Chaos Control in Nonlinear Dynamical Systems

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The presence of chaos is an integral part of the majority of nonlinear dynamical systems, which describe complex physical, chemical, biological and social processes and phenomena.

We consider the method of localization and stabilization of unstable singular points and periodic solutions of discrete and continues chaotic dynamical systems.

The idea of the method consists in construction of a certain dynamical system in the space of the greater dimension, for which the assigned unstable point (periodic trajectory) of the initial chaotic system is the projection of it's certain asymptotically stable point (orbitally stable periodic trajectory).

Localization and stabilization of unstable fixed points and unstable cycles of chaotic mappings

Let us consider a family of m-dimensional nonlinear smooth mappings

$$z_{n+1} = F(z_n, \mu), \quad z \in \mathbb{R}^m, \tag{6.8}$$

where μ is a scalar parameter and F is a smooth vector-function. Let points $z_1^*(\mu)$, $z_2^*(\mu)$, ..., $z_k^*(\mu)$ be k-periodic points of the mapping (6.8), for which equalities

$$z_2^*(\mu) = F(z_1^*(\mu), \mu),\ z_3^*(\mu) = F(z_2^*(\mu), \mu), \ldots,\ z_1^*(\mu) = F(z_k^*(\mu), \mu) \tag{6.9}$$

take place. If $k = 1$, then point $z_1^*(\mu)$ is the fixed point of the mapping (6.8). Usually there is a critical value μ_k^* of the system parameter such that the cycle (6.9) is a stable periodic trajectory of the mapping (6.8) in the domain $\mu \leq \mu_k^*$, whereas in the domain $\mu > \mu_k^*$ the cycle (6.9) is an unstable periodic trajectory of the mapping (6.8), which has in this case other regular or singular attractors.

Let us consider a $(m+1)$-dimensional mapping

$$\begin{aligned} z_{n+1} &= F^k(z_n, \mu) + \varepsilon(q_n - \mu), \\ q_{n+1} &= Q(z_n, \mu) + \beta(q_n - \mu) + \mu, \end{aligned} \tag{6.10}$$

where $\varepsilon \in \mathbb{R}^m$, $\beta \in \mathbb{R}$, $Q(z_i^*(\mu), \mu) = 0$, $i = 1, \ldots, k$. Clearly, that if the cycle (6.9) is a periodic trajectory of the mapping (6.8), then each point $(z_i^*(\mu), \mu)$, $i = 1, \ldots, k$, is a fixed point of the mapping (6.10). Let us calculate the Jacobi matrix of the mapping (6.10) at the point $(z_i^*(\mu), \mu)$

$$J(\mu) = \begin{pmatrix} \partial F^k/\partial z & \varepsilon \\ \partial Q/\partial z & \beta \end{pmatrix}.$$

All eigenvalues of the matrix $J(\mu)$ at the point μ_k^* are equal to zero if and only if

$$\sum_{i=1}^{C_{m+1}^l} J_{li}(\mu_k^*) = 0, \quad l = 1, \ldots, m+1, \tag{6.11}$$

Theorem 6.1 *If the determinant D of system of linear Eqs.* (6.11) *is not equal to zero, then there exists a value $\mu_{k1}^* > \mu_k^*$ of the system parameter such that for each $\mu \in [\mu_k^*, \mu_{k1}^*]$ the point $(z_i^*(\mu), \mu)$ is an asymptotically stable fixed point of the mapping* (6.10) *and can be localized and stabilized in the domain of $\mu \in [\mu_k^*, \mu_{k1}^*]$ by means of the iterative process* (6.10) *with initial conditions $q_0 = \mu$, $z_0 = z_i^*(\mu_k^*)$. Any other k-periodic point $z_j^*(\mu)$, $j \neq i$ of the mapping* (6.8) *also can be localized and stabilized in the domain of $\mu \in [\mu_k^*, \mu_{k1}^*]$ by means of the Eq.* (6.9).

In particular, mapping $Q(z, \mu)$ in (6.10) can be chosen in the form of

$$Q(z, \mu) = a_1(F_1^k(z, \mu) - z_1) + \cdots + a_m(F_m^k(z, \mu) - z_m),$$

where parameters a_i, $i = 1, \ldots, m$, are set to either zero or unity so as to satisfy the condition $D \neq 0$ for the determinant D of the linear system (6.11).

Example 6.1 By way of example, we consider the logistic mapping

$$z_{n+1} = \mu z_n(1 - z_n), \quad 3 < \mu \leq 4. \tag{6.13}$$

The fixed point $z^*(\mu) = 1 \quad 1/\mu$ of the mapping (6.13) is an unstable stationary point in the domain $\mu > \mu^* = 3$. Moreover, if $\mu > 3.57$, then the mapping (6.13) exhibits a chaotic behavior. Numerical experiments showed, that the fixed point $z^*(\mu)$ of the mapping (6.13) can be localized and stabilized at the intervals $3 \leq \mu \leq 3.6$ with the help of iterative process (6.10) with $\beta = -F_z = 1$ and $\varepsilon = 0.5$, calculated at the point $\mu^* = 3$. Then this fixed point can be localized and stabilized at the interval $3.6 \leq \mu \leq 4.25$ with parameters $\beta = 1.6$ and $\varepsilon = 1$, calculated at the point $\mu_1^* = 3.6$.

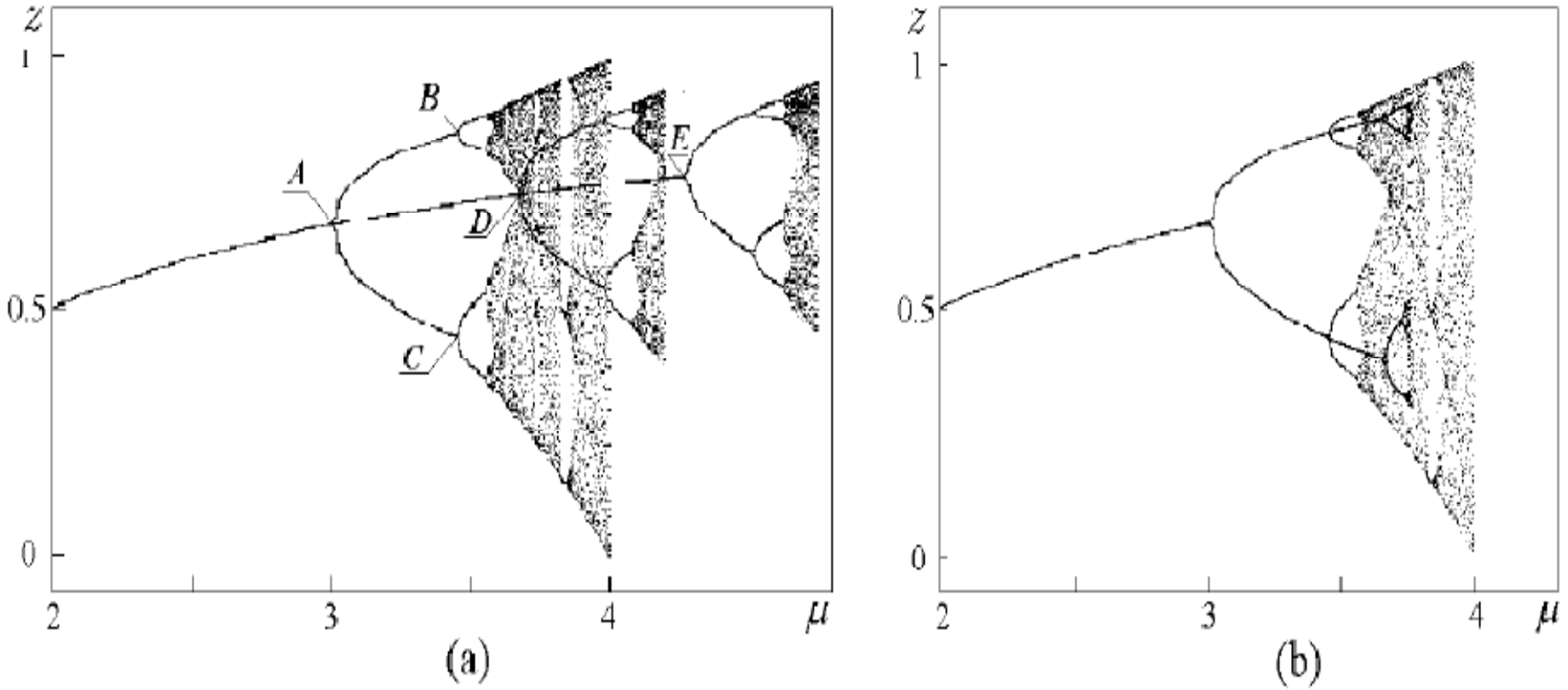

Fig. 6.1 Bifurcation diagram of logistic mapping (6.13) and the system (6.10) stabilizing it for $k = 1(a)$ and for $k = 2(b)$. Points A, B, C correspond to the period doubling bifurcations of the mapping (6.13), and points D, E correspond to the period doubling bifurcations of the stabilizing mapping (6.10). Parts of a trajectory $A-D$ and $D-E$ are obtained at the first and second iterations of the process of stabilization of the mapping (6.10) accordingly at intervals $3 \leq \mu \leq 3.6$ and $3.6 \leq \mu \leq 4.25$.

Stabilizing system in the case of 2d mappings

$$
\begin{aligned}
x_{n+1} &= f_1^k + \varepsilon_1(q_n - \mu), \\
y_{n+1} &= f_2^k + \varepsilon_2(q_n - \mu), \\
q_{n+1} &= a_1[f_1^k - x_n] + a_2[f_2^k - y_n] + \beta(q_n - \mu) + \mu,
\end{aligned}
\tag{6.15}
$$

Example 6.2 As an example we shall consider the Henon mapping

$$
\begin{aligned}
x_{n+1} &= \mu + 0.3y_n - x_n^2, \\
y_{n+1} &= x_n.
\end{aligned}
\tag{6.16}
$$

The fixed point $x^*(\mu) = y^*(\mu) = -0.35 + \sqrt{0.1225 + \mu}$ of the mapping (6.16) is an unstable stationary point in the domain $\mu > \mu^* = 0.3675$. For $\mu = \mu^*$ any of eigenvalues of the Jacobian of the mapping (6.16) is not

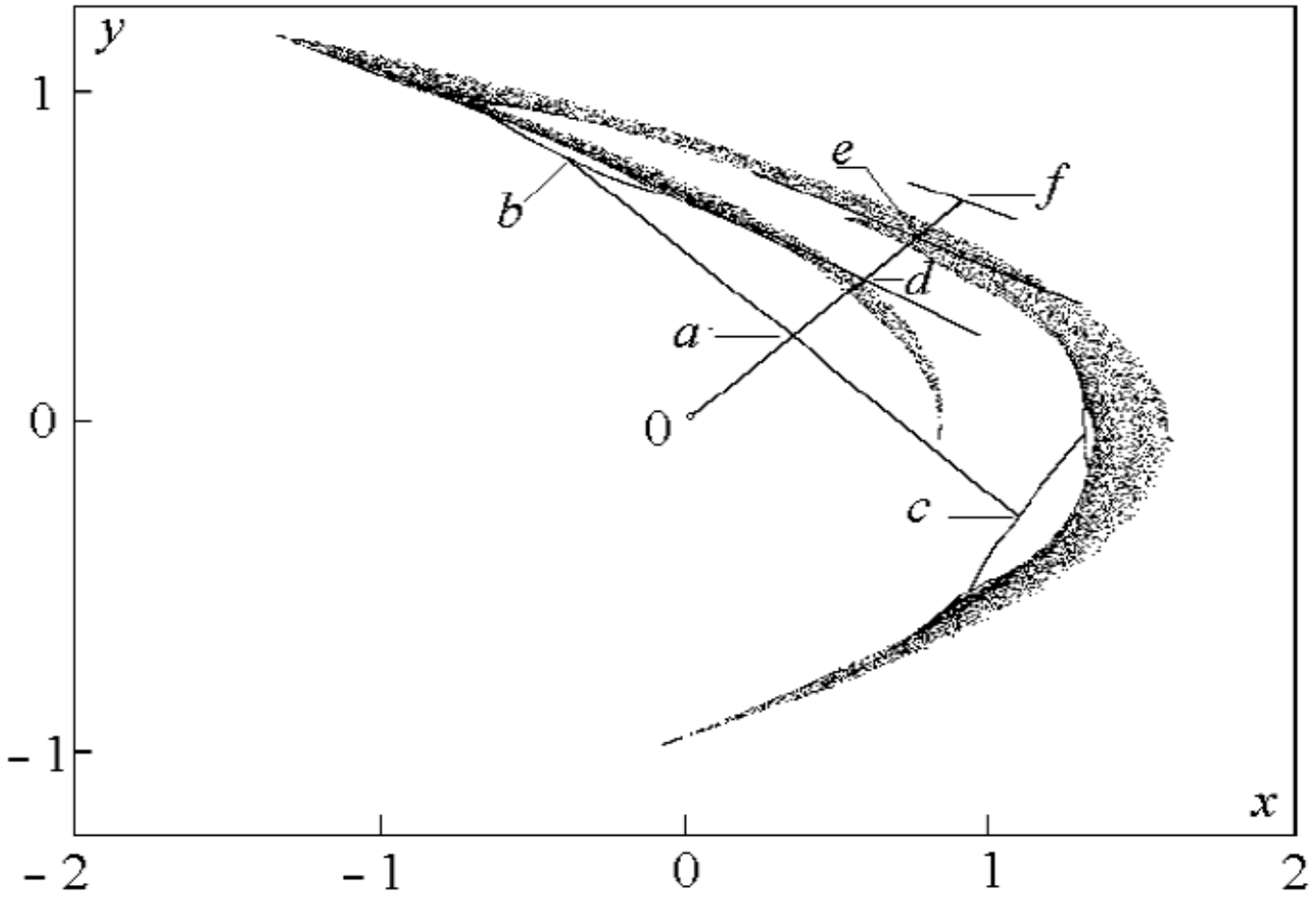

Fig. 6.2 Two-dimensional diagram of the Henon mapping (6.16) and its stabilizing system (6.15). The points a, b, c correspond to period doubling bifurcations of the mapping (6.16) and the points d, e, f correspond to period doubling bifurcations of the stabilizing mapping (6.15).

Localization and stabilization of unstable fixed points of chaotic dynamical systems

Let us consider a nonlinear dynamical system

$$\dot{x} = F(x, \mu), \quad x \in \mathbb{R}^m, \tag{6.20}$$

given by the family F of smooth mappings. Let $x^*(\mu)$ be a fixed point of system (6.20), where μ is a scalar parameter. Just as in Sec. 6.2.1, we assume that there exists a critical value μ^* of the system parameter such that point $x^*(\mu)$ is the stable fixed point of system (6.20) for $\mu \leq \mu^*$, and is an unstable fixed point for $\mu > \mu^*$.

Let us consider the $(m+1)$- dimensional dynamical system

$$\begin{aligned} \dot{x} &= F(x, \mu) + \varepsilon(q - \mu), \\ \dot{q} &= Q(x, \mu) + \beta(q - \mu), \end{aligned} \tag{6.21}$$

where $Q(x^*(\mu), \mu) = 0$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)^{\mathrm{T}}$ and $\beta \in \mathbb{R}$ are contro's

Let us calculate the Jacobian

$$J(\mu) = \begin{pmatrix} \partial F/\partial x & \varepsilon \\ \partial Q/\partial x & \beta \end{pmatrix}_{x=x^*(\mu),\, q=\mu}.$$

$\varepsilon_1, \ldots, \varepsilon_m$, β should satisfy the system of $m+1$ linear algebraic equations

$$\sum_{i=1}^{C_{m+1}^k} J_{ki}(\mu^*) = C_{m+1}^k d^k, \quad k = 1, \ldots, m+1. \tag{6.23}$$

Theorem 6.4 *If the determinant D of the system of linear Eqs. (6.23) is nonzero, then there exists a domain $\mu^* \le \mu \le \mu_1^*$ such that for any $\mu \in [\mu^*, \mu_1^*]$ real parts of all eigenvalues of matrix $J(\mu)$ are negative, and the point $(x^*(\mu), \mu)$ is an asymptotically stable fixed point of system (6.21). Hence, the fixed point $x^*(\mu)$ of the dynamical system (6.20) can be localized and stabilized in the domain $\mu \in [\mu^*, \mu_1^*]$ with the help of system (6.21) with initial conditions $x_0 = x^*(\mu^*)$, $q_0 = \mu$.*

Example 6.4 To illustrate the method, we consider the Rossler chaotic dynamical system

$$\begin{aligned} \dot{x} &= -(y+z), \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x-\mu). \end{aligned} \tag{6.24}$$

We localize and stabilize a fixed point of the system (6.24) with the help of the system

$$\begin{aligned} \dot{x} &= -(y+z) + \varepsilon_1(q-\mu), \\ \dot{y} &= x + ay + \varepsilon_2(q-\mu), \\ \dot{z} &= b + z(x-\mu) + \varepsilon_3(q-\mu), \\ \dot{q} &= -a_1(y+z) + a_2(x+ay) + a_3\big(b + z(x-\mu)\big) + \beta(q-\mu), \end{aligned} \tag{6.25}$$

The system (6.25) enables to localize and stabilize a fixed point of the system (6.24) at the interval $\mu^* \le \mu \le 4$ for values of parameters $\beta = -5$, $\varepsilon_1 = -1$, $\varepsilon_2 = 7/3$, $\varepsilon_3 = 1/6$, calculated at the point μ^*. The interval of localization and stabilization covers all domain of chaotic behaviour of the Rossler system.

Localization and stabilization of unstable cycles of chaotic dynamical systems

Let us consider a nonlinear dynamical system

$$\dot{x} = F(x, \mu), \quad x \in \mathbb{R}^m, \quad \mu \in \mathbb{R}, \tag{6.26}$$

given by a family F of smooth mappings. Let $x^*(t, \mu)$ be a closed periodic trajectory (a limit cycle) of the system (6.26), depending on the system parameter μ. Without loss of generality, we assume that there exists a critical value of the system parameter μ^* such that the trajectory $x^*(t, \mu)$ is an asymptotically orbitally stable cycle of system (6.26) for $\mu \leq \mu^*$, and the trajectory $x^*(t, \mu)$ is an unstable cycle of system (6.26) for $\mu > \mu^*$. In

Let us consider the $(m + k + 1)$-dimensional system

$$\begin{aligned} \dot{y} &= F(y, \mu) + Z(y, t, \mu) E(q - \mu e), \\ \dot{q} &= DQ(y, s, t, \mu) + \beta(q - \mu e), \\ \dot{s} &= C^{\mathrm{T}} Q(y, s, t, \mu), \end{aligned} \tag{6.27}$$

$$Q(y, s, t, \mu) = x(s, \mu) - x(0, \mu), \quad Z(y, t, \mu) = \frac{\partial y}{\partial x(0, \mu)}, \tag{6.28}$$

Theorem 6.5 *There exist constant matrices E and D of control parameters, vector C, scalar β, and also the value of system parameter $\mu_1^* > \mu^*$ such that the cycle $u^*(t,\mu)$ is an asymptotical orbitally stable limit cycle of the system* (6.27) *for all values of the parameter $\mu \in [\mu^*, \mu_1^*]$.*

Example 6.5 As an example we consider the Rossler dynamical system (6.24). If $\mu > 1.88$, then the limit cycle of the system (6.24) loses stability, and an irregular singular attractor appears in the system for $\mu = 2.35$. We localized and stabilized the limit cycle of the system (6.24) with the help of the expanded system

$$
\begin{aligned}
\dot{x} &= -(y+z) + (z_{11}\varepsilon_1 + z_{12}\varepsilon_2 + z_{13}\varepsilon_3)(q-\mu),\\
\dot{y} &= x + ay + (z_{21}\varepsilon_1 + z_{22}\varepsilon_2 + z_{23}\varepsilon_3)(q-\mu),\\
\dot{z} &= b + z(x-\mu) + (z_{31}\varepsilon_1 + z_{32}\varepsilon_2 + z_{33}\varepsilon_3)(q-\mu),\\
\dot{q} &= a_1\big(x(s)-x(0)\big) + a_2\big(y(s)-y(0)\big) + a_3\big(z(s)-z(0)\big) + \beta(q-\mu),\\
\dot{s} &= c_1a_1\big(x(s)-x(0)\big) + c_2a_2\big(y(s)-y(0)\big) + c_3a_3\big(z(s)-z(0)\big),
\end{aligned}
\tag{6.37}
$$

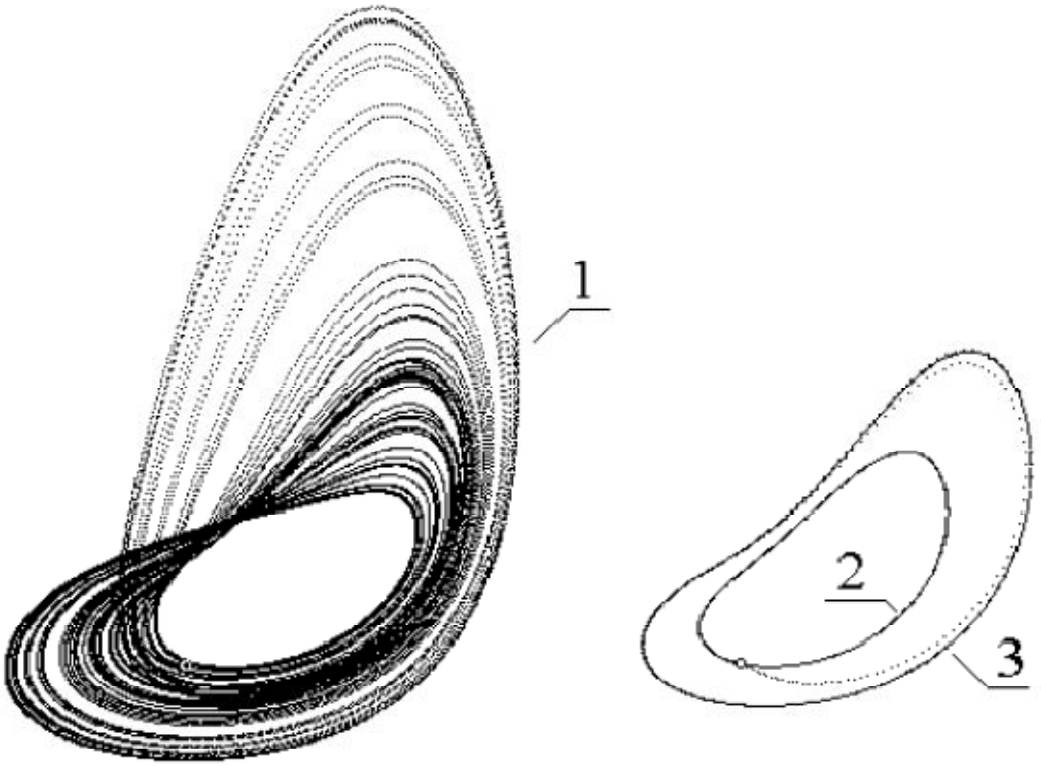

Fig. 6.3 Stabilization of an unstable limit cycle of the Rossler system.

Fig. 6.3 shows: 1 is a chaotic trajectory of the Rossler system for $\mu = 2.4$; 2 is a stable limit cycle of the Rossler system for the critical value $\mu^* = 1.85$; 3 is a projection of the limit cycle of the system (6.37) stabilizing the unstable limit cycle of the Rossler system (6.24) for $\mu = 2.4$.

Example 6.6 As an example, we consider stabilization of unstable fixed point in the nonlinear Mackey–Glass equation

$$\dot{x} = -ax(t) + \frac{\beta_0 \theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)}, \tag{6.45}$$

where β_0, θ and n are positive constants such that $\beta_0 > a > 0$ and $nB > 2$, $6aB > \beta_0$, $B = (\beta_0 - a)/\beta_0$. The Eq. (6.45) has the unique stationary state $x^* = \theta \sqrt[n]{\dfrac{\beta_0 - a}{a}}$, which loses stability for

$$\tau > \tau^* = \frac{\arccos(-a/b)}{\sqrt{b^2 - a^2}},$$

where $b = a(nB-1)$

For stabilization of unstable fixed point x^* of the Eq. (6.45) for $\tau > \tau^*$, we consider the system

$$\dot{x}(t) = -ax(t) + \frac{\beta_0 \theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)} + \varepsilon(q(t) - \tau),$$
$$\dot{q}(t) = -a(x(t) - x^*) + \beta(q(t) - \tau). \tag{6.46}$$

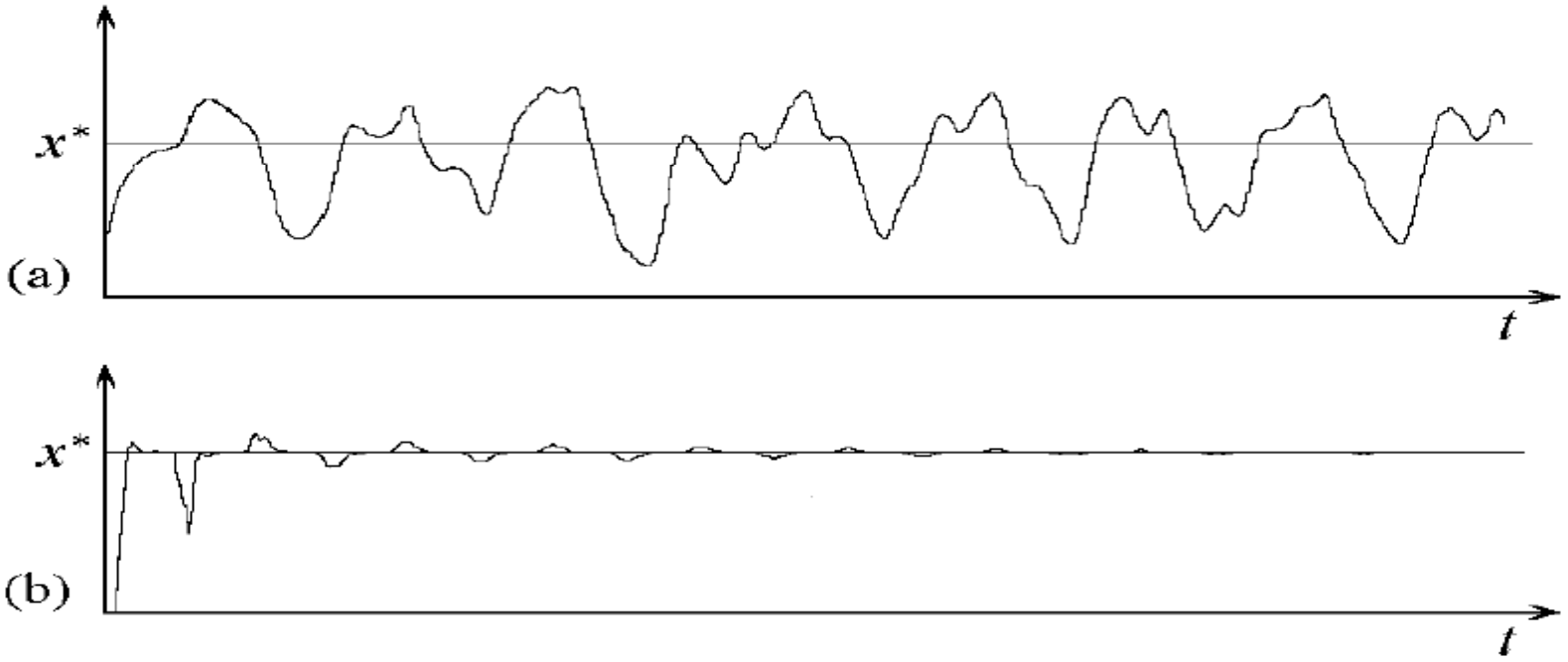

Fig. 6.4 Solutions of the Mackey–Glass Eq. (6.45) (a) and the stabilizing it system (6.46)(b) for the value of bifurcation parameter $\tau = 5$, where x^* is the stationary solution.

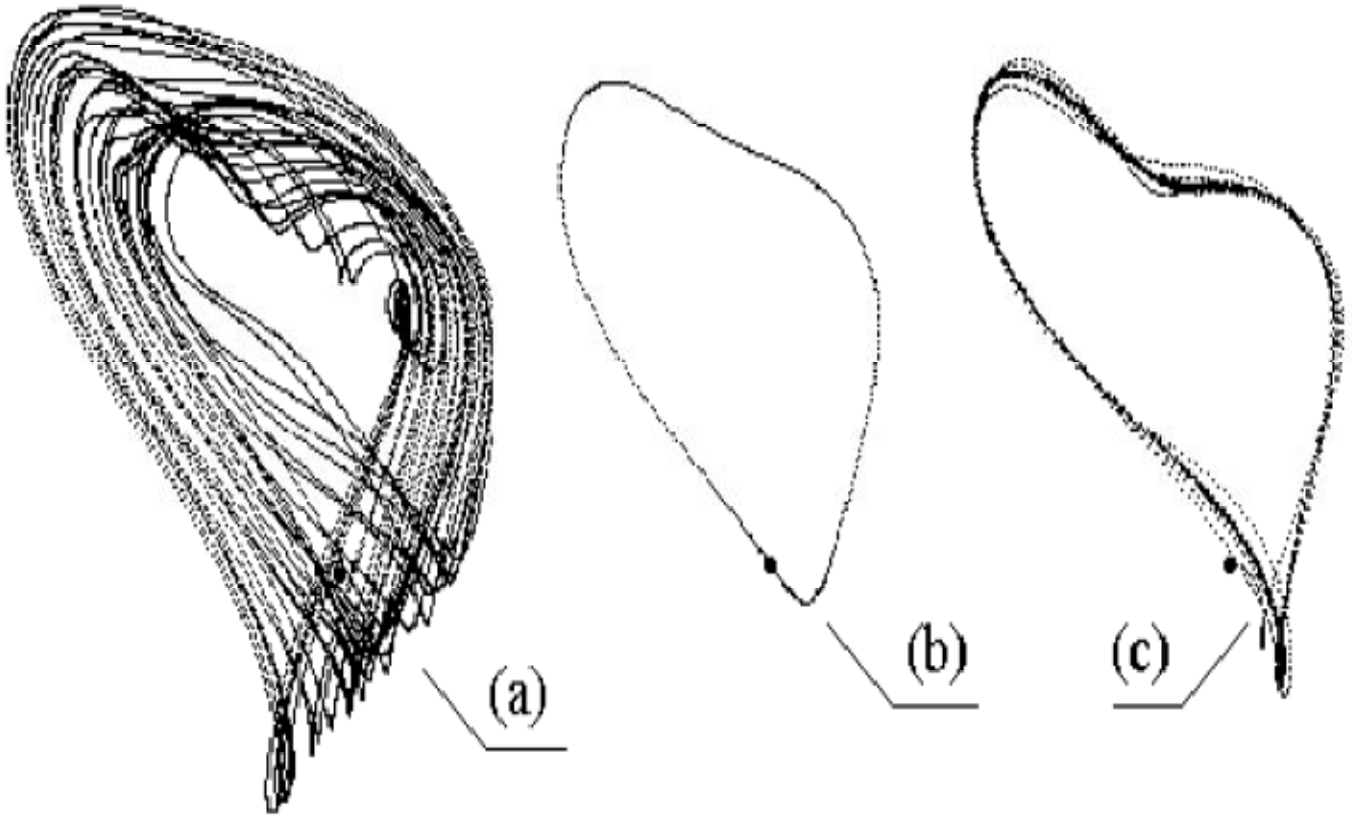

Fig. 6.5 Stabilization of an unstable periodic solution of the Mackey–Glass Eq. (6.45).

Stabilization of a thermodynamic branch in reaction–diffusion systems of equations

Stabilization of zero solution of the Kuramoto–Tsuzuki equation

$$W_t = W + (1+ic_1)W_{xx} - (1+ic_2)W|W|^2, \quad W(x,t) = u(x,t)+iv(x,t)$$
$$0 \le x \le l, \quad 0 \le t < \infty, \tag{6.54}$$
$$W(x,0) = W_0(x), \quad W_x(0,t) = W_x(l,t) = 0.$$

Expanded stabilizing system

$$\begin{gathered}
u_t = u + u_{xx} - c_1 v_{xx} - (u^2+v^2)(u - c_2 v) + \varepsilon\varphi, \\
v_t = v + c_1 u_{xx} + v_{xx} - (u^2+v^2)(u + c_2 v) + \varepsilon\psi, \\
\varphi_t = u + \beta\varphi, \quad \psi_t = v + \beta\psi, \\
0 \le x \le l, \quad 0 \le t < \infty, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \\
\varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \\
u_x(0,t) = u_x(l,t) = v_x(0,t) = v_x(l,t) = 0, \\
\varphi_x(0,t) = \varphi_x(l,t) = \psi_x(0,t) = \psi_x(l,t) = 0.
\end{gathered} \tag{6.58}$$

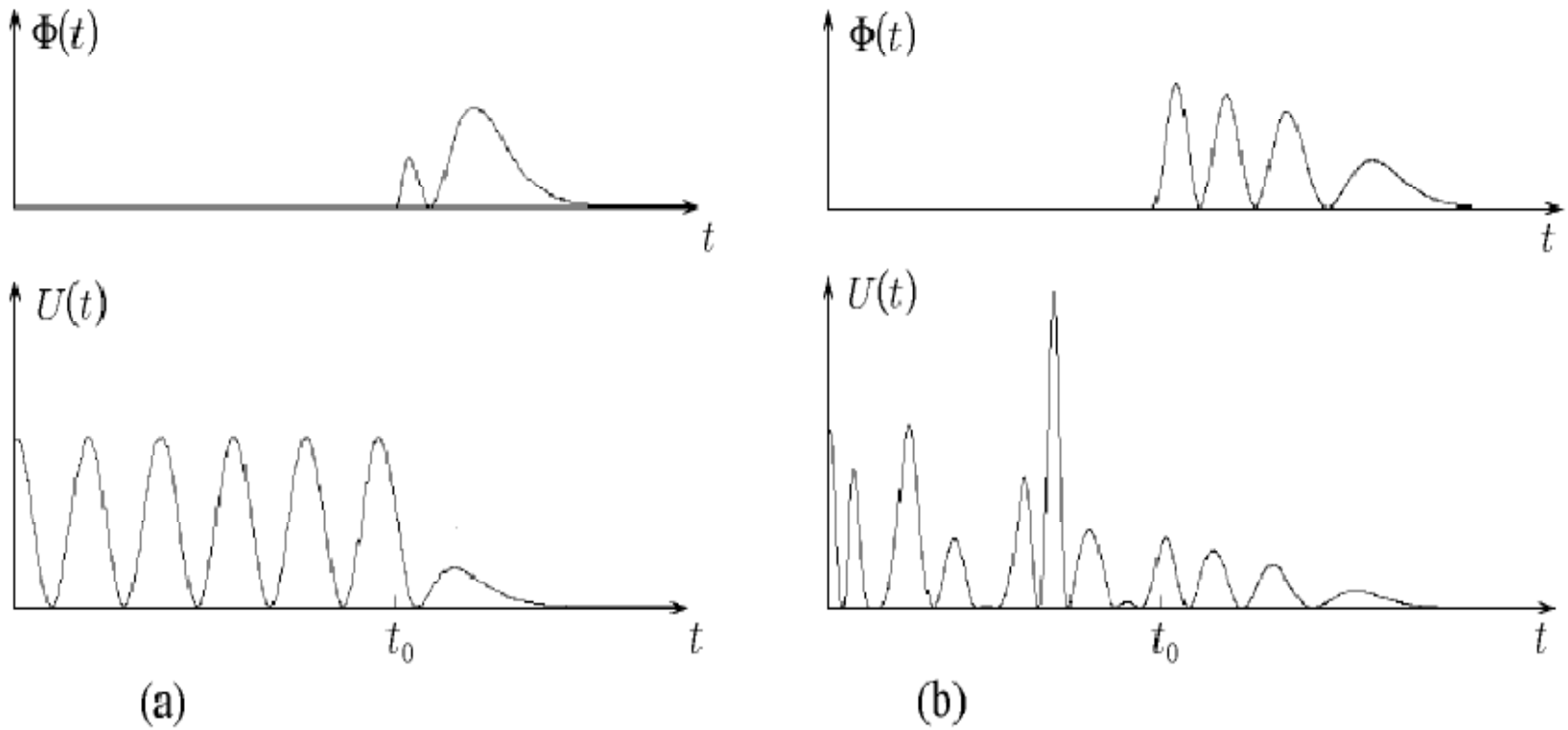

Fig. 6.6 Stabilization of the zero solution of the problem (6.54) in a case when its solution is: a) stable spatially homogeneous self-oscillating mode ($l = \pi$, $c_1 = 5$, $c_2 = -2$, $\varepsilon = -4$, $\beta = -3$) and b) spatially inhomogeneous nonperiodic mode (diffusion chaos) ($l = \pi$, $c_1 = 5$, $c_2 = -5$, $\varepsilon = -4$, $\beta = -3$).

$$U(t) = \int_0^l u^2(x,t)\,dx \text{ and } \Phi(t) = \int_0^l \varphi^2(x,t)\,dx.$$

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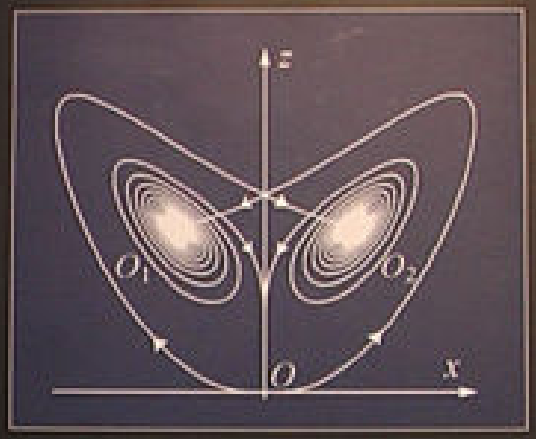

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