g-2
Lattice status and prospects

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Precision Theory for Precise Measurements
Quy Nhon, Viet Nam
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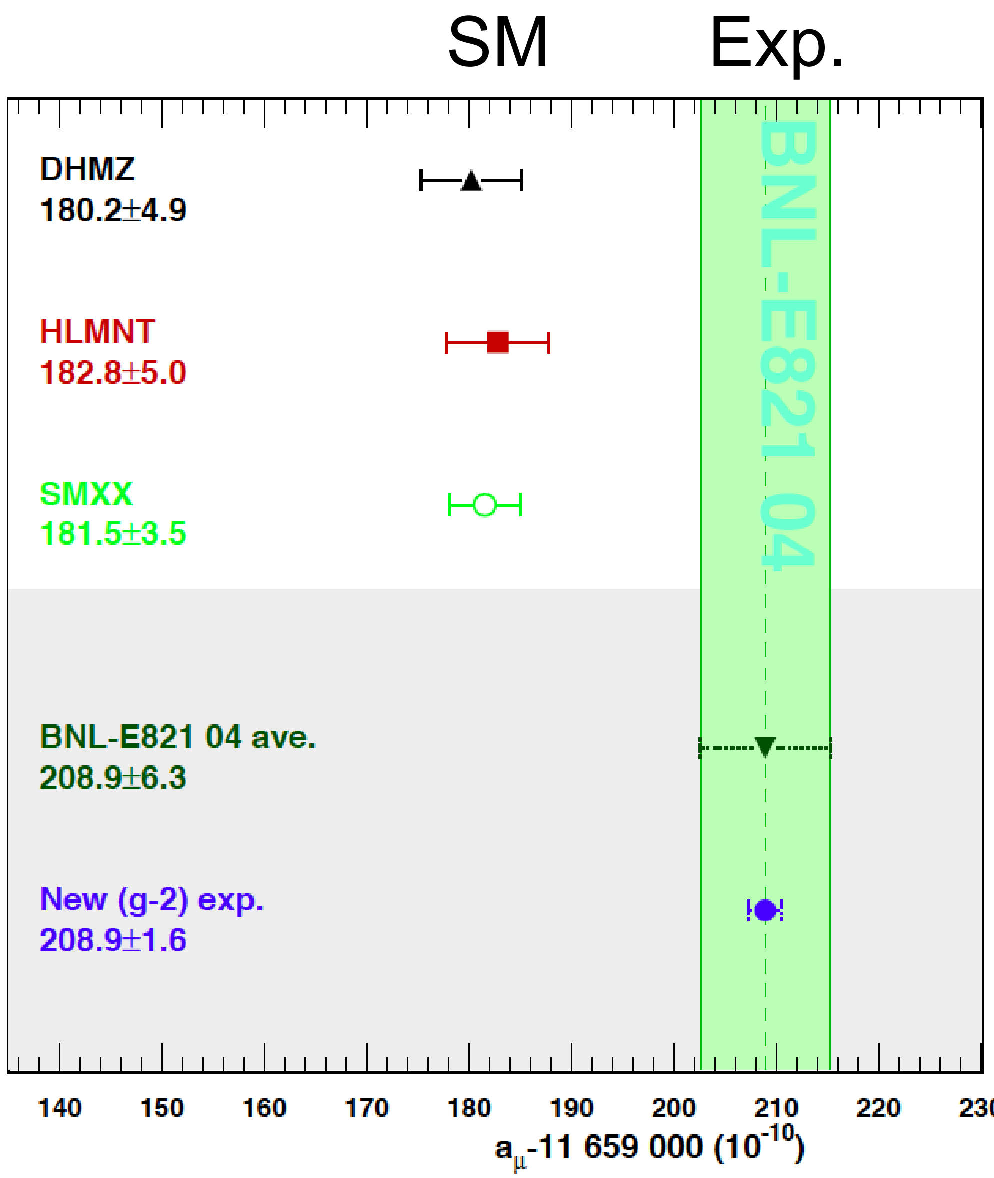
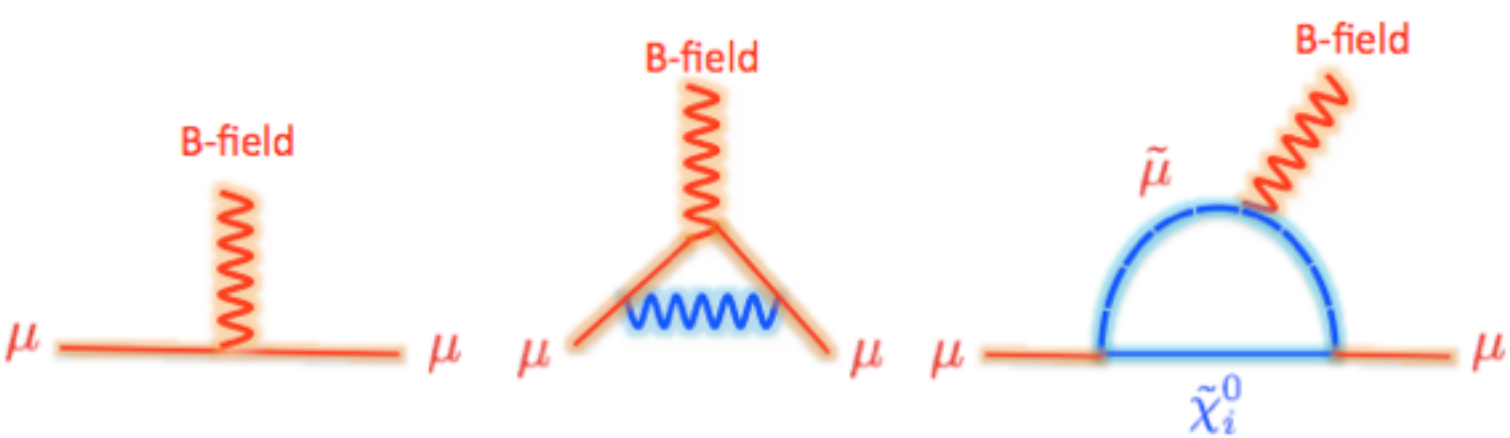
BNL-E821

Standard-model theory disagrees with the results of g-2 of the muon experiment BNL E821 by several σ.

Could be experimental or theoretical error, or it could be new physics.



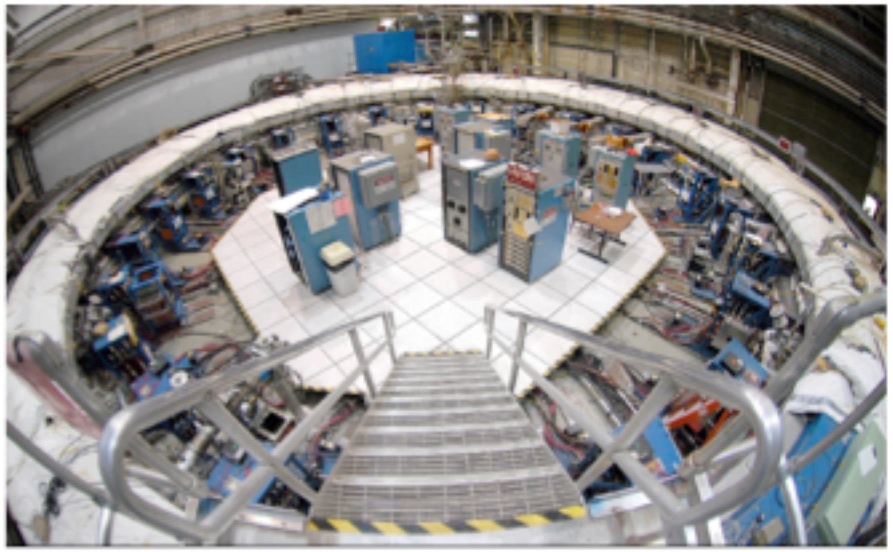


"Dirac Interaction"
g=2

QED "Schwinger Interaction"
(g-2)=α/π

BSM SUSY interaction

➜ Move to Fermilab; continue with more muons.



BNL Storage Ring during data-taking in 2001

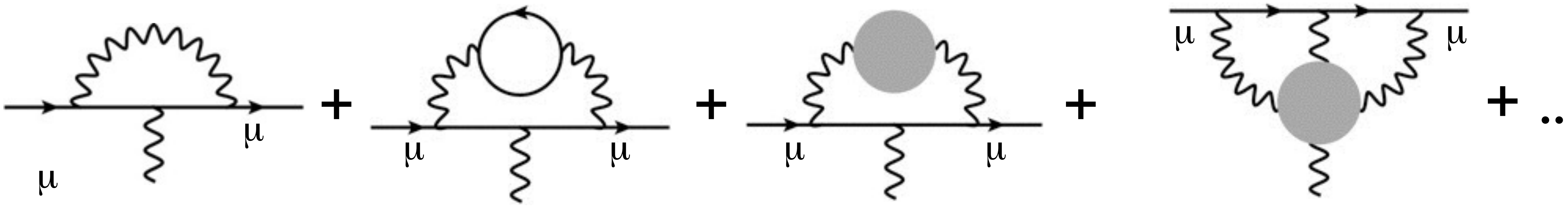
The magnet moves up the Mississippi

The magnet arrives at Fermilab

Muon g-2 in the Standard Model

Hadronic vacuum polarization (HVP):

Hadronic light-by-light (HLbL):



QED (4 loops) & EW (2 loops)

from experimental result for $e^+e^- \rightarrow$ hadrons plus dispersion relation

estimated from models such as large N_c, vector meson dominance, χPT, etc...

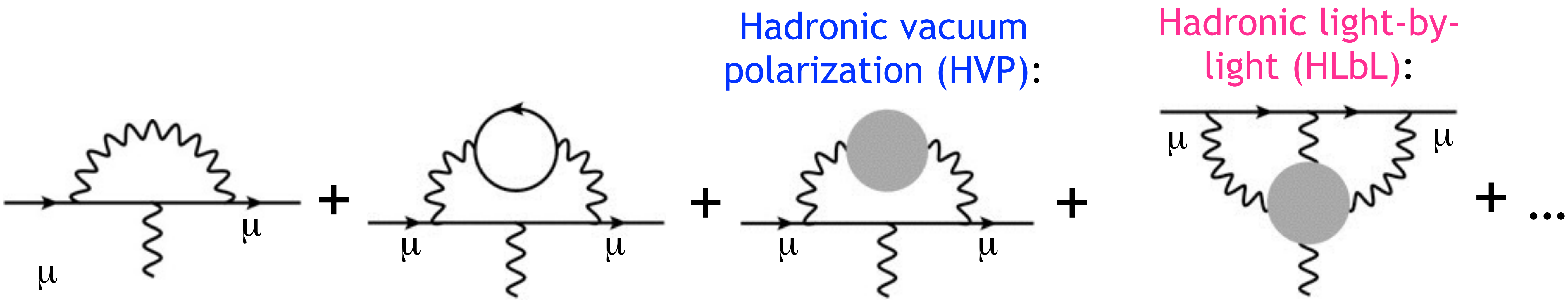
Contribution	Result ($\times 10^{11}$)		Error
QED (leptons)	$116\ 584\ 718 \pm 0.14$	$\pm 0.04_{\alpha}$	0.00 ppm
HVP(lo) [1]	$6\ 923 \pm 42$		0.36 ppm
HVP(ho)	$-98 \pm 0.9_{\text{exp}}$	$\pm 0.3_{\text{rad}}$	0.01 ppm
HLbL [2]	105 ± 26		0.22 ppm
EW	154 ± 2	± 1	0.02 ppm
Total SM	$116\ 591\ 802 \pm 49$		0.42 ppm

[1] Davier, Hoecker, Malaescu, Zhang, Eur.Phys.J. C71 (2011) 1515

[2] Prades, de Rafael, Vainshtein, 0901.0306

Thanks Ruth Van de Water for this and several other slides.

Experimental goals and lattice goals



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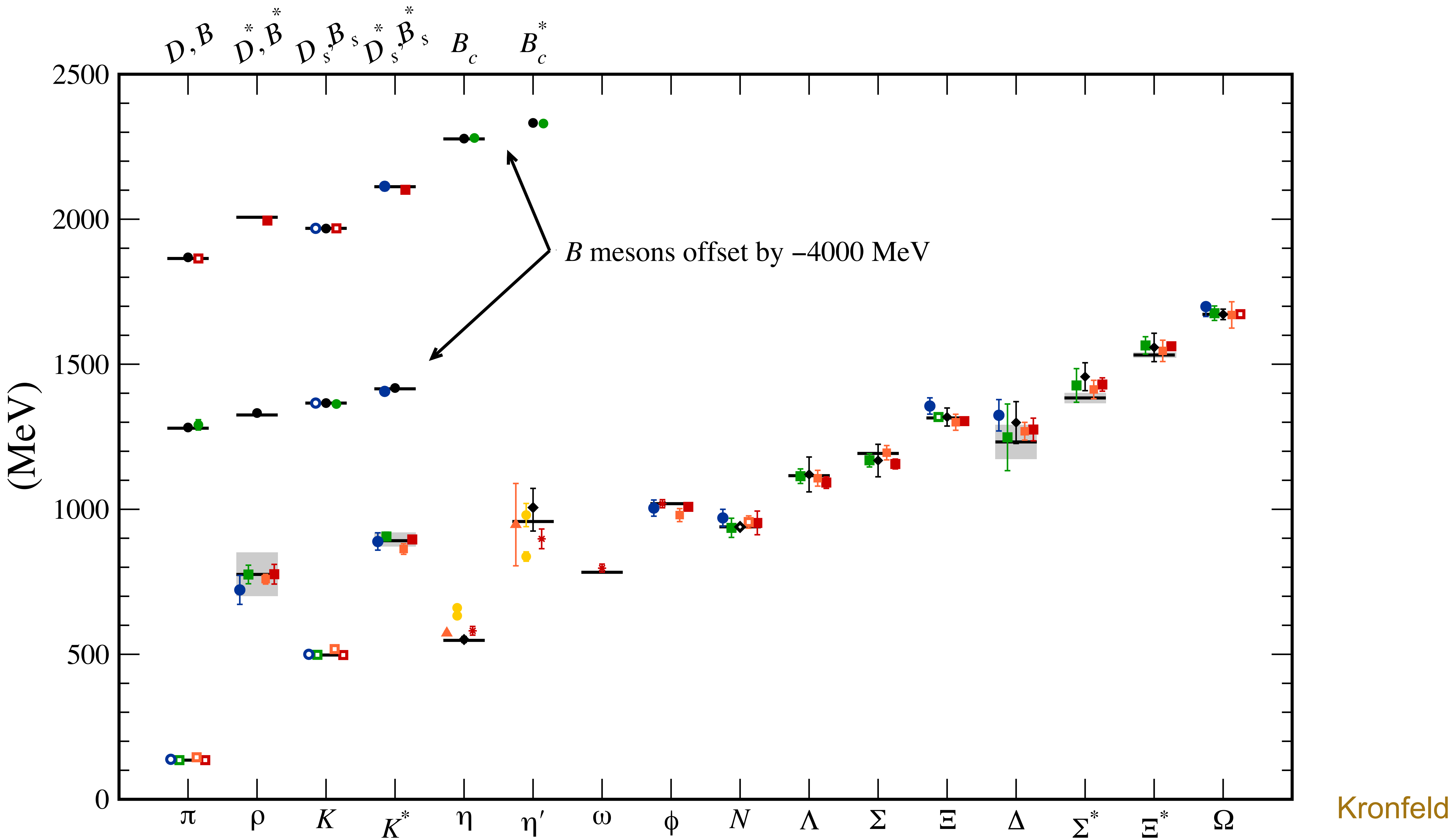
Uncertainties in hadronic corrections must be brought down to this level to fully capitalize on the experiment.

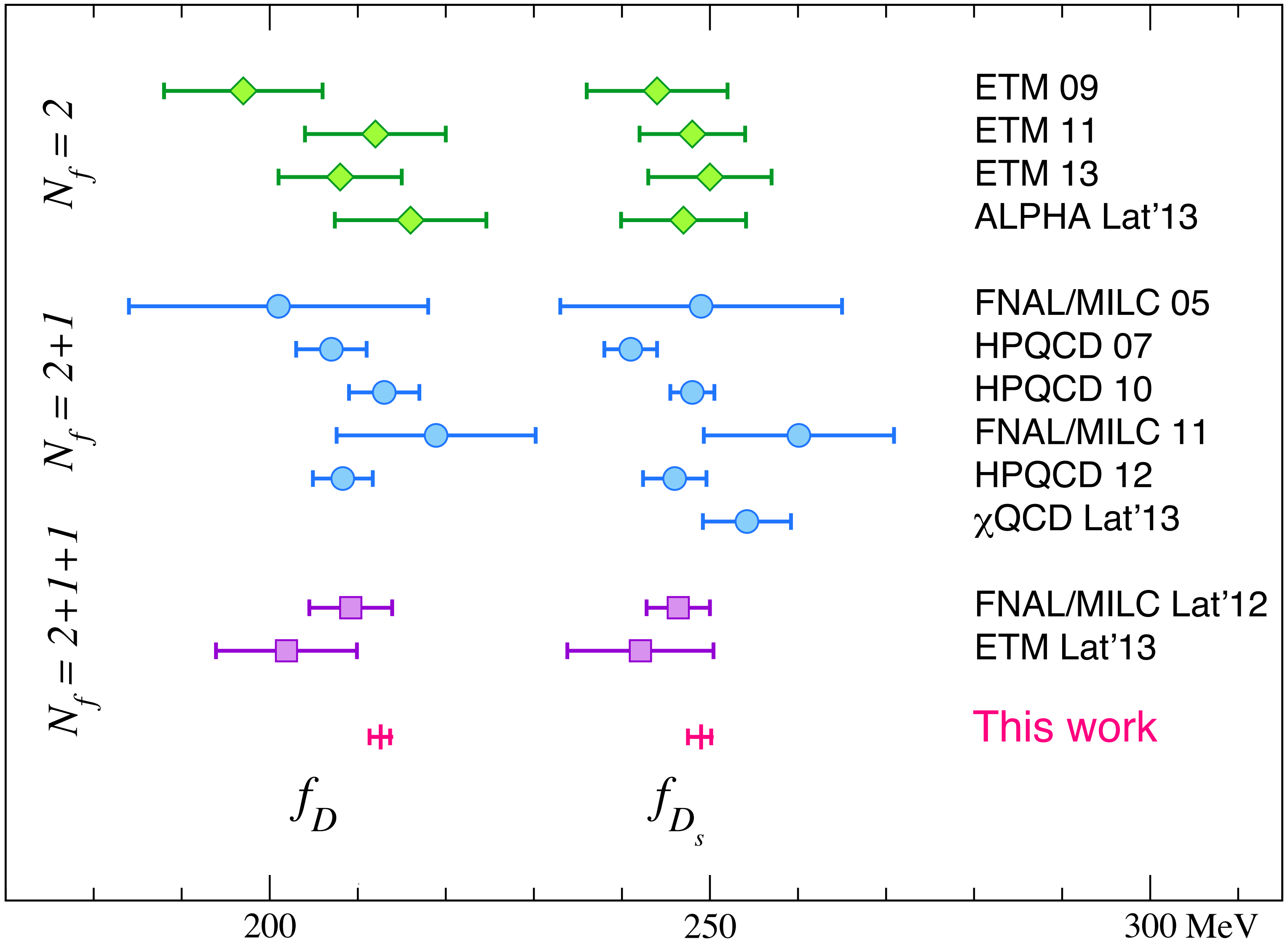
Uncertainty goal of Fermilab g-2 experiment: 0.14 ppm.

State of lattice QCD



For the last fifteen years, lattice calculations have been able to calculate the properties of sufficiently simple quantities with good understanding of the calculational uncertainties.

State of lattice QCD



Fermilab/MILC Phys. Rev. D90 (2014) no.7, 074509 arXiv:1407.3772v2.

$$f_{D^+} = 212.6(0.4)\left(^{+1.0}_{-1.2}\right)\ \text{MeV}$$ 0.5%

$$f_{D_s} = 249.0(0.3)\left(^{+1.1}_{-1.5}\right)\ \text{MeV}$$ 0.5%

$$f_{D_s}/f_{D^+} = 1.1712(10)\left(^{+29}_{-32}\right)$$ 0.3%

We're looking for 0.2% precision in muonic HVP.
That's more ambitious than we usually achieve so far, but some interesting quantities are sub-1% already.

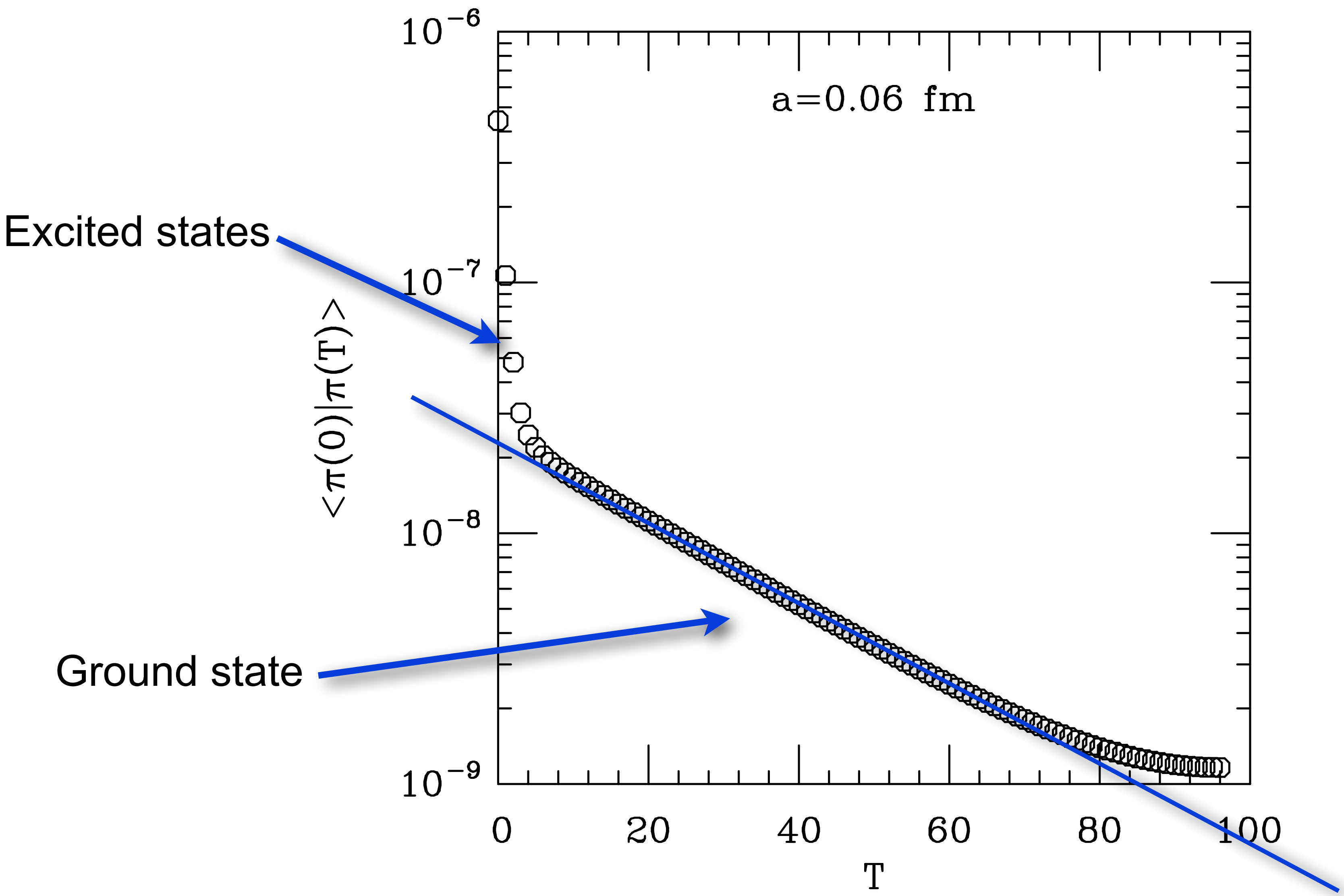
What is “simple”?

- Simplest: stable mesons.
  - Over the last ten years, many key quantities. Hadronically stable mesons, especially:
    - Heavy and light meson decay constants,
    - Semileptonic decays,
    - Meson-antimeson mixing.
  - Make possible important determinations of 8 CKM matrix elements, 5 quark masses, the strong coupling constant.
- Now: $\pi\pi$ systems, nucleons.

Coming US experimental program

- Next five years: lattice calculations are needed *throughout* the entire future US experimental program.
  - g-2.
  - LHCb, Belle-2: continued improvement of CKM results.
  - mu2e, LBNE, Nova: nucleon matrix elements.
  - Underground LBNE: proton decay matrix elements.
  - LHC, Higgs decays: lattice provides the most accurate α_s and m_c now, and m_b in the future.

How?



$$\langle \overline{\psi}\gamma_5\psi(t=0) \mid \overline{\psi}\gamma_5\psi(t)\rangle = C\exp(-Mt) + \text{excited states.}$$

If the two quarks were a u and a $\overline{u}$, the slope would give M_π, C would be proportional to F_π^2.

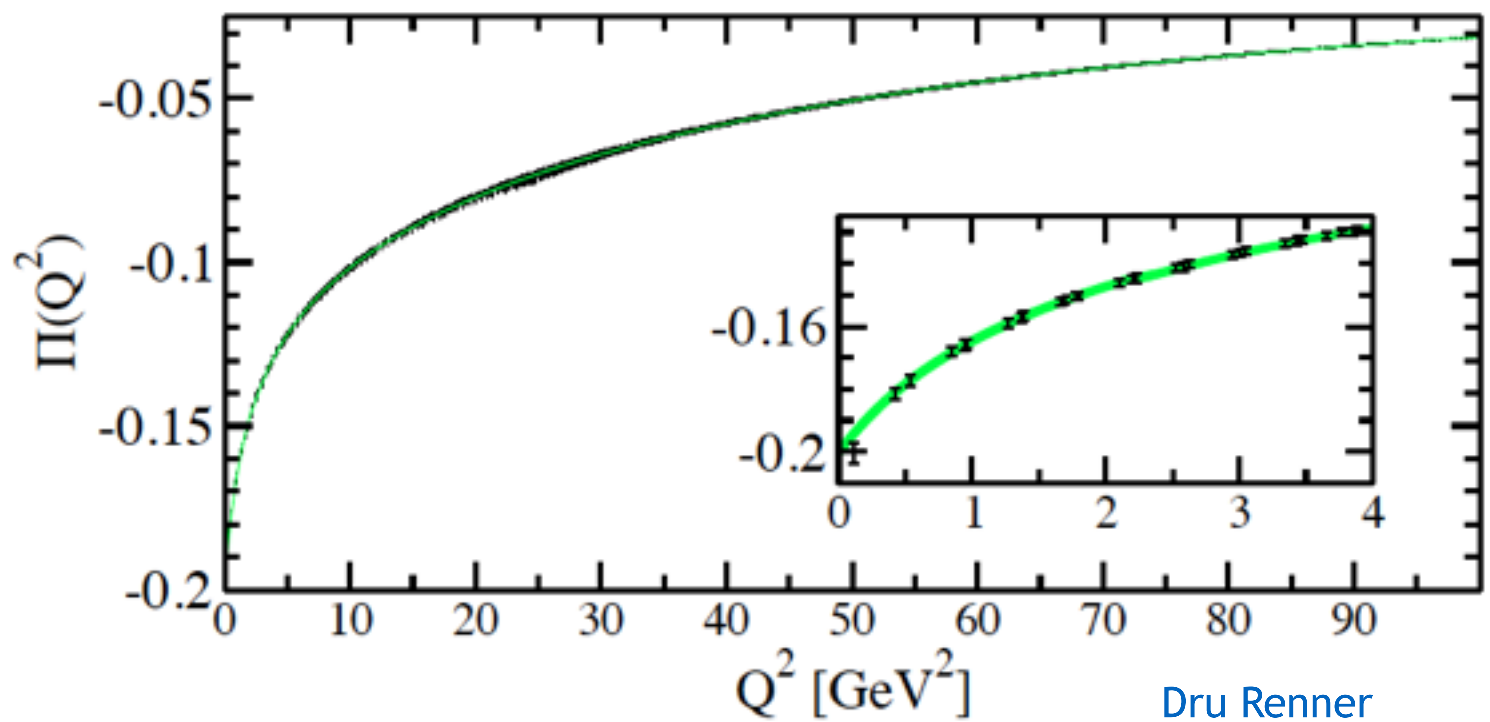
Vacuum polarization: general setup

- Calculate a_μ^{HVP} directly in from the Euclidean space vacuum polarization function:

$$a_\mu^{\mathrm{HVP(LO)}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2) \left[\Pi(Q^2) - \Pi(0)\right]$$

$$i\Pi_{\mu\nu}(q^2) =$$

- $\Pi(q^2)$ is a simple correlation function of two electromagnetic currents.
- In Euclidean space, $\Pi(q^2)$ has a smooth q^2 dependence with no resonance structure.



Dru Renner

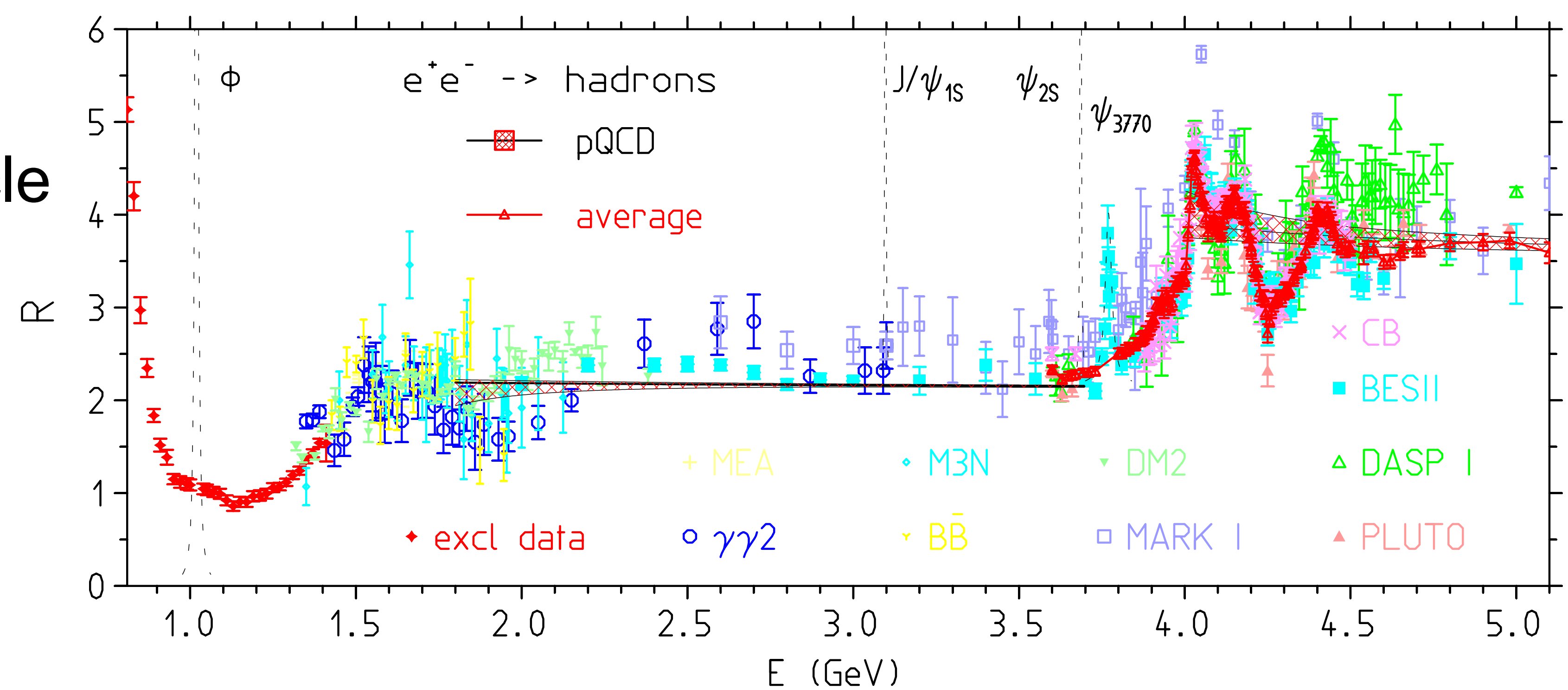
HVP from $e^+e^-\to$hadrons

- Standard-Model value for a_μ^{HVP} obtained from experimental measurement of
  $\sigma_{total}(e^+e^-\to\text{hadrons})$ via optical theorem:

$$a_\mu^{\rm HVP} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} {\rm d}s \frac{R(s)K(s)}{s^2} \qquad R \equiv \frac{\sigma_{\rm total}(e^+e^- \to {\rm hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

  - (Away from quark thresholds., use four-loop pQCD.)

- Includes >20 multi-particle channels with up to six final-state hadrons.
- Multi-hadron channels represent a small absolute contribution to a_μ^{HVP}, but contribute a significant fraction of the total uncertainty.



[Jegerlehner and Nyffeler, Phys.Rept. 477 (2009) 1-110]

Notable lattice work on vacuum polarization

- Blum, Phys. Rev. Lett. 91 (2003) 052001. Blum's formula.
- Aubin & Blum, Phys. Rev. D75 (2007) 114502.
- Feng et al., Phys. Rev. Lett. 107 (2011) 081802 .
- Hotzel et al., Lattice 2013.
- Boyle et al., Phys. Rev. D85 (2012) 074504.
- Della Morte et al., JHEP 1203 (2012) 055. Twisted BC.
- Aubin et al., Phys. Rev. D86 (2012) 054509. Pade approximants.
- RBC/UKQCD, PRL116, 232002 (2016). First disconnected diagrams.
- HPQCD, 1601.03071; PR D89, 114501 (2014). Moments method.

Hadronic vacuum polarization: Blum's formula

T. Blum, Phys. Rev. Lett. 91, 052001 (2003), hep-lat/0212018.

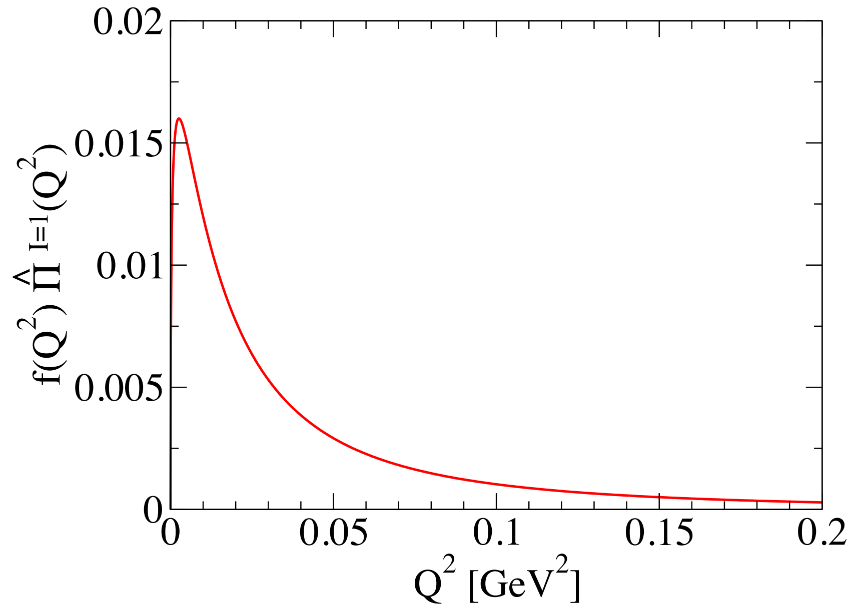
$$a_{\mu,\mathrm{HVP}}^{(\mathrm{f})} = \frac{\alpha}{\pi} \int_0^\infty dq^2 f(q^2)(4\pi\alpha Q_\mathrm{f}^2)\hat{\Pi}_\mathrm{f}(q^2) \qquad \hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$$

The four-dimensional integral yielding the hadronic vacuum polarization depends dynamically only on q^2.
The angular dependence of the kinematics factors may be integrated out with four-dimensional angular coordinates.
Result: *f* is given by:

$$f(q^2) \equiv \frac{m_\mu^2 q^2 A^3 (1 - q^2 A)}{1 + m_\mu^2 q^2 A^2}$$

$$A \equiv \frac{\sqrt{q^4 + 4m_\mu^2 q^2} - q^2}{2m_\mu^2 q^2}$$

Combined support of dynamical and kinematical factors maxes around m_μ^2, as expected.

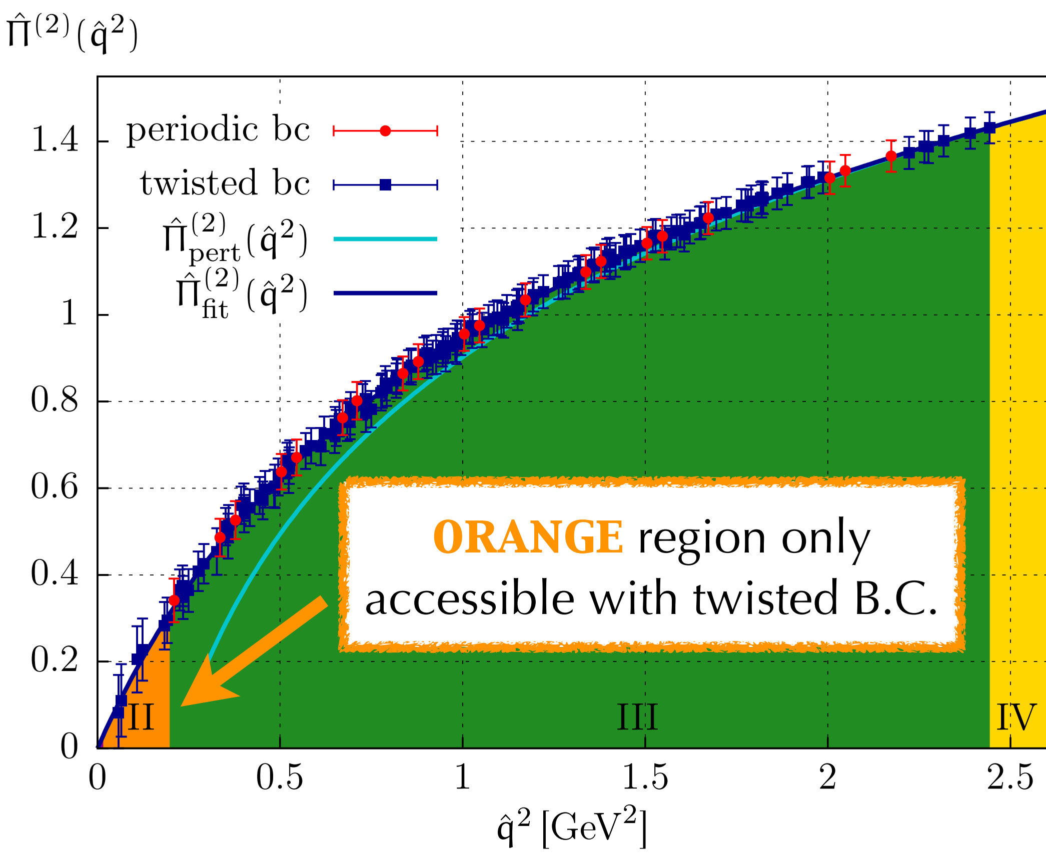


Golterman, Maltman, and Peris, PhysRevD. 90.074508, arXiv:1405.2389.

Algorithmic improvements

Twisted boundary conditions [Della Morte et al., JHEP 1203 (2012) 055].

- Because of finite spatial lattice size (volume=L^3), simulations with periodic boundary conditions can only access discrete momentum values in units of ($2\pi/L$) [**RED** points].
  - ➡ Lattice data sparse and noisy in low-q^2 region where contribution to a_μ^{HVP} is largest.
- Introduce twisted B.C. for fermion fields to access momenta below ($2\pi/L$) [**BLUE** points].



- **Padé approximants** [Aubin et al.,Phys.Rev. D86 (2012) 054509].
- Even with twisted B.C., contributions to a_μ^{HVP} from $\Pi(q^2)$ for momenta below the range directly accessible in current lattice simulations are significant.
  - Must assume functional form for q^2 dependence and extrapolate $q^2 \to 0$.
- Use model-independent fitting approach based on analytic structure of $\Pi(q^2)$ to eliminate systematic associated with vector-meson dominance fits.

Thanks Ruth Van de Water for this and several other slides.

Algorithmic improvements: the moments method

HPQCD, 1601.03071, PRD89, 114501 (2014)

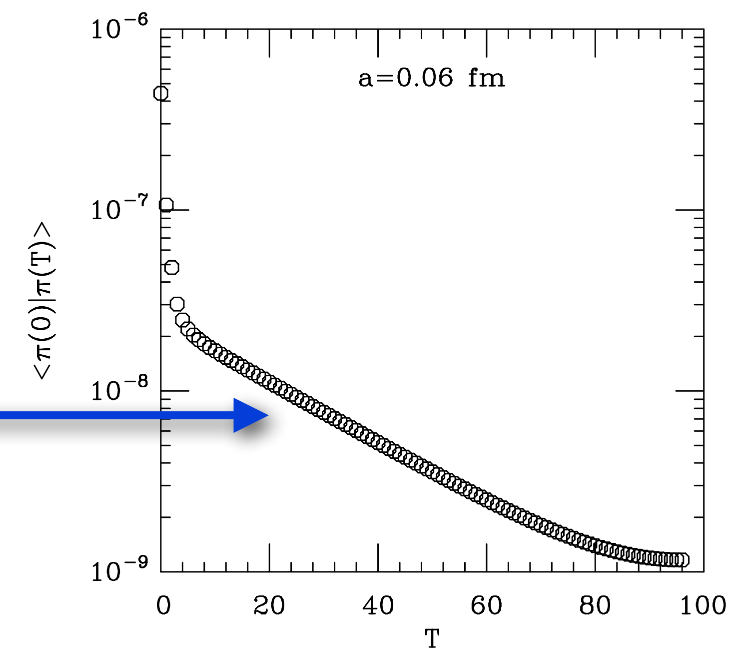
Subtractions required in the renormalized vacuum polarization function, yield numerically unstable expressions.
New approach: use spatial moments of the current-current correlates — can be obtained in terms of the derivatives in q^2, no subtractions necessary.

$$G_{2n} \equiv a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_V^2 \langle j^i(\vec{x},t) j^i(0)\rangle = (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2)\Big|_{q^2=0}.$$

$$\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$$

Very easy to calculate numerically to high precision.

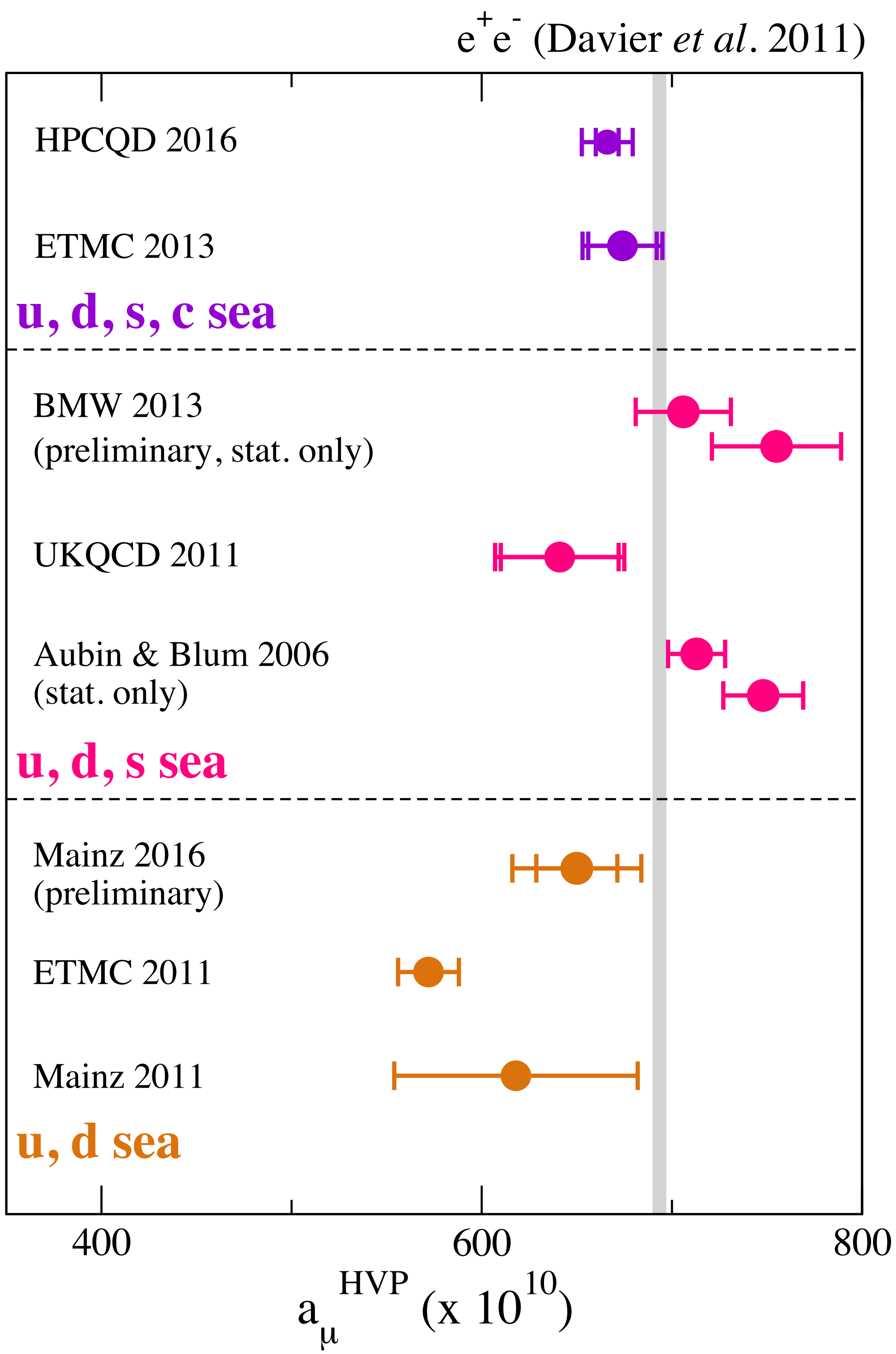
Just sum up the values of the correlation function weighted by t^{2n}.



Defining $$\hat{\Pi}(q^2) = \sum_{j=1}^{\infty} q^{2j} \Pi_j$$

We can write $$\Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}$$

to express a_μ in terms of the moments $$a_\mu = 4\alpha^2 \int_0^\infty d(q^2) f(q^2)[\Pi(q^2) - \Pi(q^2=0)]$$



- **Fermilab (*Van de Water lead*), HPQCD, & MILC collaborating on follow-up calculation which will reduce uncertainty** by increasing statistics & adding finer physical-mass ensemble.
- Sub-percent precision will require inclusion of isospin breaking & QED.

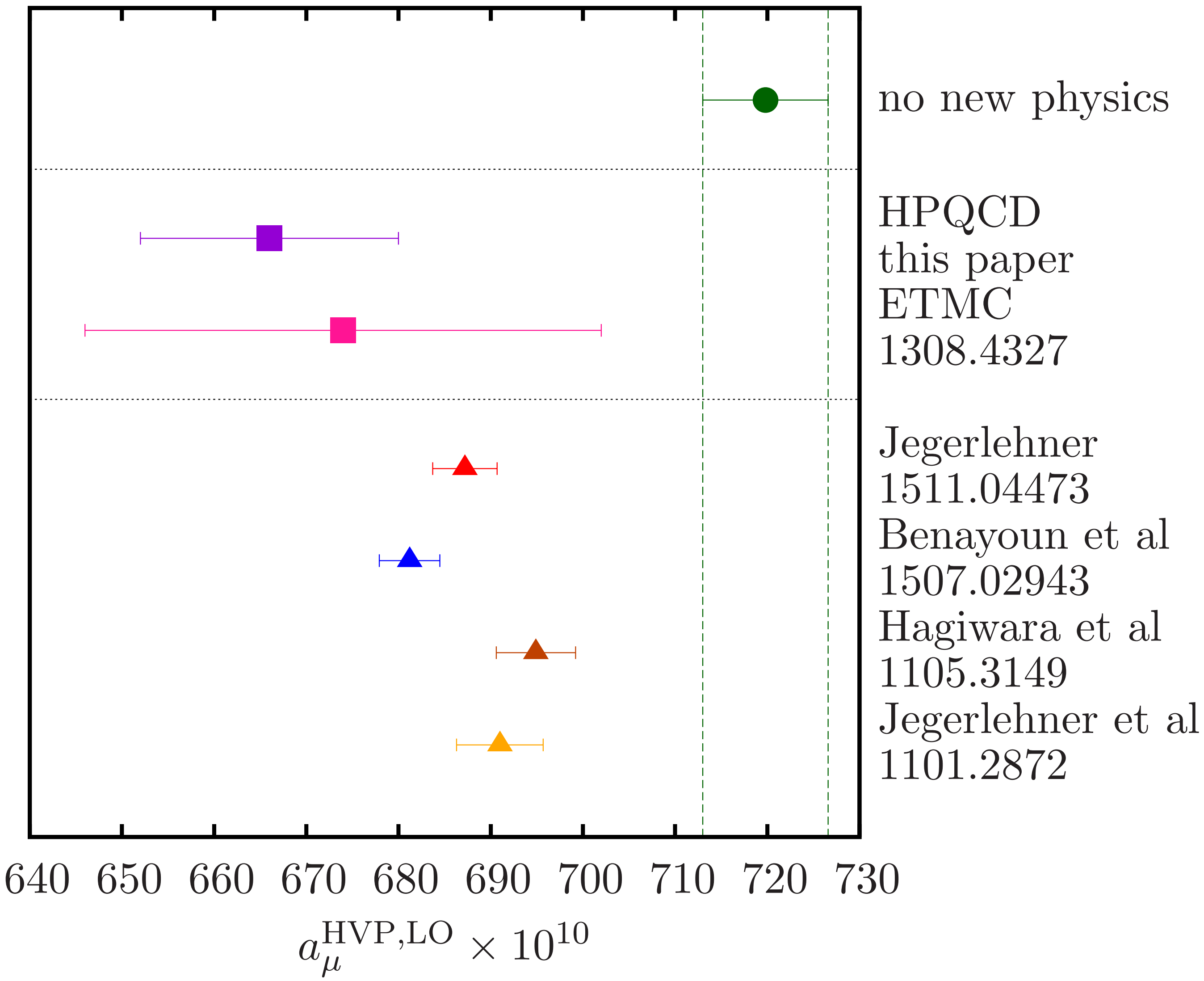


FIG. 5: Our final result for $a_\mu^{\mathrm{HVP,LO}}$ from lattice QCD compared to an earlier lattice result (also with u, d, s and c quarks) from the ETM Collaboration [13], and to results using experimental cross-section information [5–8]. We also compare with the result expected from the experimental value for a_μ assuming that there are no contributions from physics beyond the Standard Model.

VP uncertainty future

TABLE III: Error budget for the connected contributions to the muon anomaly a_μ from vacuum polarization of u/d quarks.

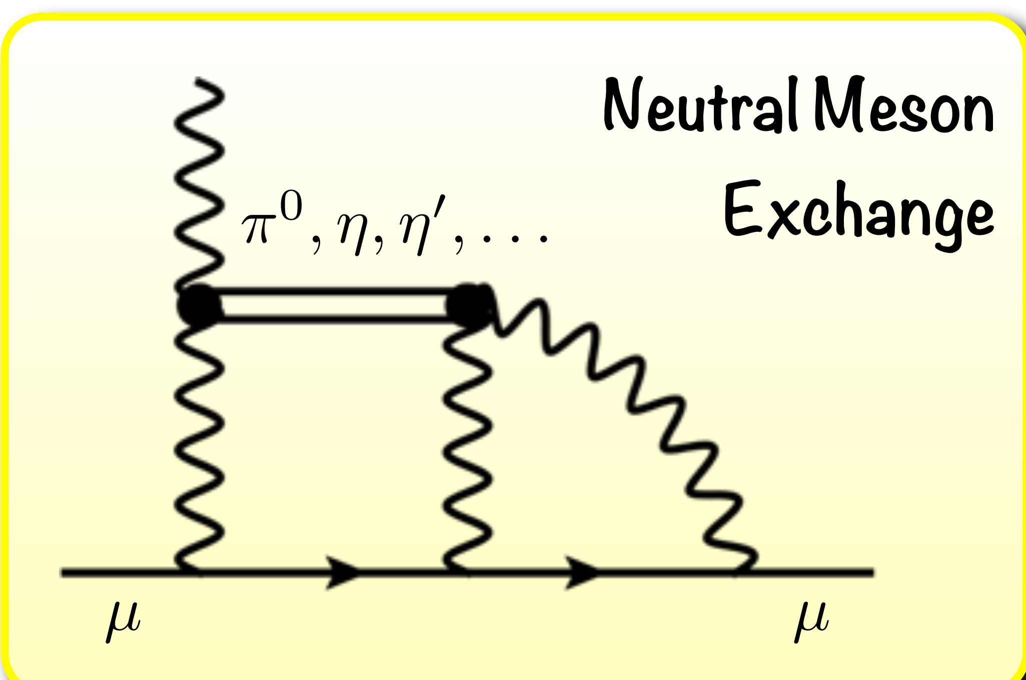
	$a_\mu^{\mathrm{HVP,LO}}(u/d)$
QED corrections:	1.0 %
Isospin breaking corrections:	1.0 %
Staggered pions, finite volume:	0.7 %
Valence m_ℓ extrapolation:	0.4 %
Monte Carlo statistics:	0.4 %
Padé approximants:	0.4 %
$a^2 \to 0$ extrapolation:	0.3 %
Z_V uncertainty:	0.4 %
Correlator fits:	0.2 %
Tuning sea-quark masses:	0.2 %
Lattice spacing uncertainty:	< 0.05 %
Total:	1.8 %

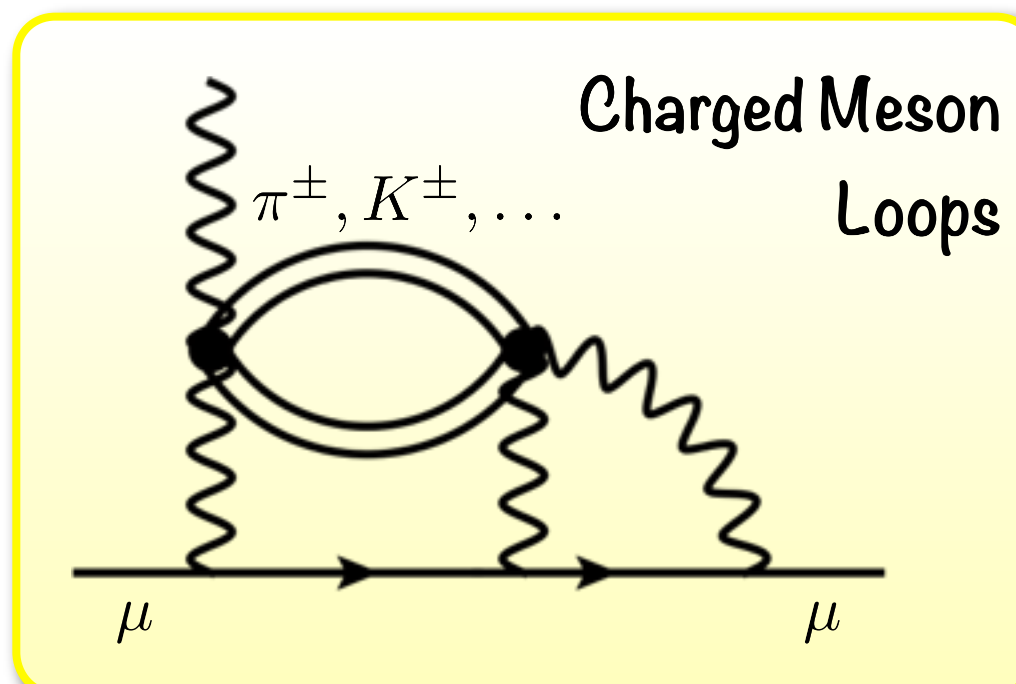
Uncertainty dominated by isospin breaking, EM.
These effects have not been included at all yet.
Including them should reduce these dramatically.

Remaining uncertainty from a half dozen effects, must be ground down by numerical brute force
— no magic bullet.

HLbL: estimation from hadronic models

- Hadronic light-by-light contribution cannot be expressed in terms of experimental quantities and must be obtained from theory [cf. Jegerlehner and Nyffeler, Phys.Rept. 477 (2009) 1-110 and Refs. therein]
  - All recent calculations compatible with constraints from large-N_c and chiral limits
  - All normalize dominant π^0-exchange contribution to measured $\pi^0 \to \gamma\gamma$ decay width
  - Differ for form factor shape due to different QCD-model assumptions such as vector-meson dominance, chiral perturbation theory, and the large N_c limit





The Glasgow consensus for HLbL

Prades, de Rafael, Vainshtein, 0901.0306, :0909.0953v1.

- Quoted error for a_μ^{HLbL} is based on model estimates, but does not cover spread of values.
  - π^0-exchange contribution estimated to be ~10 times larger than others.
  - Largest contribution to uncertainty ($\pm 1.9\times 10^{-10}$) attributed to charged pion and kaon loop contributions.

$$a^{\mathrm{HLbL}} = (10.5 \pm 2.6) \times 10^{-10}$$

Need $1.4\text{x}10^{-10}$ to match planned experimental precision.

➡ Error could easily be underestimated (and comparable to that from HVP!),
and is not systematically improvable.

Notable lattice work on light-by-light scattering

- Hayakawa et al., PoS LAT2005 (2006) 353; Blum et al., PoS LATTICE2012 (2012) 022; ... Propose dynamical photons method.
- Cohen et al., PoS LATTICE2008 (2008) 159.
- Feng et al., Phys.Rev.Lett. 109 (2012) 182001.
- Rakow, Lattice 2008.
- Blum et al., PRD93, 014503 (2016); L. Jin, Lattice 2016.

Hadronic LbL by brute force?

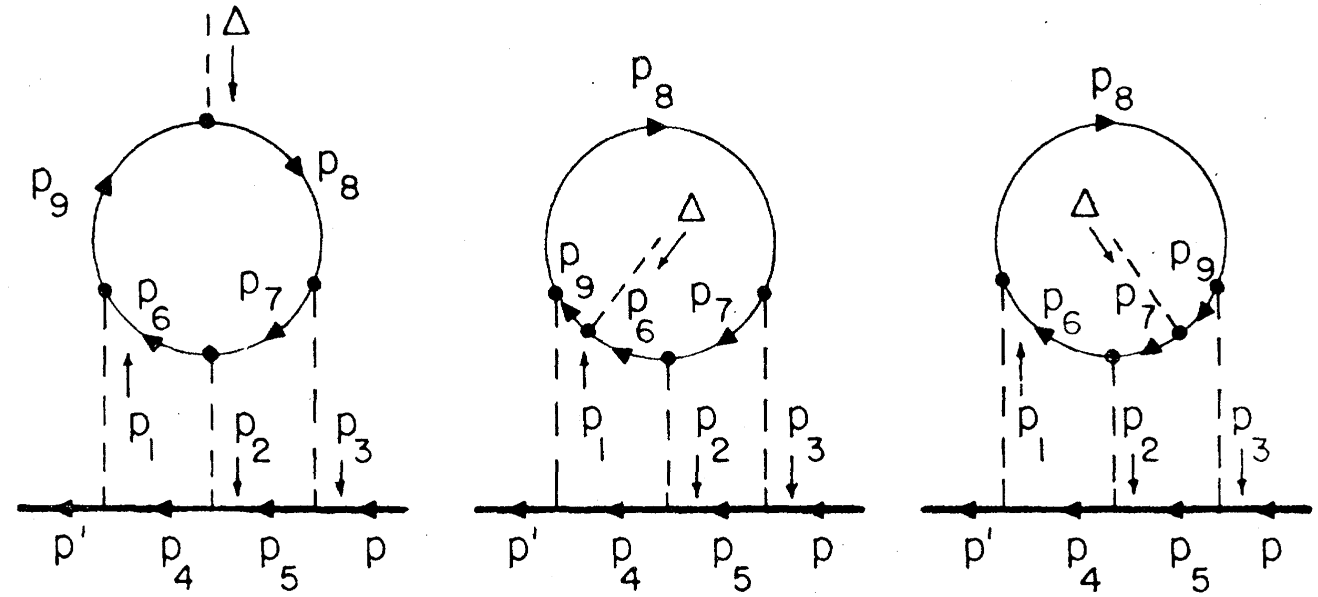


FIG. 1. Feynman diagrams containing subdiagrams of photon-photon scattering type. The heavy, thin, and dotted lines represent the muon, electron, and photon, respectively. There are three more diagrams obtained by reversing the direction of the electron loop.

Aldins, Brodsky, Durfner, & Kinoshita

$$M=\frac{e^2}{(2\pi)^8}\int d^4p_1 d^4p_3\, p_1{}^{-2}p_2{}^{-2}p_3{}^{-2}$$
$$\times \epsilon^\mu \Pi_{\kappa\rho\sigma\mu}(-p_1,\,p_2,\,p_3,\,-\Delta)\bar{u}(p')\gamma^\kappa(\not{p}_4-m_\mu)^{-1}$$
$$\times\gamma^\rho(\not{p}_5-m_\mu)^{-1}\gamma^\sigma u(p)\,,\quad (2.2)$$

Five exterior photon momenta to integrate. At each point, ~64 terms (e.g., photon gamma matrices).

To do LbL with ordinary methods naively takes orders of magnitude more CPU time than simpler calculations.

and $\Pi_{\kappa\rho\sigma\mu}$ is the polarization tensor of fourth rank representing the photon-photon scattering

$$\Pi_{\kappa\rho\sigma\mu}(-p_1,\,p_2,\,p_3,\,-\Delta)$$
$$=\frac{-ie^4}{(2\pi)^4}\int d^4p_6\,\mathrm{Tr}[\gamma_\kappa(\not{p}_6-m_e)^{-1}\gamma_\rho(\not{p}_7-m_e)^{-1}$$
$$\times\gamma_\sigma(\not{p}_8-m_e)^{-1}\gamma_\mu(\not{p}_9-m_e)^{-1}$$
$$+(\text{five other terms})-(\text{regularization terms})].\ (2.3)$$

Replace with LQCD calculation.

Dynamical photon method

Hayakawa et al., PoS LAT2005 (2006) 353

- Method introduced by Blum and collaborators in which one computes the full hadronic amplitude, including the muon and photons, nonperturbatively.
- Treat photon field in parallel with gluon field and include in gauge link, so the simulation and analysis follows a conventional lattice-QCD calculation.
- In practice, must insert a single valence photon connecting the muon line to the quark loop “by hand” into the correlation function, then perform correlated nonperturbative subtraction to remove the dominant $O(\alpha^2)$ contamination.

Mixed dynamical and analytic photons…

Blum et al., PRD93, 014503 (2016)

- **New method combines dynamical QCD gauge-field configurations with exact analytic formulae for photon propagators.**
  - Exploits stochastic methods for position-space sums to control computational cost.
  - Obtain **≲10% statistical errors at the physical pion mass** in ballpark of Glasgow consensus value $a_\mu^{\mathrm{HLbL,GC}} \times 10^{10} = 10.5(2.6)$.

$$a_\mu^{\mathrm{HLbL}} \times 10^{10} = \begin{cases} 11.60(0.96)_{\mathrm{stat.}} & \text{connected} \\ -6.25(0.80)_{\mathrm{stat.}} & \text{disconnected} \end{cases}$$

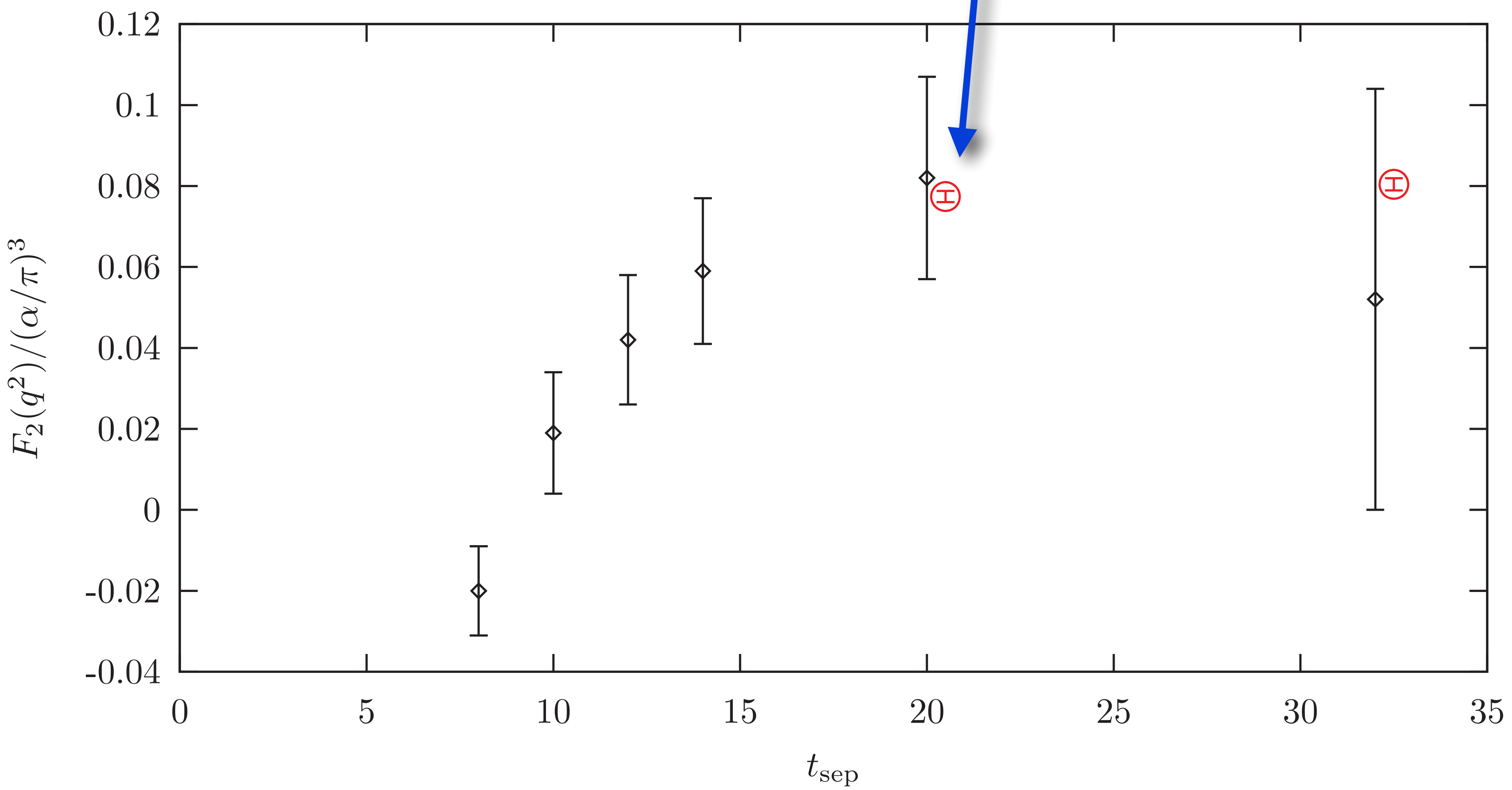
Statistical errors at least are in the ballpark required.

L. Jin, Lattice 2016; preliminary update of Blum *et al*.

- Full study of systematic errors including lattice-spacing and finite-volume effects still needed — dynamical photons have power-law volume corrections instead of exponential in the pion mass (the usual case). **Initial results encouraging!**

… yield a great improvement in statistics

Statistical precision of new method (red) is an order of magnitude better than the previous method (black).



Summary

- Lattice calculations of light by light and vacuum polarization diagrams for muon g-2 are making rapid progress,
- but work remains to be done.
  - Hadronic vacuum polarization is needed to 0.2%, better than anything achieved so far by lattice.
  - Hadronic light-by-light requires new methods which are under development.
- Precision demands on theory from experiment are daunting — it's a race to see if theory will deliver everything experiment needs by 2018, the year of first results from the new experiment.