Gluon Saturation
and BK evolution

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Content of the third lecture

- How to go beyond the BFKL approach

  the ideas that led to the Color Glass Condensate picture
- The evolution of the CGC wave function

  the JIMWLK equation and the Balitsky hierarchy
- A mean-field approximation: the BK equation

  solutions: QCD traveling waves

  the saturation scale and geometric scaling
- Example of phenomenological success

  forward particle production in proton-nucleus collisions

Going beyond the BFKL approach

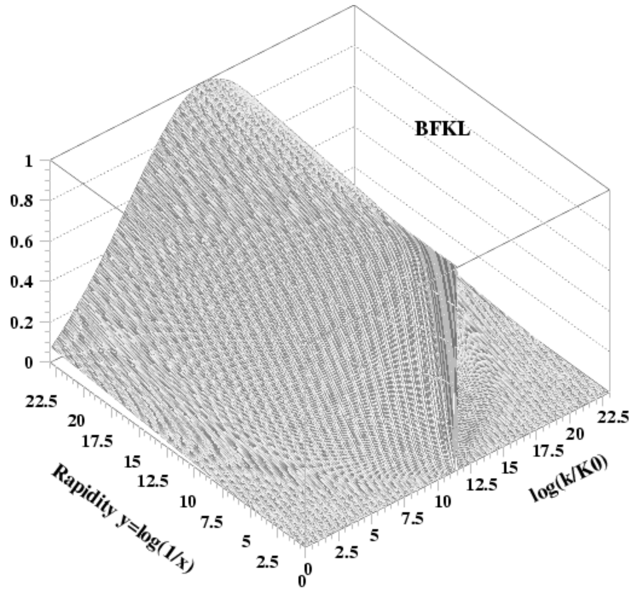
The problem with BFKL

- the growth of scattering amplitudes with energy

this leads to unitarity violations, for instance for the total cross-section, the Froissart bound $\sigma_{tot}(s) \leq \ln^2(s/m_\pi)$ cannot be verified at high energies

what did we do wrong ? use a perturbative treatment when we shouldn't have

- the growth of gluon density with increasing rapidity



even if this initial condition is a fully perturbative wave function (no gluons with small $k_\perp$)

the BFKL evolution populates the non perturbative region

this so-called infrared diffusion invalidates the perturbative treatment

Proposals to go beyond BFKL

summing $\alpha_s^n \ln^n(1/x)$ terms isn't enough, high-density effects are missing
to deal with this many body problem, one needs effective degrees of freedom

- the modified leading logarithmic approximation (MLLA)

in this approach, hadronic scattering at high energies is described by the exchange of quasi-particles called Reggeized gluons (or Reggeons) Bartels, Ewertz, Lipatov, Vacca

the BFKL approximation corresponds to the exchange of two Reggeons (a Pomeron), the idea of the MLLA is to include multiple exchanges

- the color glass condensate (CGC)

in this approach, the small-x part of the hadronic wave function is described by classical fields

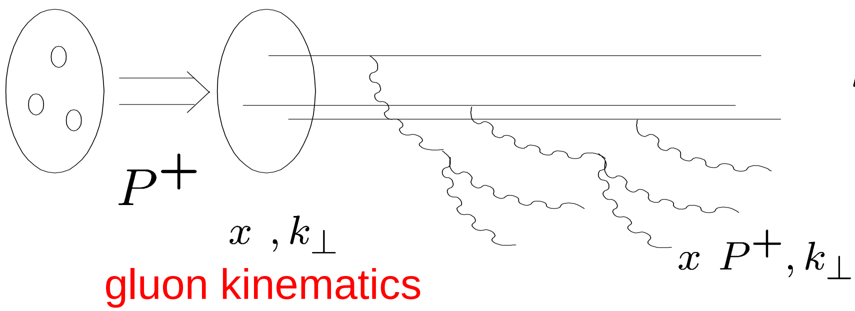
the BFKL growth is due to the approximation that gluons in the wave function evolve independently

when the gluon density is large enough, gluon recombination becomes important
the idea of the CGC is to take into account this effect via strong classical fields $\mathcal{A} \sim 1/g_s$

the CGC sums both $\alpha_s^n \ln^n(1/x)$ and $g_s^n \mathcal{A}^n$

The saturation phenomenon

- gluon recombination in the hadronic wave function



$\rho \sim \frac{x f(x, k_\perp^2)}{\pi R^2}$ gluon density per unit area

it grows with decreasing x

$\sigma_{rec} \sim \alpha_s / k^2$ recombination cross-section

recombinations important when $\rho\ \sigma_{rec} > 1$

the saturation regime: for $k^2 < Q_s^2$ with $Q_s^2 = \frac{\alpha_s x f(x, Q_s^2)}{\pi R^2}$

- the saturation regime of QCD

is weakly coupled Q_s^2 grows with decreasing *x*, and eventually $\alpha_s(Q_s^2) \ll 1$

is non-linear the gluon density evolves with *x* in a non-linear way

in general, observables are non-linear functions of the gluon density (and also depend on n-gluon correlations)

is easier to reach with nuclei $Q_s^2 \propto A^{1/3}$

for a given value of k^2, the saturation regime in a nuclear wave function extends to a higher value of x compared to a hadronic wave function

The Color Glass Condensate

- the CGC: an effective theory to describe the saturation regime

McLerran and Venugopalan (1994)

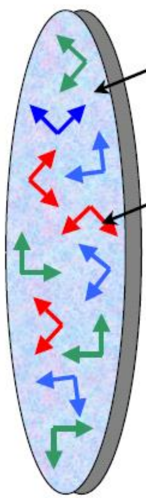
lifetime of the fluctuations in the wave function ~ $xP^+/k_\perp^2$ $\Rightarrow$ $\begin{cases} \text{high-x partons} \equiv \text{static sources } \rho \\ \text{low-x partons} \equiv \text{dynamical fields } \mathcal{A} \end{cases}$

short-lived fluctuations

$$|\text{hadron}\rangle = |qqq\rangle + |qqqg\rangle + \ldots + |qqq\ldots\ldots ggggg\rangle \quad \Rightarrow \quad |\text{hadron}\rangle = \int D\rho\, \Phi_x[\rho]\, |\rho\rangle \equiv |\text{CGC}\rangle$$

effective wave function for the dressed hadron

separation between the long-lived high-x partons and the short-lived low-x gluons



$\rho^a(x_T)$ valence partons as static random color source

A_μ small x gluons as large classical fields

classical Yang-Mills equations

$$\left(D_\nu F^{\nu\mu}\right)^a = \delta^{\mu +}\delta(x^-)\rho^a(x_\perp)$$

the evolution of $|\Phi_x[\rho]|^2$ with *x* is a renormalization group equation

which sums both $\begin{cases} \alpha_s^n \ln^n(1/x) \\ g_s^n \mathcal{A}^n \end{cases}$

this effective description of the hadronic wave function applies only to the small-x part

Basic features

- from color charge to color field

the CGC is moving in the x^- direction and the gauge is $A^+ = 0$

the current is $J^\mu(x^\nu) = \delta^{\mu-}\rho(x^+, \mathbf{x}) \sim 1/g_s$

solving Yang-Mills equations gives

$$A^\mu(x^\nu) = \delta^{\mu-}\alpha(x^+, \mathbf{x}) \sim 1/g_s \qquad \rho(x^+, \mathbf{x}) = -\nabla^2\alpha(x^+, \mathbf{x})$$

- from the CGC wave function to CGC averages

in practice we deal with CGC averages such as $\langle S \rangle_x = \int D\rho \left|\Phi_x[\rho]\right|^2 S[\rho]$

from $\left|\Phi_x[\rho]\right|^2$, one can obtain the unintegrated gluon distribution, as well as any n-parton distributions

when computing the unintegrated gluon distribution $\langle h|\mathcal{A}\mathcal{A}|h\rangle = \langle \mathcal{A}\mathcal{A}\rangle_x$

occupation numbers become large in the saturation regime $\langle \mathcal{A}\mathcal{A}\rangle_x \propto 1/\alpha_s$

we recover the BFKL equation in the low-density regime $g_s\mathcal{A} \ll 1$

the growth of the saturation scale is about $Q_s^2 \propto x^{-0.3}$

Saturation and multiple scatterings

- scattering off the CGC

so far we only discussed the hadronic/nuclear wave function, but during a collision that probes the saturation regime, multiple scatterings occur when $g_s \mathcal{A} \sim 1$

parton saturation

(i.e. the fact that the gluon distribution in the wave function evolves in a non-linear way)

and multiple scatterings

(i.e. the fact that observables are non-linear functions of the gluon distribution)

are of equal importance

to be consistent, both should be included

what I will discuss: how the wave function $|\Phi_x[\rho]|^2$ evolves with x

how do we "measure" it with well-understood probes

The evolution of the CGC wave function: the B-JIMWLK equations

The JIMWLK equation

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

- a functional equation for the *x* evolution of $\left|\Phi_x[\alpha]\right|^2$:

$$\frac{d}{d\ln(1/x)}\left|\Phi_x[\alpha]\right|^2 = H^{JIMWLK} \otimes \left|\Phi_x[\alpha]\right|^2 \qquad \alpha = T^c\alpha_c$$

$$H^{JIMWLK}\otimes|\Phi_x[\alpha]|^2 = \int \frac{d^2\mathbf{x}}{2\pi}\frac{d^2\mathbf{y}}{2\pi}\frac{d^2\mathbf{z}}{2\pi}\frac{(\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\frac{\delta}{\delta\alpha_c(\mathbf{x})}\left[1+W_A^\dagger(\mathbf{x})W_A(\mathbf{y})\right.$$

$$\left.-W_A^\dagger(\mathbf{x})W_A(\mathbf{z})-W_A^\dagger(\mathbf{z})W_A(\mathbf{y})\right]^{cd}\frac{\delta}{\delta\alpha_d(\mathbf{y})}|\Phi_x[\alpha]|^2$$

with $\displaystyle\frac{\delta}{\delta\alpha_c(\mathbf{x})} \equiv \lim_{x^+\to\infty}\frac{\delta}{\delta\alpha_c(x^+,\mathbf{x})}$

the Wilson lines $W_A[\alpha](\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int dx^+\tilde{T}^c\alpha_c(x^+,\mathbf{x})\right\}$ sum powers of $g_s\ \alpha\sim 1$

adjoint representation $(\tilde{T}^c)_{ab} = -if^{cab}$

the JIMWLK equation gives the evolution of the wave function for small enough *x*

the equivalent Balitsky equations are obtained by considering the scattering of simple test projectiles (dipoles) off the CGC

Scattering off the CGC

- eigenstates ?

their interaction should conserve their spin, polarisation, color, momentum …

in the high-energy limit, the eigenstates are simple when the partons transverse momenta are Fourier transformed to transverse coordinates

in the large-Nc limit, further simplification: the eigenstates are only made of dipoles

$\left|(\mathbf{x},\mathbf{y})\right\rangle$ **x**, **y** : transverse coordinates

$\left|(\mathbf{x}_1,\mathbf{y}_1);\ldots;(\mathbf{x}_N,\mathbf{y}_N)\right\rangle$

other quantum numbers aren't explicitely written, the eigenvalues depend only on the transverse coordinates

- eigenvalues

$$\hat{S}\left|(\mathbf{x}_1,\mathbf{y}_1);\ldots;(\mathbf{x}_N,\mathbf{y}_N)\right\rangle=\left\langle S_{\mathbf{x}_1\mathbf{y}_1}\ldots S_{\mathbf{x}_n\mathbf{y}_n}\right\rangle_{CGC}\left|(\mathbf{x}_1,\mathbf{y}_1);\ldots;(\mathbf{x}_N,\mathbf{y}_N)\right\rangle$$

$$T_{\mathbf{xy}}=1-S_{\mathbf{xy}}$$

$$\left\langle S\right\rangle_{CGC}=\int D\alpha\,\left|\Phi_x[\alpha]\right|^2 S[\alpha]$$

$q\bar{q}$ scattering amplitude off the (dense) target CGC

the interaction with the CGC conserves transverse position

Dipoles as test projectiles

- dipoles are ideal projectiles to probe small distances

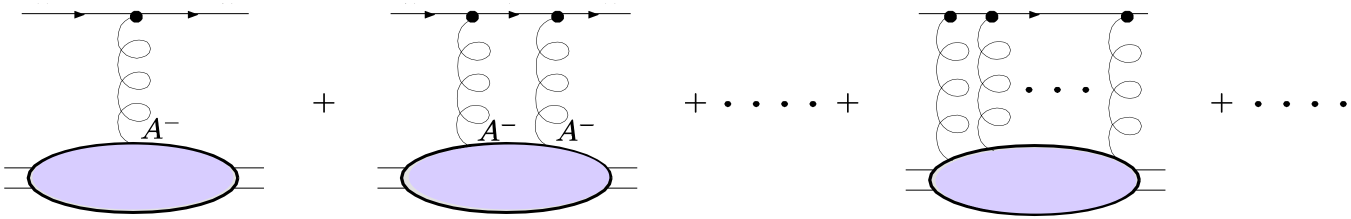
the $q\bar{q}$ dipole: $$T_{\mathbf{uv}}[\alpha] = 1 - \frac{1}{N_c} Tr(W_F^{+}(\mathbf{v}) W_F(\mathbf{u}))$$

u : quark space transverse coordinate

v : antiquark space transverse coordinate

scattering of the quark: $$W_F[\alpha](\mathbf{x}) = \mathcal{P}\exp\left\{ig_s \int dx^+ T^c \alpha_c(x^+, \mathbf{x})\right\}$$

α dependence kept implicit in the following



JIMWLK equation → evolution equation for the dipole correlators

$$\left\langle T_{\mathbf{u}_1\mathbf{v}_1} \ldots T_{\mathbf{u}_n\mathbf{v}_n} \right\rangle_x = \int D\alpha \left|\Phi_x[\alpha]\right|^2 T_{\mathbf{u}_1\mathbf{v}_1}[\alpha] \ldots T_{\mathbf{u}_n\mathbf{v}_n}[\alpha]$$

The Balitsky hierarchy

- equations for dipoles scattering of the CGC

Balitsky (1996)

$$\frac{d}{dY}\left\langle T_{\mathbf{uv}}\right\rangle_Y=\bar{\alpha}\int\frac{d^2z}{2\pi}\frac{(\mathbf{u}-\mathbf{v})^2}{(\mathbf{u}-\mathbf{z})^2(\mathbf{z}-\mathbf{v})^2}\left(\left\langle T_{\mathbf{uz}}\right\rangle_Y+\left\langle T_{\mathbf{zv}}\right\rangle_Y-\left\langle T_{\mathbf{uv}}\right\rangle_Y-\left\langle T_{\mathbf{uz}}T_{\mathbf{zv}}\right\rangle_Y\right)=\left\langle f(T_{\mathbf{uv}})\right\rangle_Y$$

the BFKL kernel

$$Y=\ln(1/x)$$

$$\frac{d}{dY}\left\langle T_{\mathbf{uz}}T_{\mathbf{zv}}\right\rangle_Y=\left\langle f(T_{\mathbf{uz}})T_{\mathbf{zv}}\right\rangle_Y+\left\langle T_{\mathbf{uz}}f(T_{\mathbf{zv}})\right\rangle_Y+O\left(\frac{1}{N_c^2}\right)$$

an hierarchy of equation involving correlators with more and more dipoles

- in the large Nc limit, the hierarchy is restricted to dipoles

general structure: $$\frac{d}{\bar{\alpha}dY}T_Y^{(n)}\propto \underbrace{T_Y^{(n)}}_{\text{BFKL}}-\underbrace{T_Y^{(n+1)}}_{\text{saturation}}$$

solving the B-JIMWLK equations gives $\left\langle T_{\mathbf{xy}}\right\rangle_Y$, $\left\langle T_{\mathbf{xz}}T_{\mathbf{zy}}\right\rangle_Y$ …
which can then be used to compute observables

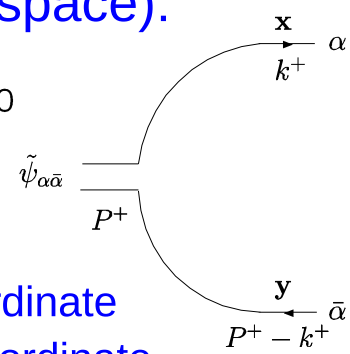
we will now derive the first equation of the hierarchy

Recall the dipole wave function

- the valence component $|d\rangle = |q\bar{q}\rangle$ (using the mixed space):

$$|d\rangle = \int d^2x\, d^2y\, \tilde{\psi}(\mathbf{x}-\mathbf{y})\, |q(\mathbf{x});\bar{q}(\mathbf{y})\rangle$$

$\ln(P^+/k^+) \sim 0$



$\tilde{\psi}(r)$ such that only perturbative sizes $r << 1/\Lambda_{QCD}$ are included

$\mathbf{x}$: quark space transverse coordinate

$\mathbf{y}$: antiquark space transverse coordinate

other degrees of freedom are not explicitely written

- the dipole dressed with one gluon $|d\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle$

$$|d\rangle = \int d^2x\, d^2y\, \tilde{\psi}(\mathbf{x}-\mathbf{y}) \Bigg[\left(1 - g_S^2\, C_F N_c \int d^2z\, |\varphi(\mathbf{x},\mathbf{y},\mathbf{z})|^2 \right)^{1/2} |q(\mathbf{x});\bar{q}(\mathbf{y})\rangle$$

color index

$$+ g_S \int d^2z\, \varphi(\mathbf{x},\mathbf{y},\mathbf{z}) T^a |q(\mathbf{x});\bar{q}(\mathbf{y}); g(\mathbf{z},a)\rangle \Bigg]$$

$$\frac{\mathbf{x}-\mathbf{z}}{(\mathbf{x}-\mathbf{z})^2} - \frac{\mathbf{y}-\mathbf{z}}{(\mathbf{y}-\mathbf{z})^2}$$

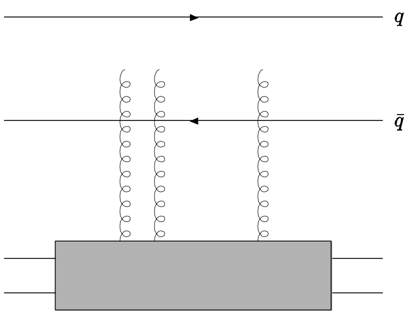
from QCD at order g_S

the dipole rapidity: $Y_d = \ln(P^+/k'^+) > 0$

Elastic scattering of the dipole

- the dipole scattering is described by Wilson lines

$$\hat{S}\left|q(\mathbf{x});\bar{q}(\mathbf{y})\right\rangle = Tr(W_F^+(\mathbf{y})W_F(\mathbf{x}))\left|q(\mathbf{x});\bar{q}(\mathbf{y})\right\rangle$$

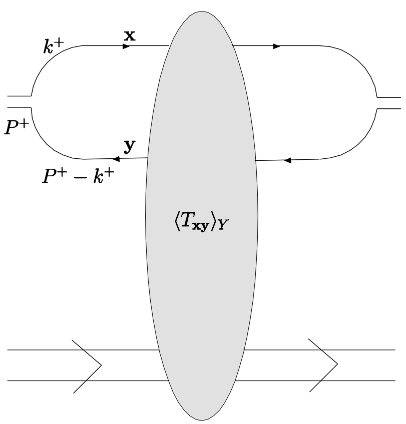


scattering of the antiquark　　　scattering of the quark

for the qqg component: $W_F(\mathbf{x})\quad W_F^+(\mathbf{y})\quad W_A(\mathbf{z})$

let's compute the elastic scattering amplitude $A_{el}(Y)$ in the dipole-CGC collision where $Y = Y_{CGC} + Y_d$ is the total rapidity

- in the frame in which $Y = Y_{CGC}$ and $Y_d = 0$



$$-iA_{el}(Y) = \int d^2x d^2y\, \left|\tilde{\psi}\,(\mathbf{x}-\mathbf{y})\right|^2 \int D\alpha\, \left|\Phi_Y[\alpha]\right|^2 T_{\mathbf{xy}}[\alpha]$$

$$= \int d^2x d^2y\, \left|\tilde{\psi}\,(\mathbf{x}-\mathbf{y})\right|^2 \left\langle T_{\mathbf{xy}}\right\rangle_Y$$

$$T_{\mathbf{xy}}[\alpha] = 1 - \frac{1}{N_c} Tr(W_F^+(\mathbf{y})W_F(\mathbf{x}))$$

Frame independence of $A_{el}(Y)$

- in the frame in which $|d\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle$ and $Y_{CGC} < Y$

using the following identity to get rid of the adjoint Wilson line

$q\bar{q}g$ = 1/2 $q\bar{q}q\bar{q}$ -1/(2Nc) $q\bar{q}$

$$Tr(W_F^+(\mathbf{y})T^a W_F(\mathbf{x})T^b)W_A^{ab}(\mathbf{z}) = \frac{1}{2}Tr(W_F^+(\mathbf{y})W_F(\mathbf{z}))Tr(W_F^+(\mathbf{z})W_F(\mathbf{x})) - \frac{1}{2N_c}Tr(W_F^+(\mathbf{y})W_F(\mathbf{x}))$$

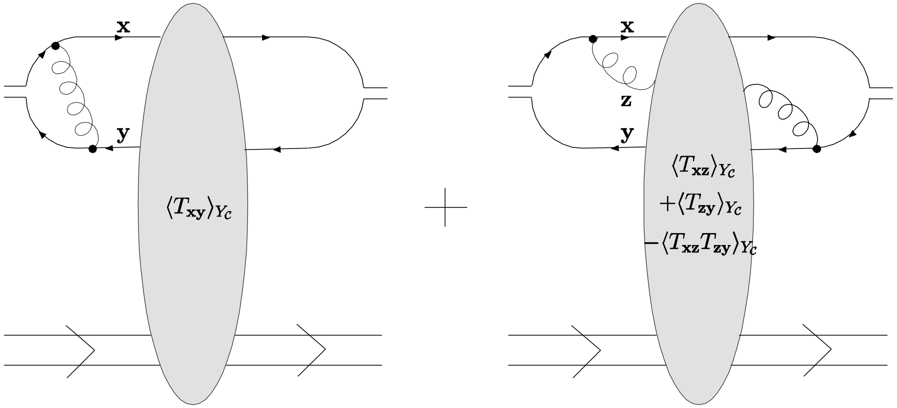
one obtains

$$-iA_{el}(Y) = \int d^2x d^2y \left|\tilde{\psi}(\mathbf{x}-\mathbf{y})\right|^2 \Big[\langle T_{\mathbf{xy}}\rangle_{Y_{CGC}}$$

$$+\bar{\alpha}(Y - Y_{CGC})\int \frac{d^2z}{2\pi}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\Big(\langle T_{\mathbf{xz}}\rangle_{Y_{CGC}} + \langle T_{\mathbf{zy}}\rangle_{Y_{CGC}} - \langle T_{\mathbf{xy}}\rangle_{Y_{CGC}} - \langle T_{\mathbf{xz}}T_{\mathbf{zy}}\rangle_{Y_{CGC}}\Big)\Big]$$

- frame independence:

our two expressions for $A_{el}(Y)$ should be identical, this requirement in the limit Y - $Y_{CGC} \equiv dY \to 0$ gives the first Balitsky equation



A mean field approximation: the Balitsky-Kovchegov equation

The BK equation

- obtain by neglecting correlations

the BK equation is a closed equation for $\left\langle T_{\mathbf{xy}}\right\rangle_Y$ obtained by assuming $\left\langle T_{\mathbf{xz}} T_{\mathbf{zy}}\right\rangle_Y = \left\langle T_{\mathbf{xz}}\right\rangle_Y \left\langle T_{\mathbf{zy}}\right\rangle_Y$

$$\frac{d}{dY}\left\langle T_{\mathbf{xy}}\right\rangle_Y = \bar{\alpha} \int \frac{d^2z}{2\pi} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left(\left\langle T_{\mathbf{xz}}\right\rangle_Y + \left\langle T_{\mathbf{zy}}\right\rangle_Y - \left\langle T_{\mathbf{xy}}\right\rangle_Y - \left\langle T_{\mathbf{xz}}\right\rangle_Y \left\langle T_{\mathbf{zy}}\right\rangle_Y\right)$$

let's consider impact-parameter independent solutions $\langle T_{\mathbf{xy}}\rangle_Y = N_Y(r = |\mathbf{x}-\mathbf{y}|)$

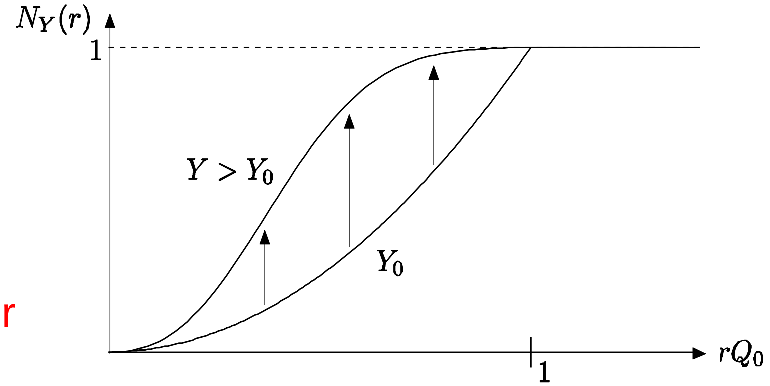
r = dipole size

- solutions: qualitative behavior

at small *Y*, N_Y is small, and the quadratic term can be neglected, the equation reduces then to the linear BFKL equation and N_Y rises exponentially with *Y*

as N_Y gets close to 1 (the stable fixed point of the equation), the non-linear term becomes important, and $\frac{d}{dY}N_Y \to 0$, N_Y saturates at 1

with increasing *Y*, the unitarization scale get bigger



Coordinate vs momentum space

- let' s go to momentum space

$$f_Y(k) = \int \frac{d^2r}{2\pi r^2}\, e^{i\mathbf{k}.\mathbf{r}} N_Y(r) \quad \Longrightarrow \quad \frac{d}{dY} f_Y(k) = \bar{\alpha} \int \frac{dk'^2}{k'^2}\left[\frac{k'^2 f_Y(k') - k^2 f_Y(k)}{|k^2 - k'^2|} + \frac{k^2 f_Y(k)}{\sqrt{4k'^4 + k^4}}\right] - \bar{\alpha} f_Y^2(k)$$

due to conformal invariance, the linear part of the equation is the same for $f_Y(k)$ and $\frac{N_Y(r)}{r^2}$

coordinate space	momentum space
$N_Y(r)$ dipole scattering amplitude	$f_Y(k)$ unintegrated gluon distribution
linear BFKL equation	linear BFKL equation
$\partial_Y N_Y(r) = \bar{\alpha}\ \chi(\partial_L)\, N_Y(r)$	$\partial_Y f_Y(k) = \bar{\alpha}\ \chi\left(-\partial_{\tilde{L}}\right) f_Y(k)$
$L = -\log(r^2 Q_0^2)$	$\tilde{L} = \log(k^2/Q_0^2)$
genuine saturation:	no real saturation:
$N_Y(r) < 1$ for $r \to \infty$	$f_Y(k) < \log(1/k)$ for $k \to 0$

both equation are in the same universality class as the F-KPP equation

The F-KPP equation

- same features as the BK equation

Fisher, Kolmogorov, Petrovsky, Piscounov

$$\underbrace{\partial_t u(x,t)}_{\text{time derivative}} = \underbrace{\partial_x^2 u(x,t)}_{\text{space derivative}} + \underbrace{u(x,t)}_{\text{linear term meaning exponential growth}} - \underbrace{u^2(x,t)}_{\text{non linear term meaning saturation}}$$

- dictionary F-KPP $\rightarrow$ BK

$t \longrightarrow \bar{\alpha} Y$

$x \longrightarrow L$ or $\tilde{L}$

$u \longrightarrow N$ or f

when expanding $\chi\left(-\partial_{\tilde{L}}\right)$ to second order,
the equations are the same (in momentum space)
in spite of small difference, same universality class

the precise forms of the space derivatives and of the non-linear term doesn't matter

these equations belong to the same universality class

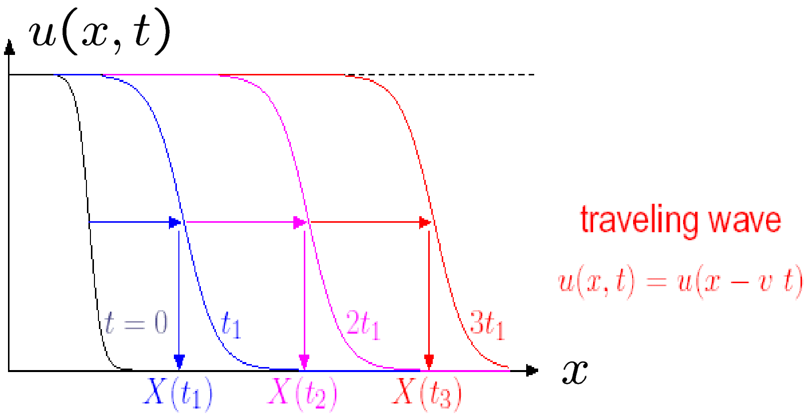
F-KPP and BK asympotic (in *t* or *Y*) solutions are the same: traveling waves

Traveling wave solutions

- what is a traveling wave

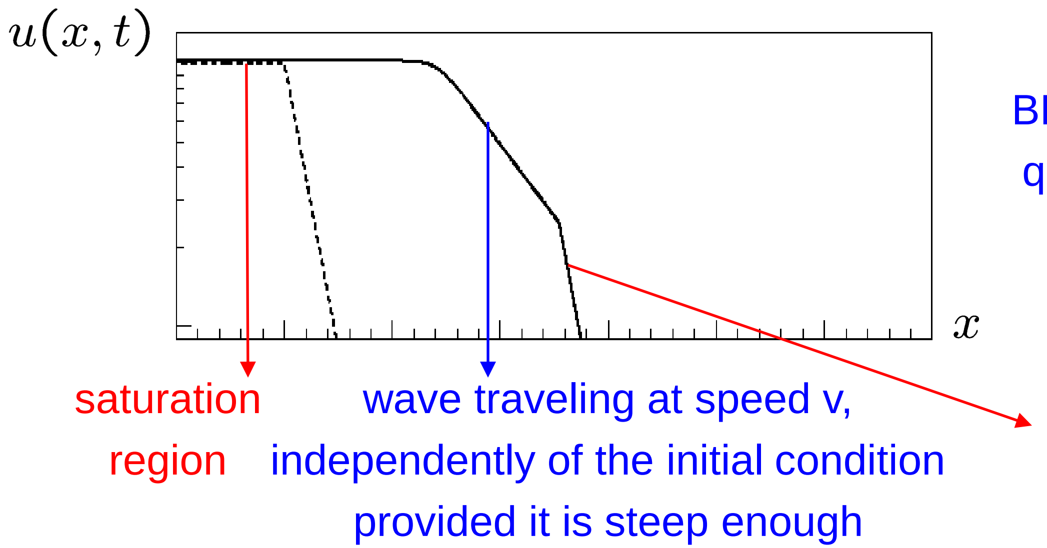
$$u(x,t) = u(s \equiv x - vt)$$

the speed of the wave v is determined only by the linear term of the equation



position: $X(t) = X_0 + vt$

- the formation of the traveling wave



BK solutions: same rapidity evolution

quantitative features can be derived

$$N_Y(r) = f(L - v_c\ \bar{\alpha}Y)$$

Recall the BFKL solutions

- a superposition of waves with speeds $v(\gamma) = \chi(\gamma)/\gamma$

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$N_Y^{lin}(r) = \int_{1/2-i\infty}^{1/2+i\infty} \frac{d\gamma}{2i\pi} e^{-\gamma\left(L - \bar{\alpha}\frac{\chi(\gamma)}{\gamma}Y\right)} n_0(\gamma)$$

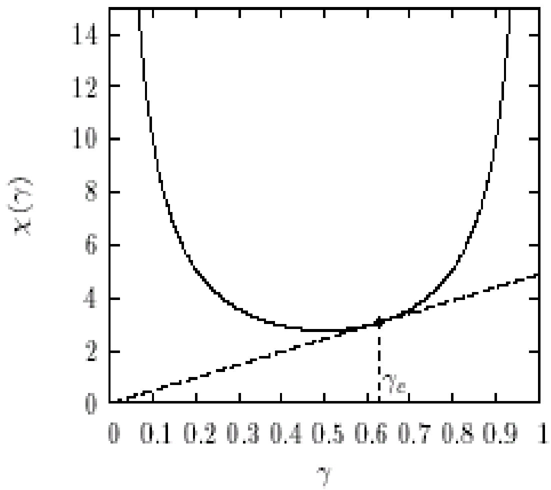
initial condition in Mellin space

- the minimal speed $v_c = \min\left(\chi(\gamma)/\gamma\right)$

is obtained for $\gamma = \gamma_c \simeq 0.6275$ and its value is $v_c = \chi(\gamma_c)/\gamma_c \simeq 4.88$

this will be the speed of the asymptotic traveling wave

$$N_Y(r) \sim e^{-\gamma_c(L - v_c\bar{\alpha}Y)}$$



- the saturation exponent γ_c

for the traveling wave to form, the initial condition $N_{Y_0}(r) \sim e^{-\gamma_0 L}$ must feature $\gamma_0 > \gamma_c$

QCD traveling waves

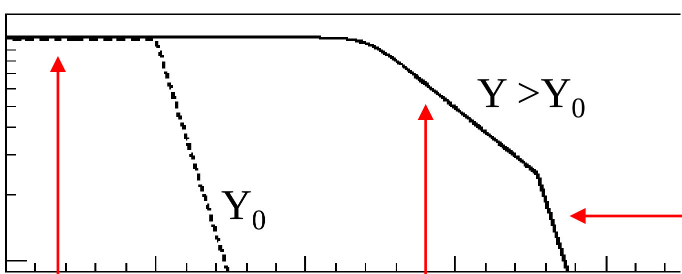
- the initial condition is steep enough

Munier and Peschanski (2004)

in QCD $\gamma_0 = 1$ $\quad N_{Y_0}(r) \sim e^{-L} = Q_0^2\ r^2$

- asymptotic solutions of BK

$N_Y(r)$



$N_Y(r) \propto e^{-L}$

$L = -\log(r^2 Q_0^2)$

$N_Y(r) = 1$ $\quad N_Y(r) \propto e^{-\gamma_c(L - v_c\bar{\alpha}Y)}$

same features for $f_Y(k)$ except in the saturation region

$f_Y(k) = \log(1/k)$ for $k \to 0$

- the saturation scale

called geometric scaling

$N_Y(r) = \text{cste}$ $\Rightarrow$ $L = v_c\bar{\alpha}Y$ $\Rightarrow$ $r = 1/Q_s(Y)$

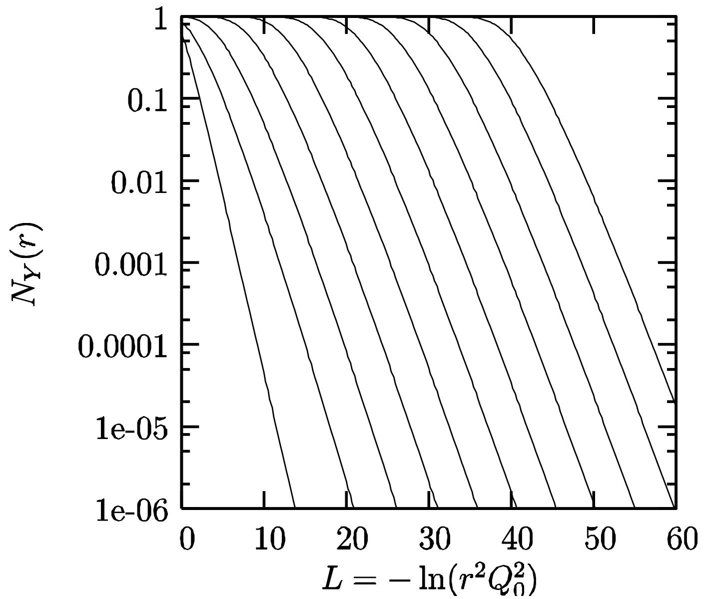
with $Q_s^2(Y) = Q_0^2\ e^{v_c\bar{\alpha}Y} = Q_0^2\ x^{-v_c\bar{\alpha}}$ $\quad$ then $N_Y(r) \propto \left(r^2 Q_s^2(Y)\right)^{-\gamma_c}$

Numerical solutions

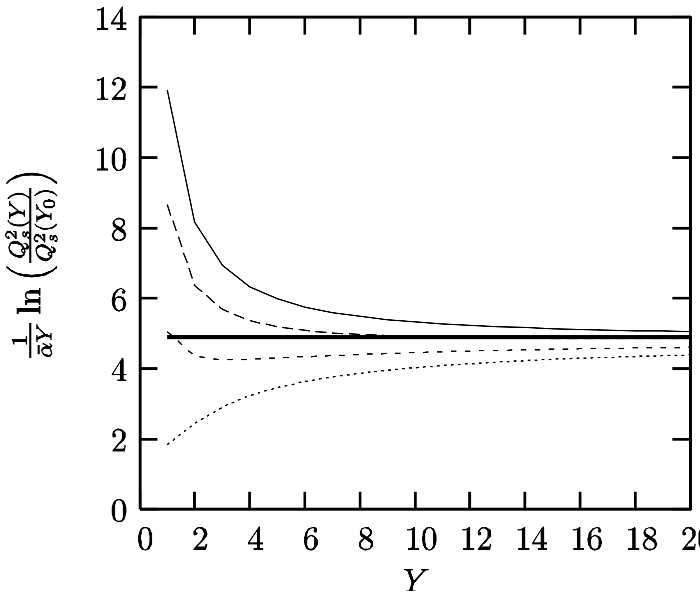
Soyez

- numerical simulations confirm the results

for the scattering amplitude



for the saturation scale



4.88

sub-asymptotic corrections (≡ geometric scaling violations) are also known:

$$\frac{1}{Y}\ \ln\left(\frac{Q_s^2(Y)}{Q_0^2}\right) = \bar{\alpha}v(Y) = \bar{\alpha}v_c - \frac{3}{2\gamma_c Y} + \frac{3}{2\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)Y^3}} + \mathcal{O}\left(\frac{1}{Y^2}\right)$$

Running coupling corrections

- running coupling corrections to the BK equation

taken into account by the substitution

$$\alpha_s(\mathbf{r}^2) = \left[-\frac{11N_c - 2N_f}{12\pi} \ln\left(\mathbf{r}^2 \Lambda^2_{QCD} \right) \right]^{-1}$$

$$\frac{\bar{\alpha}}{2\pi} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \xrightarrow[\text{Weigert}]{\text{Kovchegov}} \frac{N_c}{2\pi^2} \left[\frac{\alpha_s((\mathbf{x}-\mathbf{z})^2)}{(\mathbf{x}-\mathbf{z})^2} - 2\frac{\alpha_s((\mathbf{x}-\mathbf{z})^2)\alpha_s((\mathbf{z}-\mathbf{y})^2)}{\alpha_s((\mathbf{x}-\mathbf{y})^2)} + \frac{\alpha_s((\mathbf{z}-\mathbf{y})^2)}{(\mathbf{z}-\mathbf{y})^2} \right]$$

Balitsky $\downarrow$

$$\frac{N_c \alpha_s((\mathbf{x}-\mathbf{y})^2)}{2\pi^2} \left[\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} + \frac{1}{(\mathbf{x}-\mathbf{z})^2} \left(\frac{\alpha_s((\mathbf{x}-\mathbf{z})^2)}{\alpha_s((\mathbf{z}-\mathbf{y})^2)} - 1 \right) + \frac{1}{(\mathbf{z}-\mathbf{y})^2} \left(\frac{\alpha_s((\mathbf{z}-\mathbf{y})^2)}{\alpha_s((\mathbf{x}-\mathbf{z})^2)} - 1 \right) \right]$$

- consequences

similar to those first obtained by the simpler substitution $\frac{\bar{\alpha}}{2\pi} \to \frac{N_c\ \alpha_s((\mathbf{x}-\mathbf{y})^2)}{2\pi^2}$

running coupling corrections slow down the increase of Qs with energy

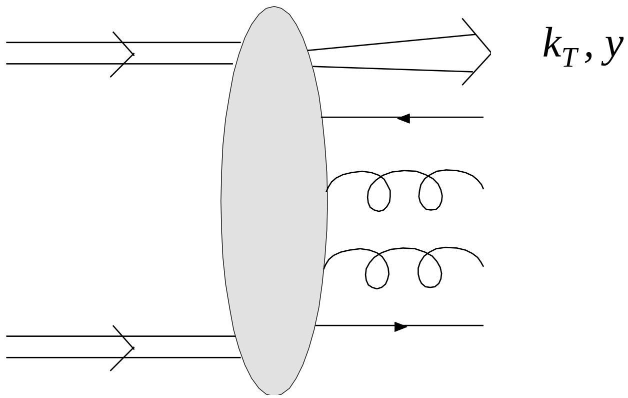
$$\ln\left(\frac{Q_s^2(Y)}{Q_0^2} \right) = \sqrt{\frac{24 N_c v_c}{11N_c - 2N_f} Y}$$

also confirmed by numerical simulations, however this asymptotic regime is reached for larger rapidities

Forward particle production in pA collisions

Forward particle production

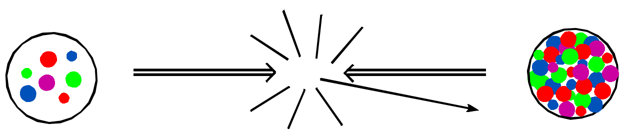
- forward rapidities probe small values of *x*



transverse momentum k_T, rapidity $y > 0$

values of x probed in the process:

$$x_1\sqrt{s} = k_T\, e^{y} \qquad\qquad x_2\sqrt{s} = k_T\, e^{-y}$$



the large-x hadron should be described by standard leading-twist parton distributions

the small-x hadron/nucleus should be described by CGC-averaged correlators

the cross-section:

$$k_T^2\frac{d\sigma}{d^2k_T dy} \propto g(x_1,k_T^2)\, f(x_2,k_T^2)$$

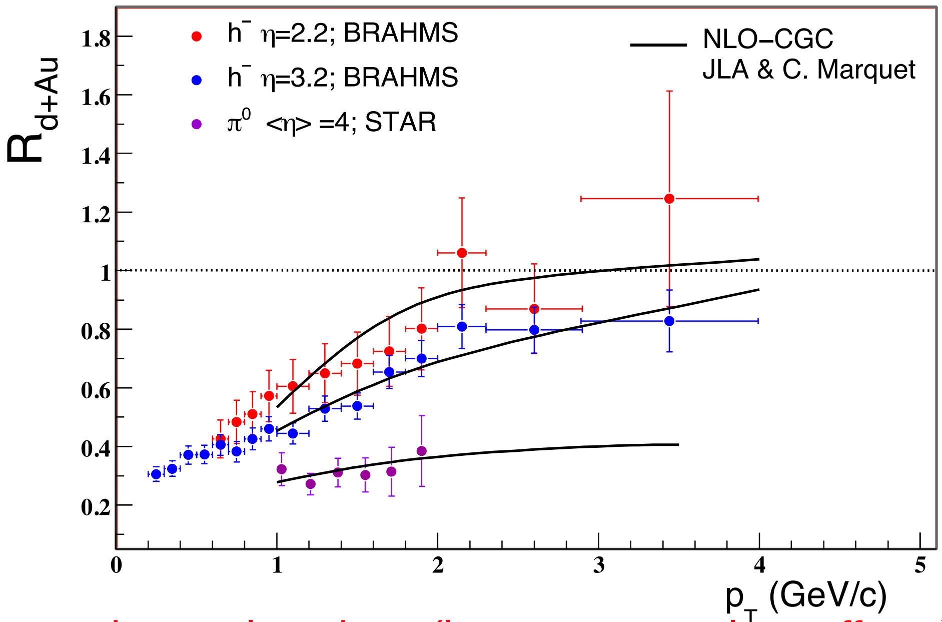
single gluon production probes only the unintegrated gluon distribution (2-point function)

Nuclear modification factor

$R_{dA} = 1$ in the absence of nuclear effects, i.e. if the gluons in the nucleus interact incoherently as in A protons

$$R_{dA} = \frac{1}{N_{coll}} \frac{\dfrac{dN^{dA\to hX}}{d^2k dy}}{\dfrac{dN^{pp\to hX}}{d^2k dy}}$$

the suppressed production ($R_{dA} < 1$) was predicted in the Color Glass Condensate picture, along with the rapidity dependence

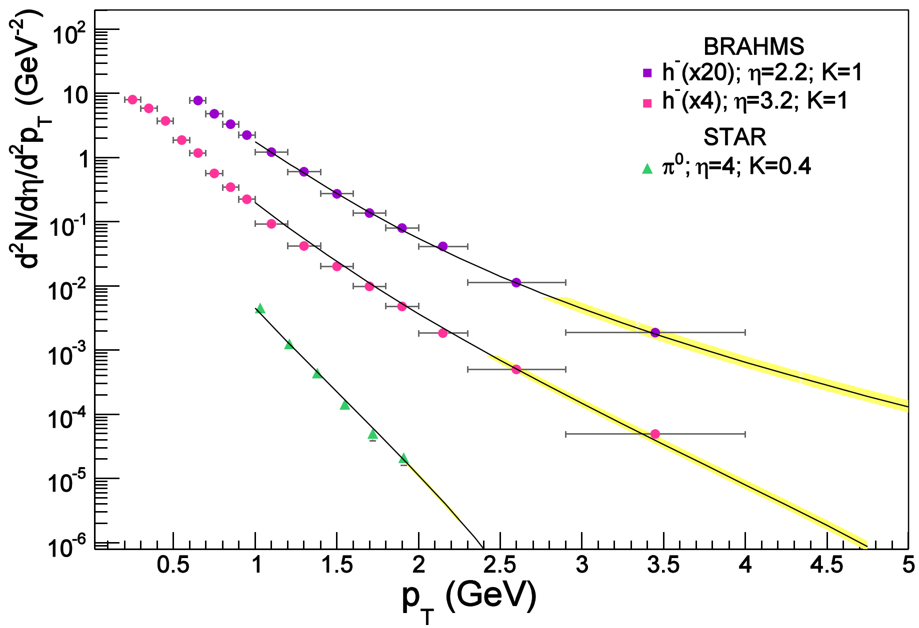


Albacete and CM (2010)

note: alternative explanations (large-x energy loss effects) have been proposed

Kopeliovich et al (2005), Frankfurt et al (2007)

NLO-BK description of d+Au data



Albacete and C.M. (2010)

the shapes and normalizations are well reproduced, except the π^0 normalization

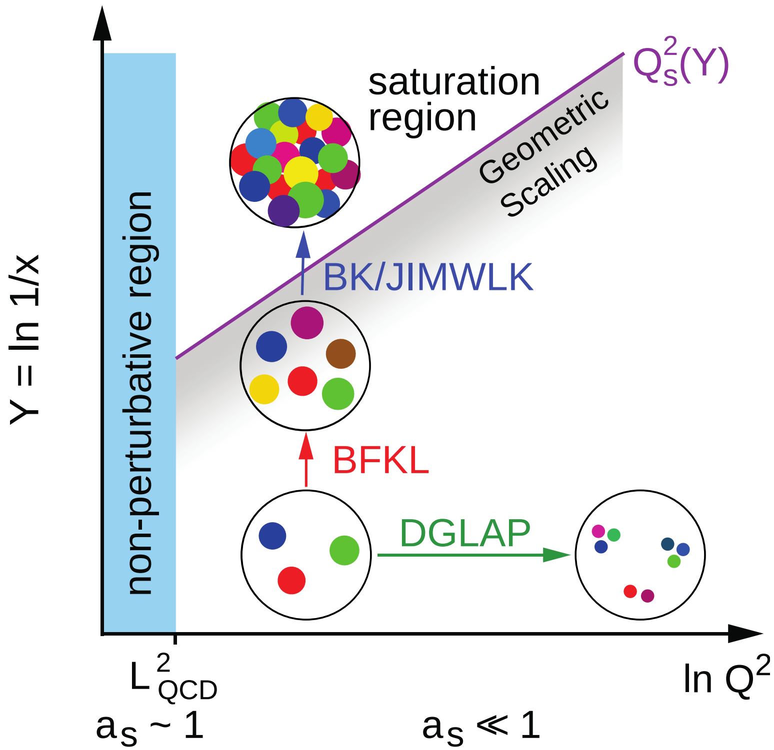
the speed of the *x* evolution and of the p_T decrease are predicted

this fixes the two parameters of the theory:

- the value of *x* at which one starts to trust (and therefore use) the CGC description
- and the saturation scale at that value of *x* $Q_s^2(x_0) = 0.4\ \mathrm{GeV}^2$ $\quad x_0 = 0.02$

Conclusions

- fundamental consequence of QCD dynamics:



at asymptotically small x:

- QCD evolution becomes non-linear
- particle production becomes non-linear
- QCD stays weakly coupled

- the Color Glass Condensate (CGC) has emerged as the best candidate to approximate QCD in the saturation regime

  both in terms of practical applicability and phenomenological success

- the energy dependence of the saturation scale, and more generally of observables, can be computed from first principles

although in practice, the predictivity will depend on the level of accuracy of the calculation (LO vs NLO, amount of non-perturbative inputs needed, …)