The University of
Manchester

Computation of Resistive Wakefields

Adina Toader and Roger Barlow

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Resistive Wakefields - Roger Barlow at CLIC08 Workshop

http://www.hep.manchester.ac.uk/u/roger/talks/2008/10.pdf

1. Longitudinal $m = 0$ wakefield
   - Long Range approximation (Chao)
   - Short Range approximation (Bane and Sands)
   - Full treatment

2. Longitudinal $m > 0$ wakefield (higher order modes)

3. Transverse wakes

4. AC conductivity

5. Implementation

Idea in a Nutshell:

- Uniform circular pipe of radius b and conductivity σ
- Assume thick pipe
- Interested in short range intrabunch fields
- Solve Maxwell's Equations in vacuum and in metal pipe
- Decompose into angular modes ($cos(m\theta)$, $m = 0, 1, 2\ldots$)
- Match the boundary conditions
- Work in frequency space as differentiation → multiplication

The longitudinal wake for $m = 0$

The Fourier Transform of the wake is the impedance:

$$\tilde{\vec{F}}(k) = \int \vec{F}(s) e^{iks} ds \qquad \vec{F}(k) = \frac{1}{2\pi} \int \tilde{\vec{F}}(k) e^{-iks} dk$$

$$\tilde{E}_z(k) = \frac{2q}{b} \frac{1}{\frac{ikb}{2} - \left(\frac{\lambda}{k} + \frac{k}{\lambda}\right)\left(1 + \frac{i}{2\lambda b}\right)}$$

where

$$\lambda(k) = \sqrt{\frac{2\pi\sigma|k|}{c}}(i + \mathrm{sgn}(k))$$

Introduce s_0, the scaling length (20 µm for 1 cm Copper)

$$s_0 = \sqrt[3]{\frac{cb^2}{2\pi\sigma}} \qquad K = s_0 k \qquad s' = \frac{s}{s_0}$$

The simplest case: Chao's formula

In the long range limit $\widetilde{E}_z(k) = \frac{2q}{b}\frac{1}{\left(-\frac{\lambda}{k}\right)}$

The Fourier Transform is well known to be: $E_z(s) = \frac{q}{2\pi b}\sqrt{\frac{c}{\sigma}}s^{-\frac{3}{2}}$

We do it the hard way by numerically integration of:

$$E_z(s) = \frac{s_0}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}[f_{even}(K)]\cos(Ks') + \mathrm{Im}[f_{odd}(K)]\sin(Ks'))dK$$

$$f_{even}(K) = \frac{1}{2}[f(K) + f(-K)] = -\frac{q}{b^2}\sqrt{K}$$

$$f_{odd}(K) = \frac{1}{2}[f(K) - f(-K)] = \frac{q}{b^2}i\sqrt{K}$$

Solution: First integrate analytically wrt x. Function becomes $-\cos(Kx)\frac{\sqrt{K}}{K}$

Integrate numerically wrt K.

Differentiate numerically wrt x.

A more accurate formula : Bane and Sands

$$\tilde{E}_z(k) = \frac{2q}{b}\frac{1}{\left(\frac{ikb}{2} - \frac{\lambda}{k}\right)}$$

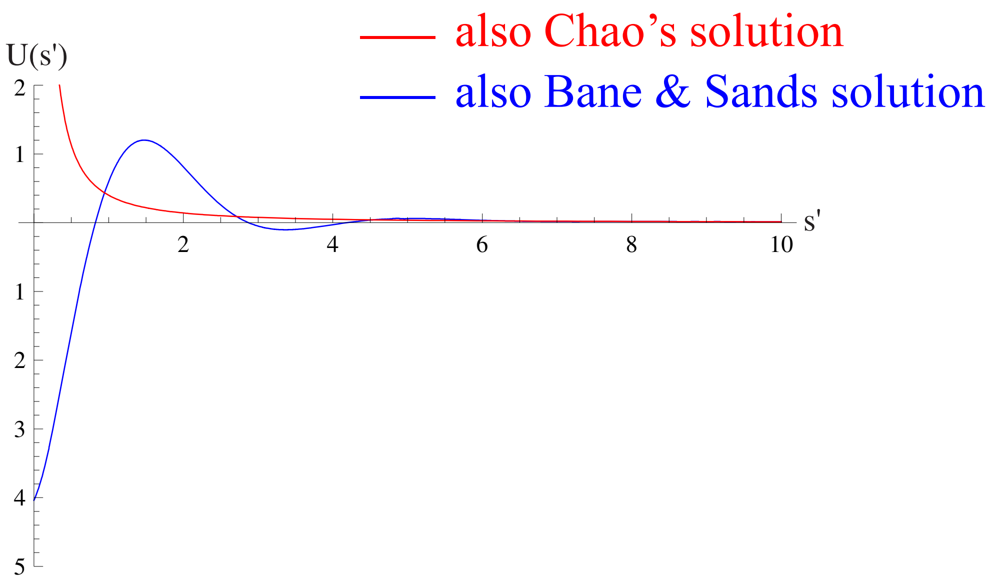
Solution well known to be: $E_z(s) = \frac{4qc}{\pi b^2}\left(\frac{e^{-s'}}{3}\cos(\sqrt{3}s') - \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{x^2 e^{-x^2 s'}}{x^6+8}dx\right)$

We have:

$$E_z(s) = \frac{s_0}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}[f_{even}(K)]\cos(Ks') + \mathrm{Im}[f_{odd}(K)]\sin(Ks'))dK$$

$$f_{even}(K) = \frac{1}{2}[f(K) + f(-K)] = -\frac{q}{b^2}\frac{\frac{2}{\sqrt{K}}}{\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}$$

$$f_{odd}(K) = \frac{1}{2}[f(K) - f(-K)] = -\frac{q}{b^2}\frac{2i\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)}{\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}$$



Shows our results. Accurately reproduces Chao and B&S.

Wake is a function of three parameters (s, b and σ), but the use of s_0 enables it to be written as a universal function $U(s')$, where $E_z(s,b)=s_0 q/b^2\, U(s/s_0)$.

The full formula

Full version, with $\xi = s_0^2 / b^2$

$$f_{even}(K) = -\frac{8q}{b^2}\frac{\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}}{4[\xi\sqrt{K} - \frac{1}{K}\left(\xi + 2\sqrt{K}\right) + K)]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}\right)^2}$$

$$f_{odd}(K) = -\frac{8q}{b^2}\frac{2i\left[\xi\sqrt{K} - \frac{1}{K}\left(\xi + 2\sqrt{K}\right) + K\right]}{4[\xi\sqrt{K} - \frac{1}{K}\left(\xi + 2\sqrt{K}\right) + K)]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}\right)^2}$$

Although this is no longer a universal curve, it can still be expressed as a function of two variables (s' and ξ) rather than the full set of three.

The B&S approximation corresponds to the function at $\xi = 0$.

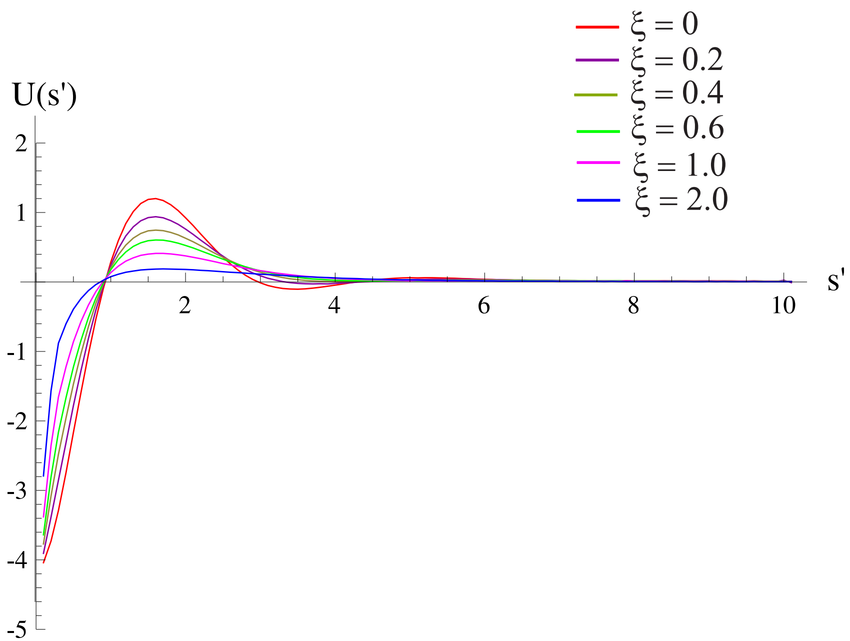


Figure shows how the function changes for different values of ξ.

Longitudinal: Higher order modes

For higher order modes, using the same technique for $m > 0$

$$\tilde{E}_z^m(k) = \frac{4I_m}{b^{2m+1}} \frac{1}{\dfrac{ikb}{m+1} - \left(\dfrac{\lambda}{k} + \dfrac{2k}{\lambda}\right)\left(1 + \dfrac{i}{2\lambda b}\right) - \dfrac{im}{kb}}$$

where I_m is the charge moment of order m: $\iint \rho(r,\theta) r^m \cos(m\theta) dr d\theta$

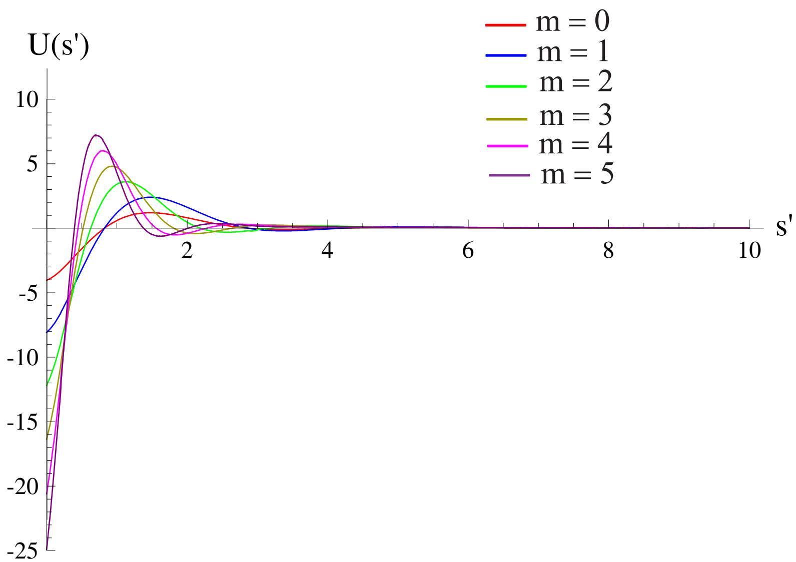
- Any angular distribution of charges at a particular radius can be described in terms of these moments.

This can be separated into odd and even parts,

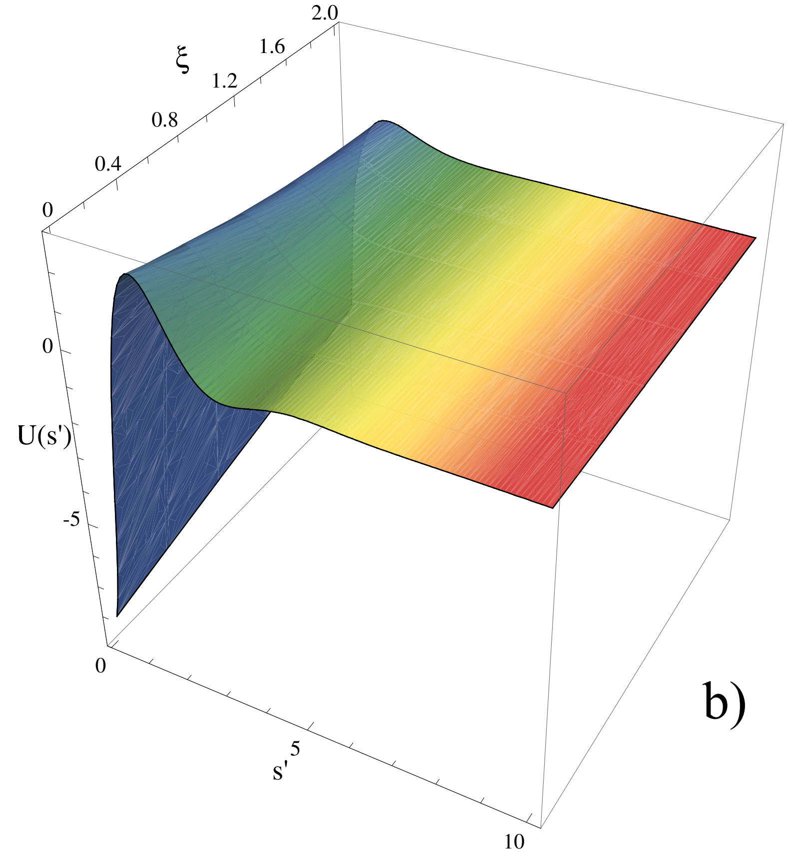
$$f_{even}(K) = -\frac{8I_m s_0}{b^{2m+2}} \frac{\xi^2 + 2\xi\sqrt{K} + \dfrac{4}{\sqrt{K}}}{4\left[\xi\sqrt{K} - \dfrac{1}{K}\left(\xi + 2\sqrt{K}\right) + 2\left(\dfrac{K}{m+1} - \xi\dfrac{m}{K}\right)\right]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \dfrac{4}{\sqrt{K}}\right)^2}$$

$$f_{odd}(K) = -\frac{8I_m s_0}{b^{2m+2}} \frac{2i\left[\xi\sqrt{K} - \dfrac{1}{K}\left(\xi + 2\sqrt{K}\right) + 2\left(\dfrac{K}{m+1} - \xi\dfrac{m}{K}\right)\right]}{4\left[\xi\sqrt{K} - \dfrac{1}{K}\left(\xi + 2\sqrt{K}\right) + 2\left(\dfrac{K}{m+1} - \xi\dfrac{m}{K}\right)\right]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \dfrac{4}{\sqrt{K}}\right)^2}$$

Longitudinal: Higher order modes



a)



b)

Higher order modes. a) various modes with $\xi = 0$;

b) the $m = 1$ wake as a function of ξ and s'.

Transverse wakes

Transverse wake also a sum over angular modes

$$\vec{F}_T(r,\theta,s) = \sum_m r^{m-1} r'^m \left(\hat{r}\cos(m\theta) - \hat{\theta}\sin(m\theta)\right) W_T^m(s)$$

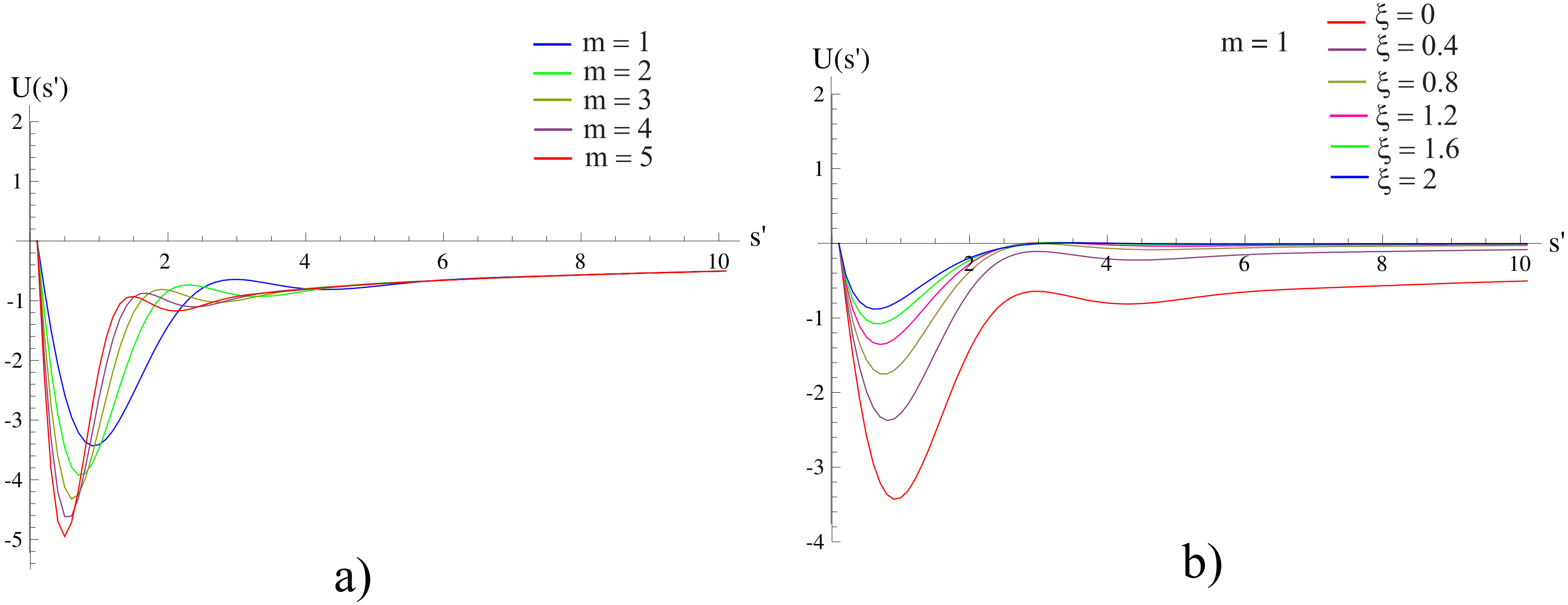
The Panofsky-Wenzel theorem $\nabla_T F = \dfrac{\partial \vec{F}_T}{\partial z}$

applies term by tem giving $W'^m_T(s) = E_z^m(s)$

- Thus the transversal wake at any order can be obtained by integrating the longitudinal wake.

Lucky we have that integral already evaluated!

Transverse wakes



Transverse wakes. a) various modes with $\xi = 0$; b) mode $m = 1$ with ξ variable.

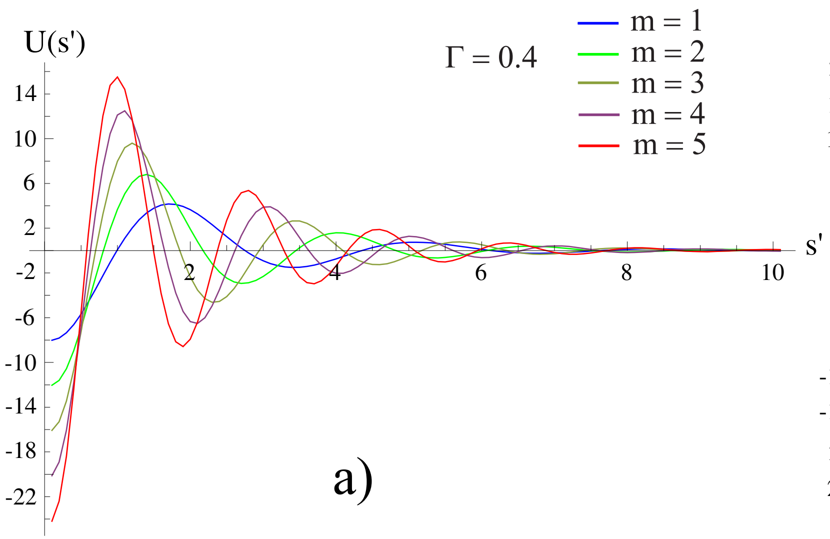
AC Conductivity

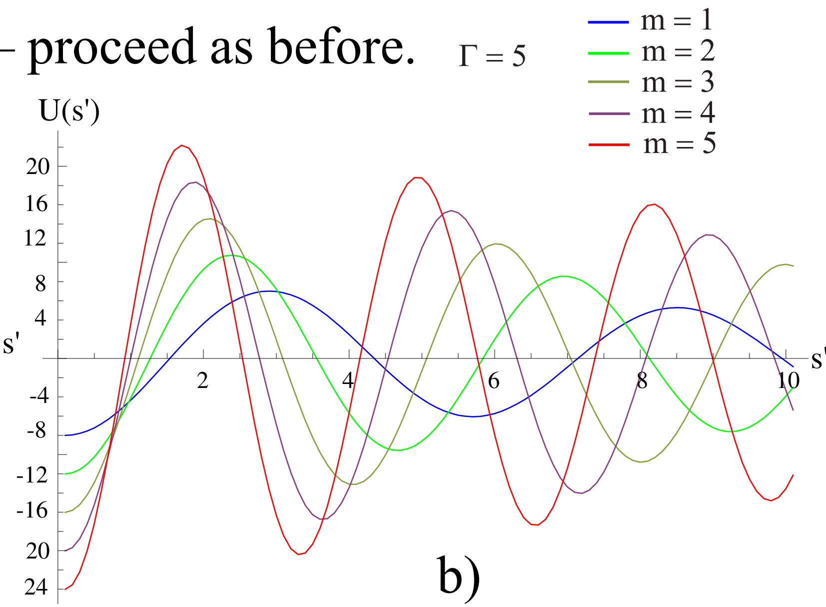
In the classical Drude model, in units of normalised wave number $K=s_0k$

$$\tilde{\sigma} = \frac{\sigma}{1 - iK\Gamma}$$

with $\Gamma = c\tau/s_0$ relaxation factor

Introduce into previous formulae – proceed as before.





AC Conductivity. Longitudinal wake for various modes ($m > 0$) for a) $\Gamma = 0.4$ and b) $\Gamma = 5$.

Implementation

- Use Mathematica to integrate even and odd functions and generate tables of values as function of s', ξ , Γ.

- Only 3 variables - 2 of them don't vary (for a given collimator).

- Write to file C++ object collimatortable(file, Gamma, xi) – portable.

- Reads complete table and interpolates to get single table for s' collimatortable::interpolate(double sprime) returns the value.

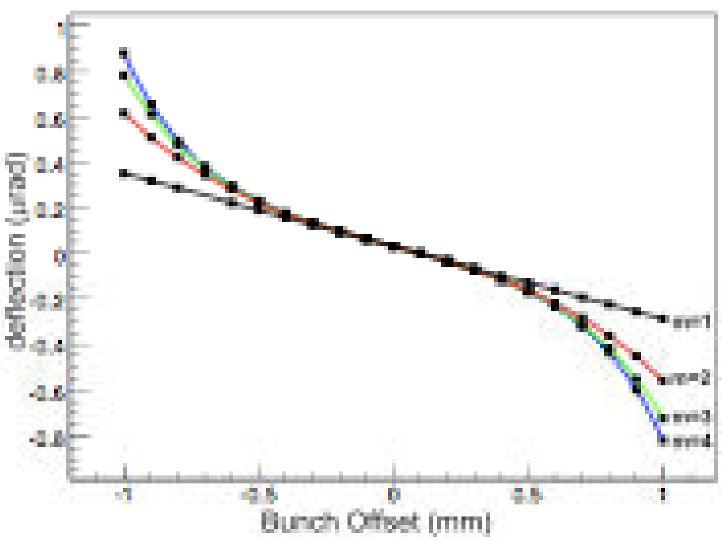
Tables and Documentation at:

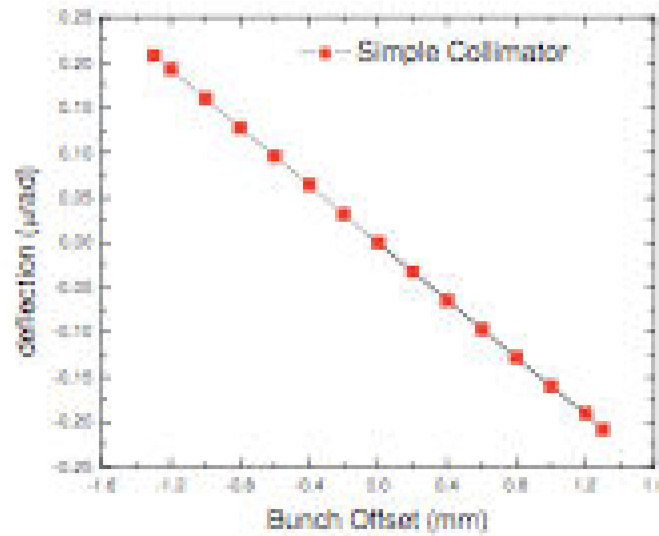
http://www.hep.manchester.ac.uk/u/adina

Examples: simple collimator

b = 1.4 mm, $\sigma = 2.33 \cdot 10^{6}$ $(\Omega m)^{-1}$ (Titanium), L = 1 m

$\Gamma = 0.49$, $\xi = 0.00014$





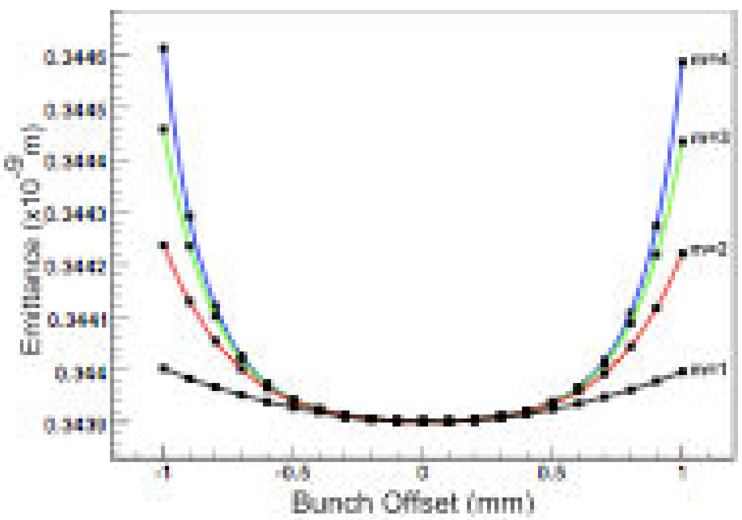
MERLIN

PLACET

Examples: simple collimator

$b = 1.4$ mm, $\sigma = 2.33*10^{6}$ $(\Omega m)^{-1}$ (Titanium), $L = 1$ m

$\Gamma = 0.49$, $\xi = 0.00014$



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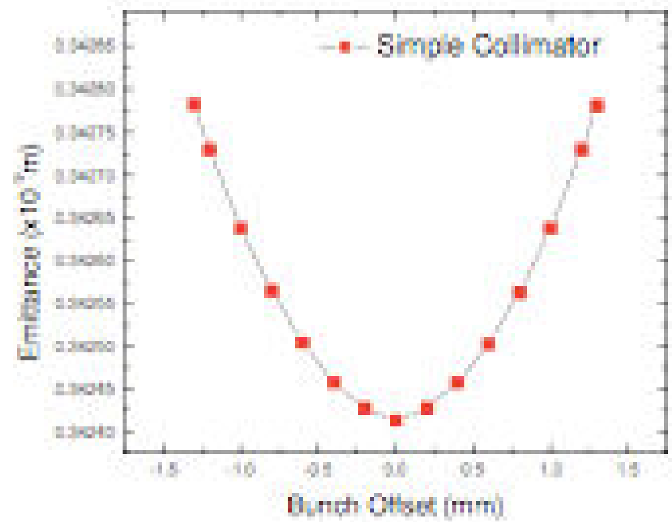

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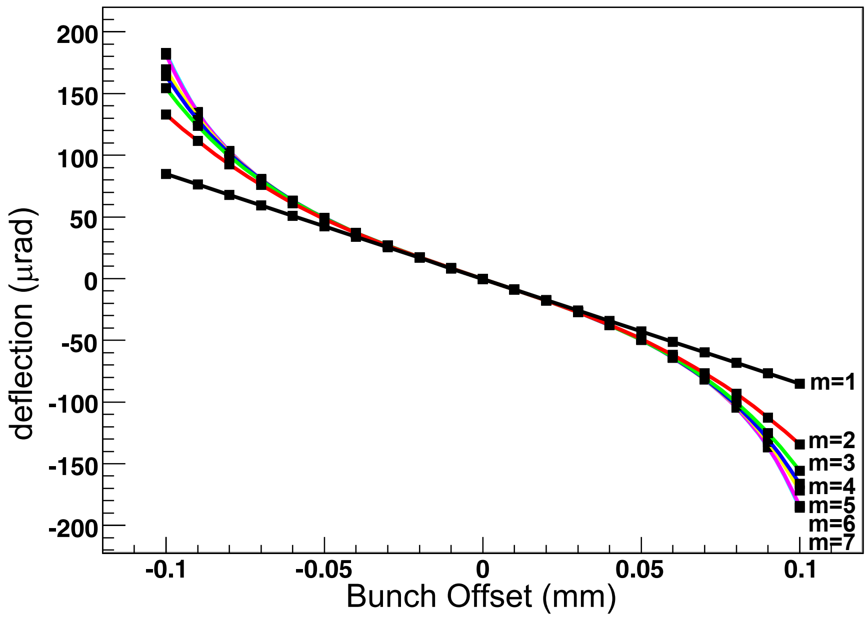
Examples: extreme collimator

$b = 0.125$ mm, $\sigma = 7.14*10^{14}$ $(\Omega m)^{-1}$ (Graphite), L = 1 m

$\Gamma = 2.85$, $\xi = 0.007$



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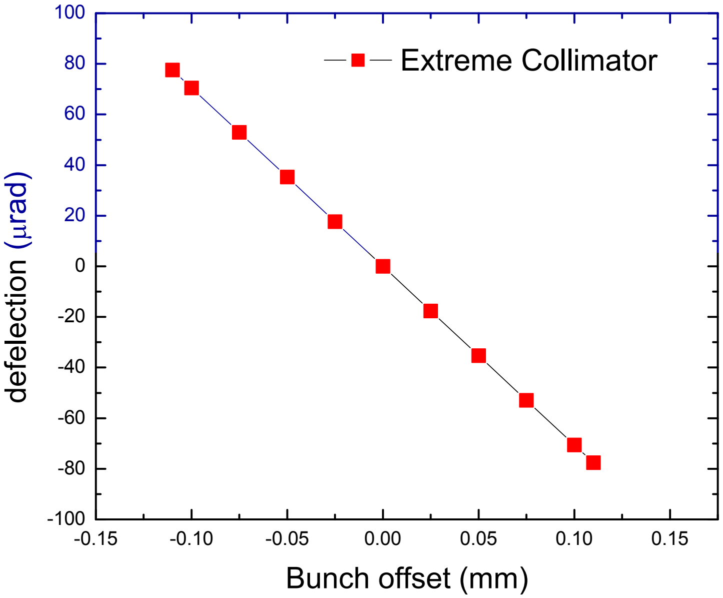

PLACET

Examples: extreme collimator

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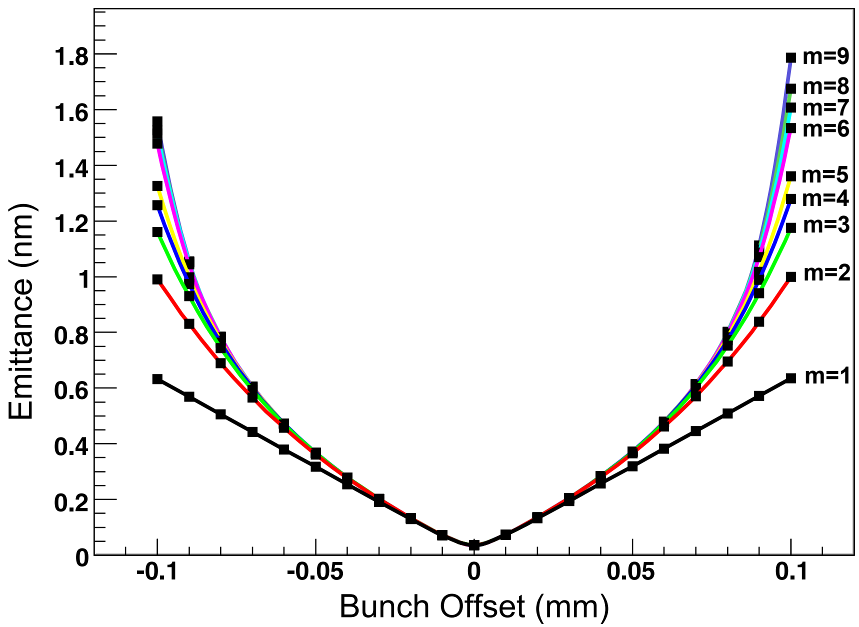

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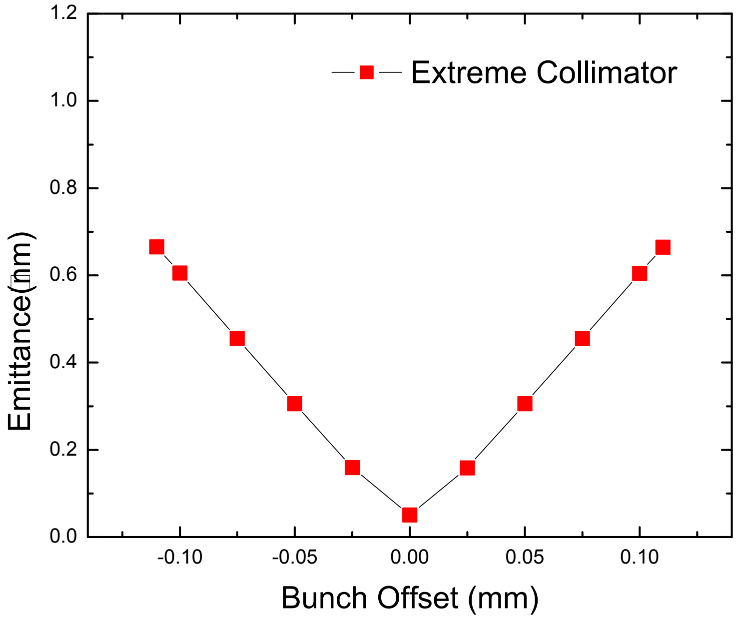

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Conclusions

- We have found an approach which unifies, simplifies and speeds up the calculations.
- It includes short range and long range wakes with no artificial division between them using pretabulated numerical Fourier Transform.

Higher order angular modes and AC conductivity effects can be easily included.

- We have shown that this can be used by PLACET and Merlin and can be easily included in other codes.

- Extend to rectangular apertures.