, suitable for numerical evaluation.
- Future work:
  - Analytic continuation of Chen's iterated integrals
  - Optimization of numerical evaluation
  - Amplitudes and cross-section

(canonical)



Thanks for your attention!

- a generic matrix linear system of 1st order ODE

$$\partial_x Y(x) = A(x) Y(x)\ , \quad Y(x_0) = Y_0$$

- in the general non-commutative case, the Magnus theorem tells us that

$$Y(x) = e^{\Omega(x,x_0)}\ Y(x_0) \equiv e^{\Omega(x)}\ Y_0$$

- with $\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x)$ and

$$\begin{aligned}
\Omega_1(x) &= \int_{x_0}^{x} d\tau_1 A(\tau_1)\ , \\
\Omega_2(x) &= \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)] \\
\Omega_3(x) &= \frac{1}{6}\int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 [A(\tau_1),[A(\tau_2),A(\tau_3)]] + [A(\tau_3),[A(\tau_2),A(\tau_1)]] \\
& \dots
\end{aligned}$$

Relation with Dyson series [Blanes, Casas, Oteo and Ros 09]

Magnus $\leftrightarrow$ Dyson series. Dyson expansion of the solution Y in terms of the *time-ordered* integrals Y_n

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x)$$

$$Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \dots \int_{x_0}^{\tau_{n-1}} d\tau_n \, A(\tau_1)A(\tau_2)\cdots A(\tau_n) \, ,$$

Then

$$Y(x) = e^{\Omega(x)} Y_0 \quad \Rightarrow \quad \sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right)$$

and

$$\begin{aligned} Y_1 &= \Omega_1 \, , \\ Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2 \, , \\ Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3 \end{aligned}$$