

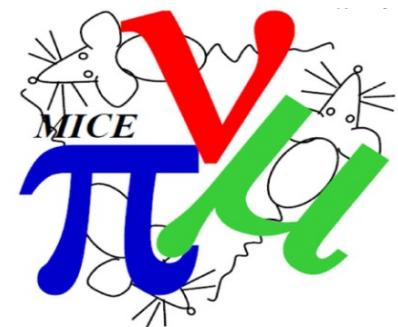
# Nonlinear Beam Optics with Kernel Density Estimation



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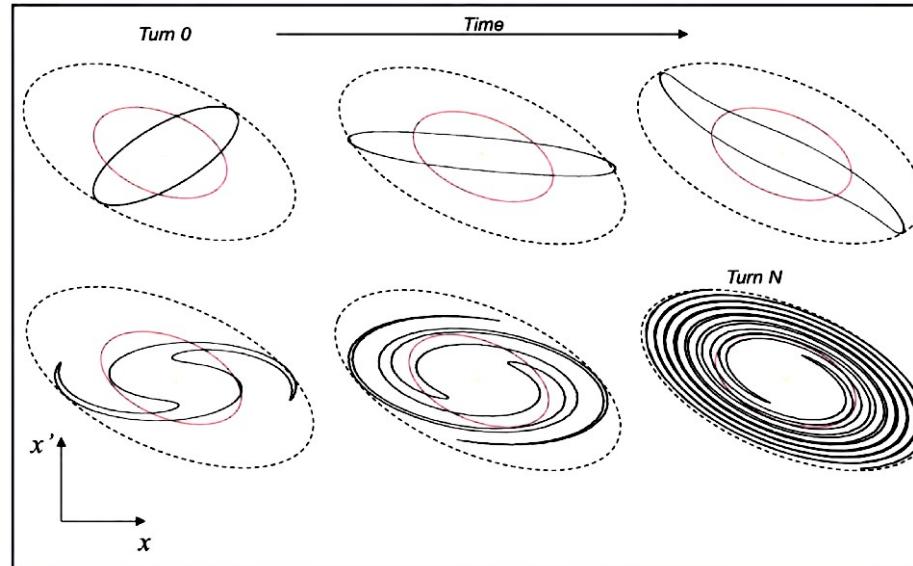
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# Motivation

- Solenoid beam optics is prone to filamentation and other nonlinear effects.



V. Kain, "Beam Transfer and Machine Protection", CERN

- Nonlinearities can cause the beam's distorted shape to fill larger ellipse leading to “apparent” RMS emittance growth.
- An alternative measure of cooling → use Kernel Density Estimation (KDE) to estimate the probability density function (PDF) and the phase-space volume occupied by the beam.

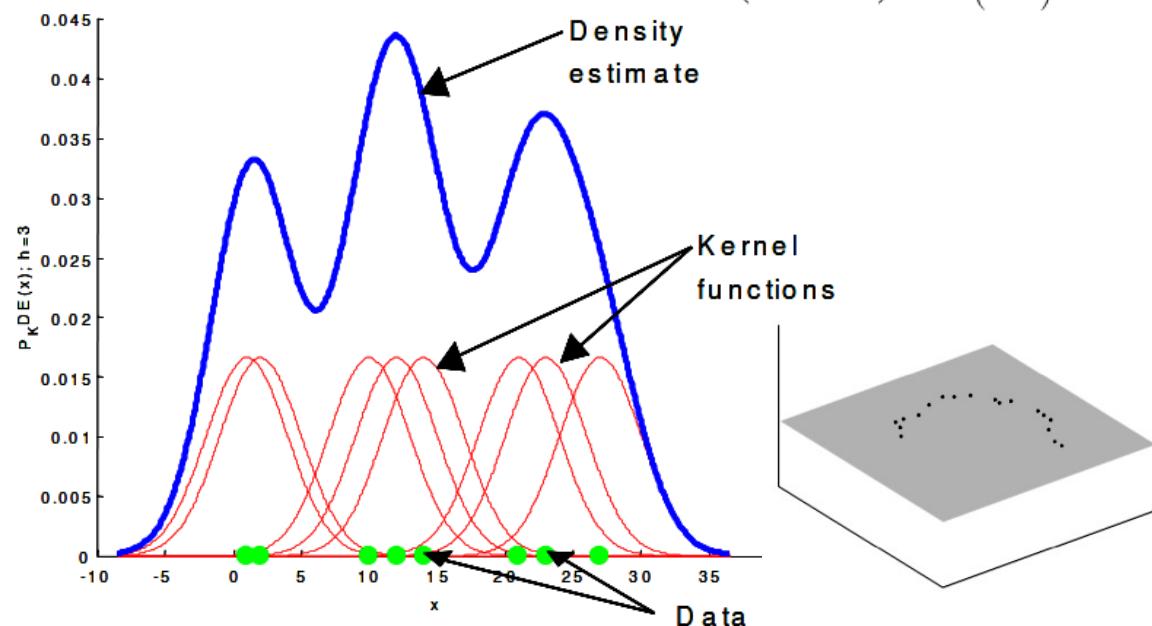
# Background

- Kernel Density Estimation → estimates probability density function of muons in phase space using kernel functions.
- Individual kernel functions,  $K$  of width or **bandwidth**  $h$  assigned to individual muons with  $x_i$  in  $d$ -dimensional phase space. Underlying PDF is revealed by summing over contributions from all N muons,

$$\hat{f}(\vec{x}) = \frac{1}{h^d N} \sum_{i=1}^N K\left(\frac{\vec{x} - \vec{x}_i}{h}\right),$$

- Gaussian kernel functions are used throughout this analysis,

$$K\left(\frac{\vec{x} - \vec{x}_i}{h}\right) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{\|\vec{x} - \vec{x}_i\|^2}{2h^2}\right).$$



M. Rousson, et. al., "Efficient Kernel Density Estimation of Shape and Intensity Priors for Level Set Segmentation", (MICCAI) (2005)

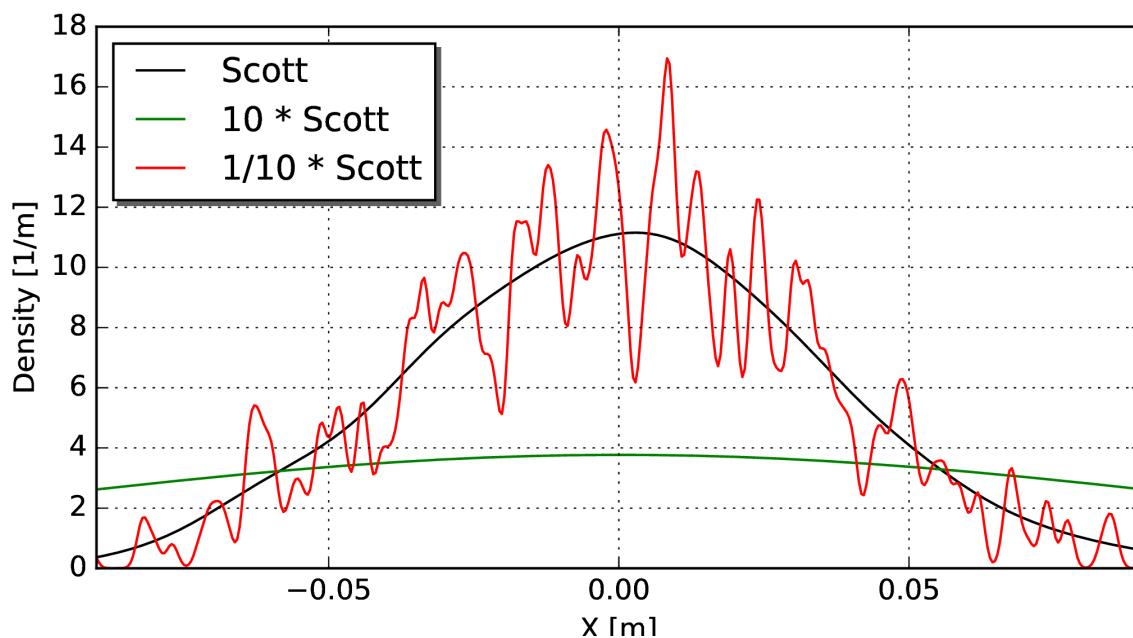
R. Gutierrez Osuna, "Kernel density estimation", CSCE 666 Pattern Analysis, Texas A&M University.

# Bandwidth Parameter

- Bandwidth parameter consists of a factor multiplied by a data scale parameter.
  - Choice of scale parameter: in one-dimension (i.e. muon x position) is the standard deviation and in two-dimensions and higher, the muon covariance matrix.
- Two factors defined in Python's Scipy module ( $n$  = number of muons and  $d$  = dimension variable):

$$\rightarrow \text{Scott's factor, } h = [n]^{-\frac{1}{d+4}}$$

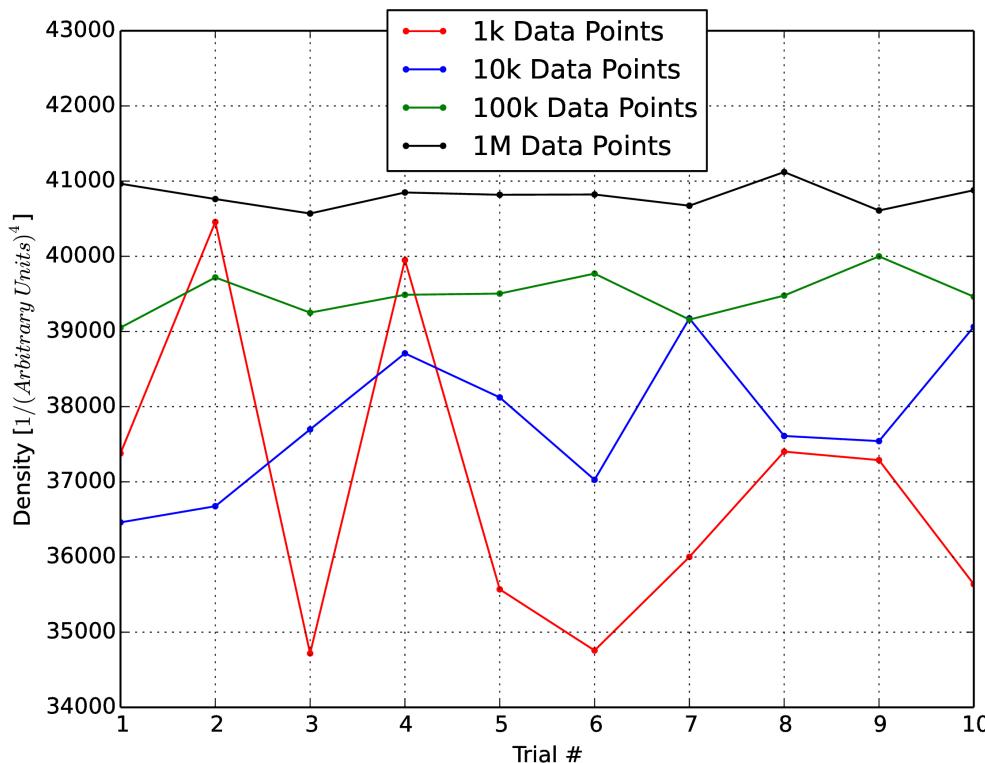
$$\rightarrow \text{Silverman's factor, } h = \left[ \frac{n(d+4)}{4} \right]^{-\frac{1}{d+4}}$$



- Both bandwidth factors follow the “rule of thumb” approach – by D. W. Scott and B. W. Silverman based on some parametric family (Gaussian in this case) and scaled to minimize the Asymptotic Mean Integrated Squared Error (measure of discrepancy between KDE and true PDF).
- The Scott's factor used throughout this analysis.
- Figure: effects of varying the bandwidth factor: **noisier density** with a smaller factor and **over-smoothed density** with a larger factor.

# Dependence on Sample Size

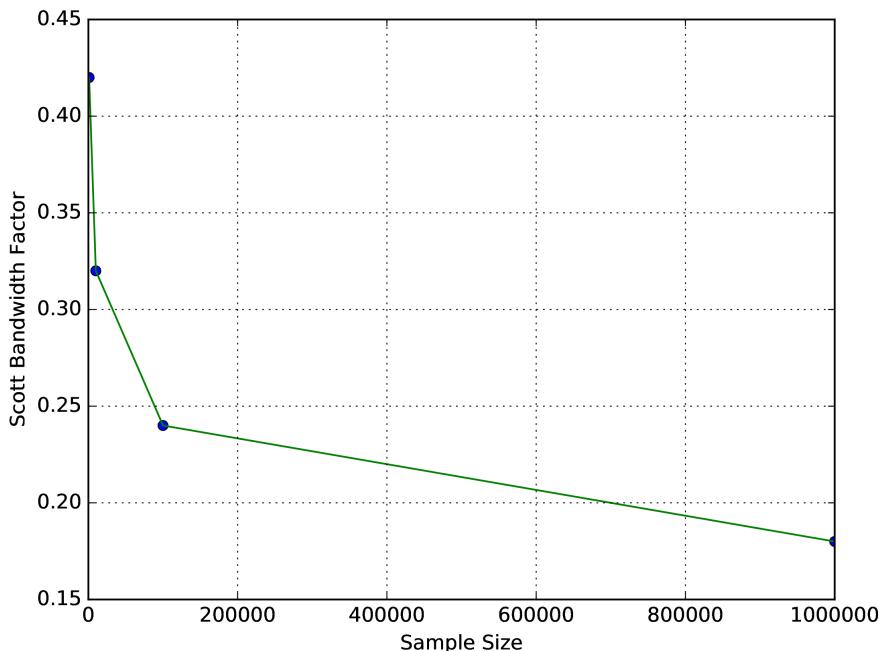
- Minimum required data points not immediately obvious → generally best if this is tested for a specific sample space:
- Randomly generated 10 four-dimensional generic Gaussian distributions of sizes:  $\sigma_x = \sigma_y = 0.03$  a.u &  $\sigma_{px} = \sigma_{py} = 0.02$  a.u.
- Estimated density for  $1\sigma$  contours of these 10 distributions using KDE.
- Repeated the above step for 4 different sample sizes.
- Result: reduced variability with sample size growth.



- Mean density increases as sample size grows. Possible reasons:
  - ✓ Bandwidth parameter (slide 4) becomes smaller as sample size grows hence leading to different densities.
  - ✓ The contour is fixed to  $1\sigma$  of the distribution (in 4D, the number of data points in  $1\sigma$  is ~9% of the total number of data points). As the number of points inside this  $1\sigma$  contour grows, the probability of finding a data point within the contour (PDF) increases.
  - ✓ KDE should approach true PDF as sample size grows.

# Bandwidth's Dependence on Sample Size

- Bandwidth factor depends on:
  - sample size, n.
  - the standard deviation and covariance matrix of the data set under study (respectively in one- and multi-dimensional spaces).
- To verify dependence on sample size, the bandwidth factors (Scott's factor from slide 4) for different sample sizes were computed. Result: larger n leads to a smaller bandwidth factor.

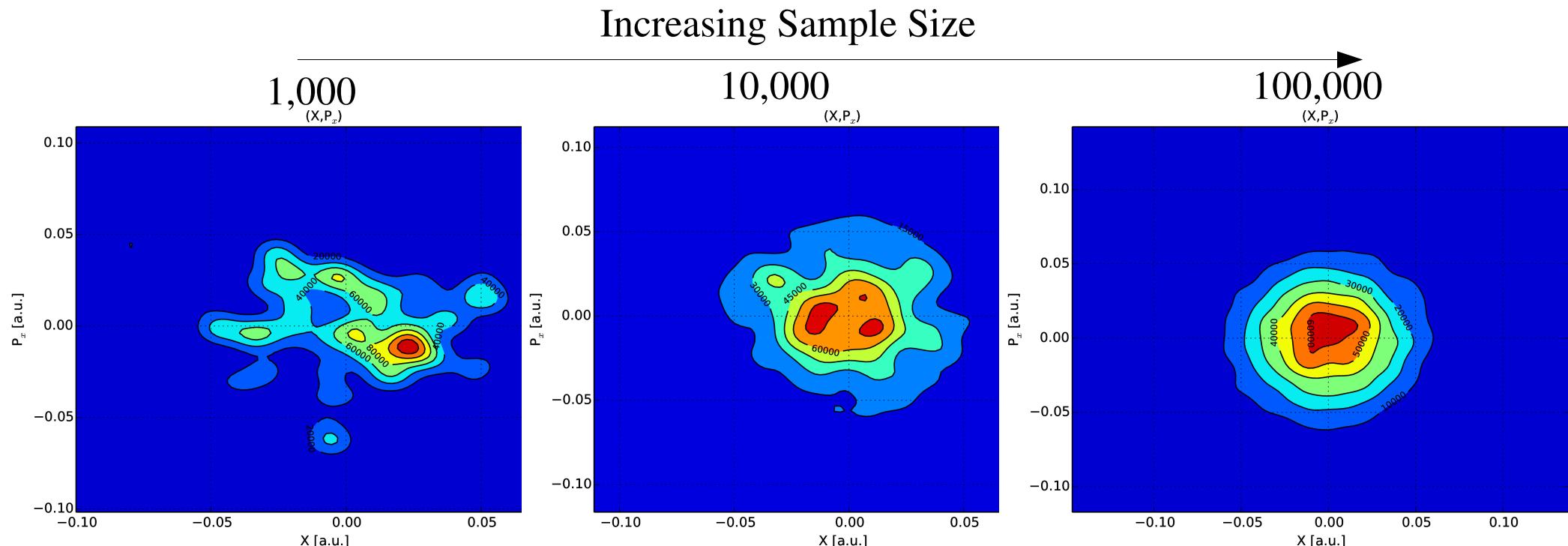


Sample Size	Scott's Bandwidth Factor
1,000	0.42
10,000	0.32
100,000	0.24
1,000,000	0.18

Side note: in multi-dimensions, covariance matrix elements are multiplied by the bandwidth factor → each coordinate has a separate bandwidth **parameter**. I study the effect of varying the bandwidth **factor not the parameter**.

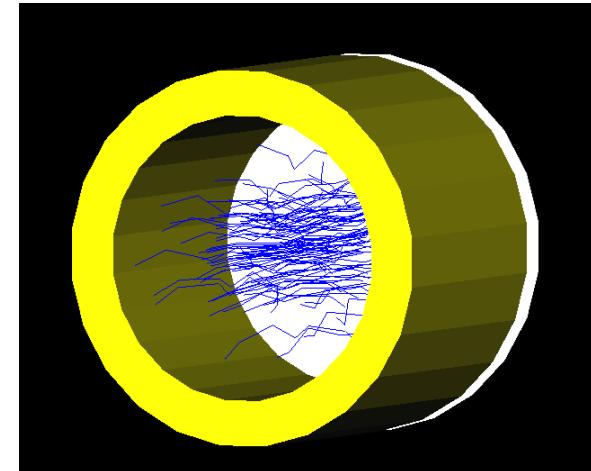
# Bandwidth's Dependence on Sample Size Cont.

- Tested the bandwidth's dependence on the number of data points using a generic Gaussian distribution (first trial from slide 5).
- Fixed bandwidth factor to 0.2:
  - ✓ Too small for 1k sample size (left) → density too noisy.
  - ✓ Appropriate for 100k sample size (right) → density not too noisy and not over-smoothed.
  - ✓ As sample size grows, the fixed bandwidth becomes more and more appropriate.



# Bandwidth's Effect on Density

- Motivation:
  - ✓ Fix the sample size (10,000) and study the effects of varying the bandwidth factor on density.
  - ✓ Study the sensitivity of KDE to long tails.
- Approach:
  - ✓ Generated a G4beamline Gaussian muon beam and passed it through a solenoid.
  - ✓ The regions in phase space to which KDE is applied: the entrance (upstream) and the center (downstream) of the solenoid.

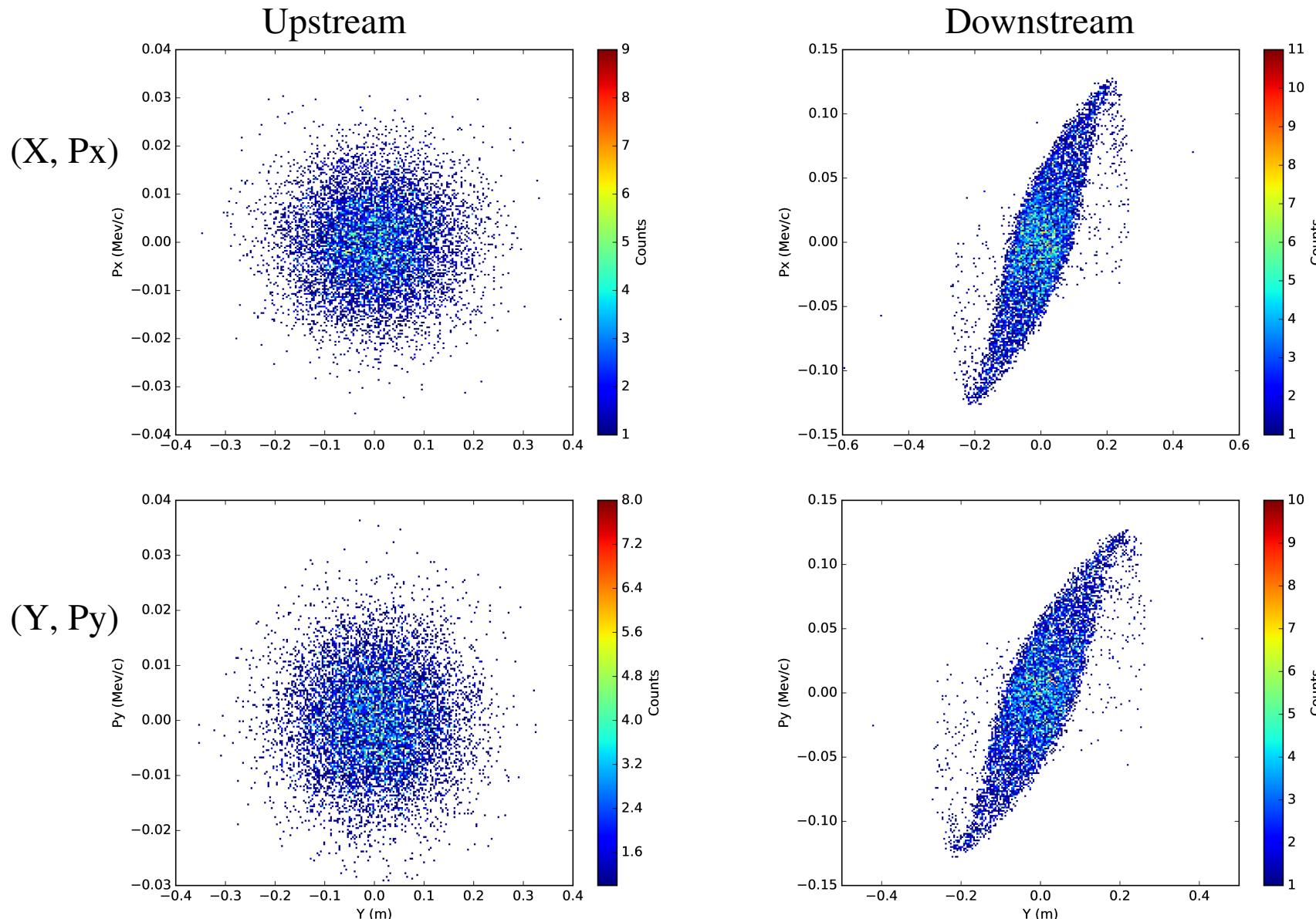


Simulation Parameters	Values
Number of events	10,000
Coil current	227.9 A/mm <sup>2</sup>
Solenoid Length	500 mm
$\sigma_x$	90 mm
$\sigma_y$	90 mm
$\sigma_{px}$	0.045 MeV/c
$\sigma_{py}$	0.045 MeV/c
Reference Momentum	200 MeV/c

Side Note: **Intentionally** produced the non-linear tails via an increase in beam size to study KDE of long-tailed distributions and the effects of bandwidth on the density in extreme conditions.

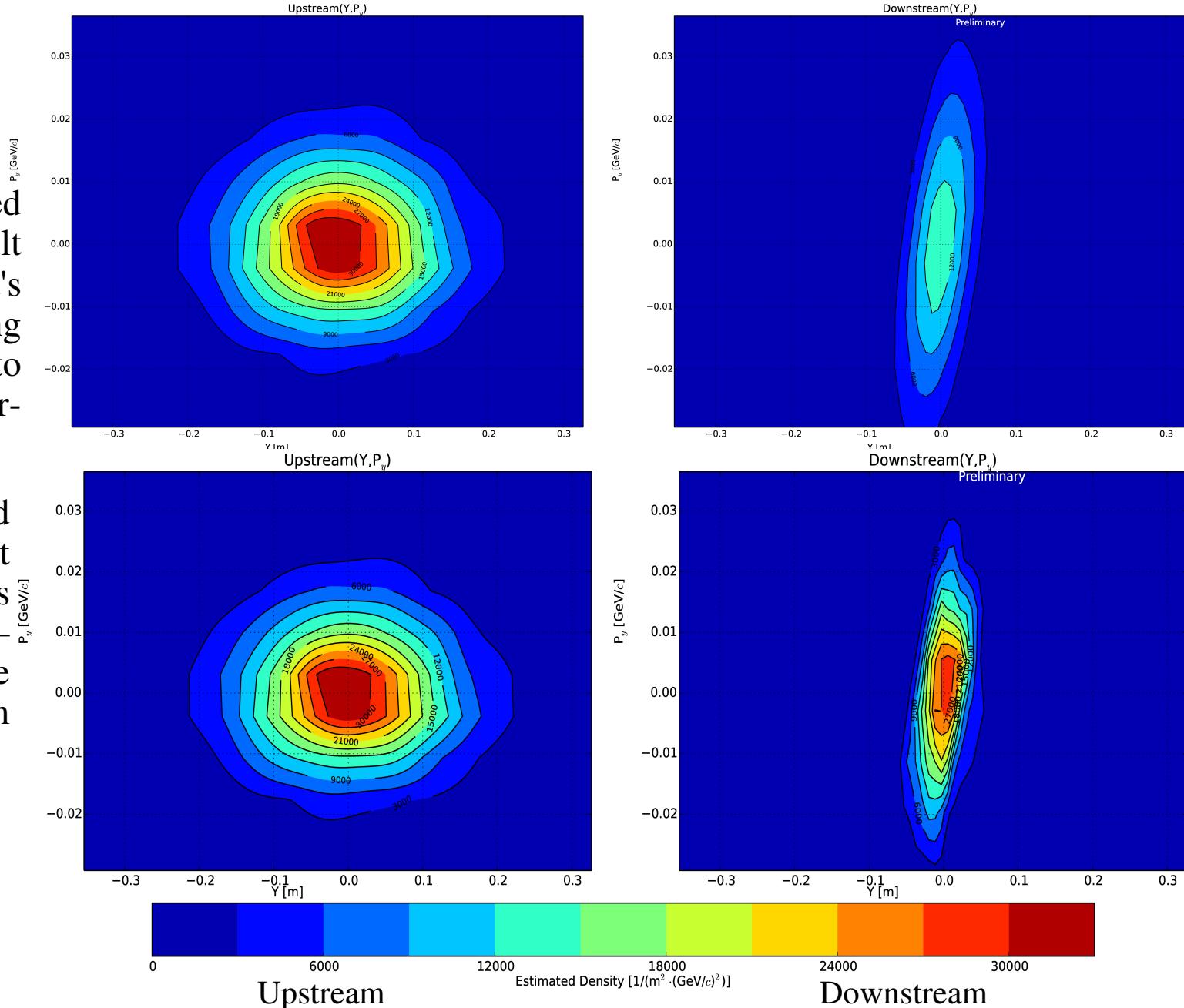
# Bandwidth's Effect on Density Cont. – Phase Space

- Clear display of the long-tails and the non-linear effects in the distribution, downstream.



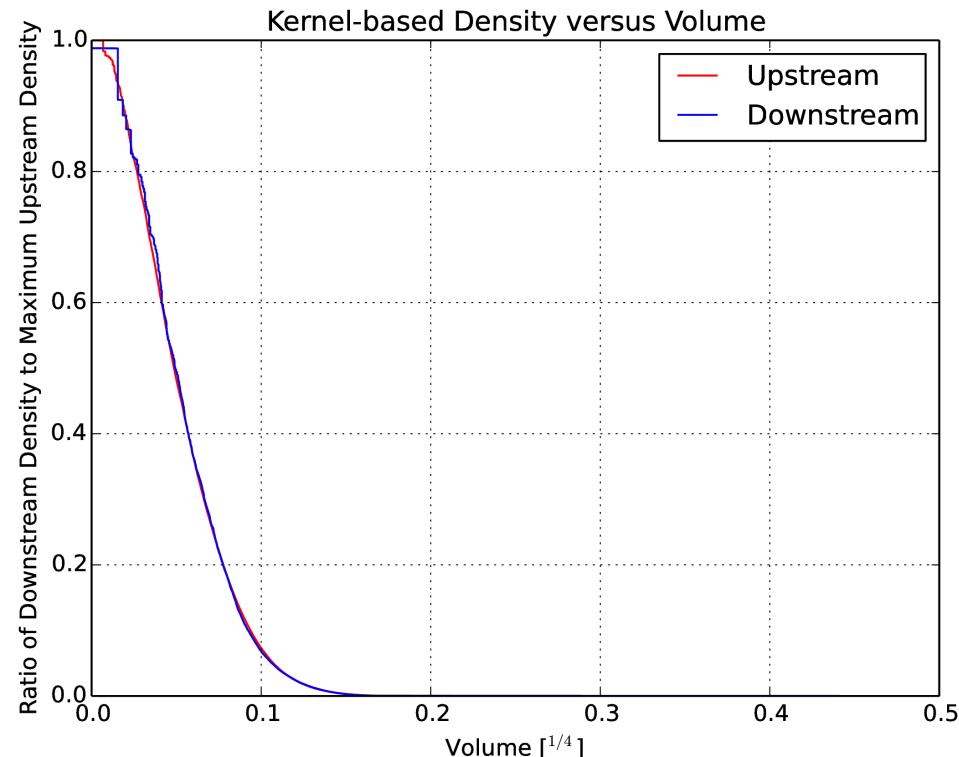
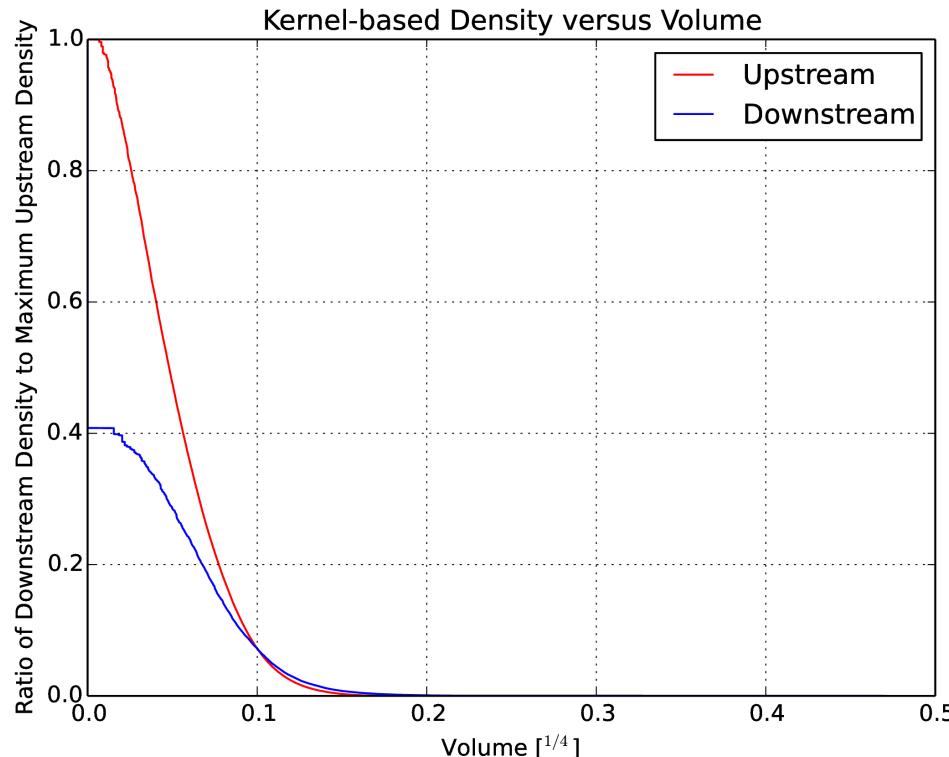
# Bandwidth's Effect on Density – Contours

- Top: density obtained with the default bandwidth factor, Scott's – demonstrates heating from upstream to downstream and is over-smoothed.
- Bottom: density obtained with the default bandwidth factor, Scott's multiplied by 0.42 – better represents the tailed phase space from last slide.



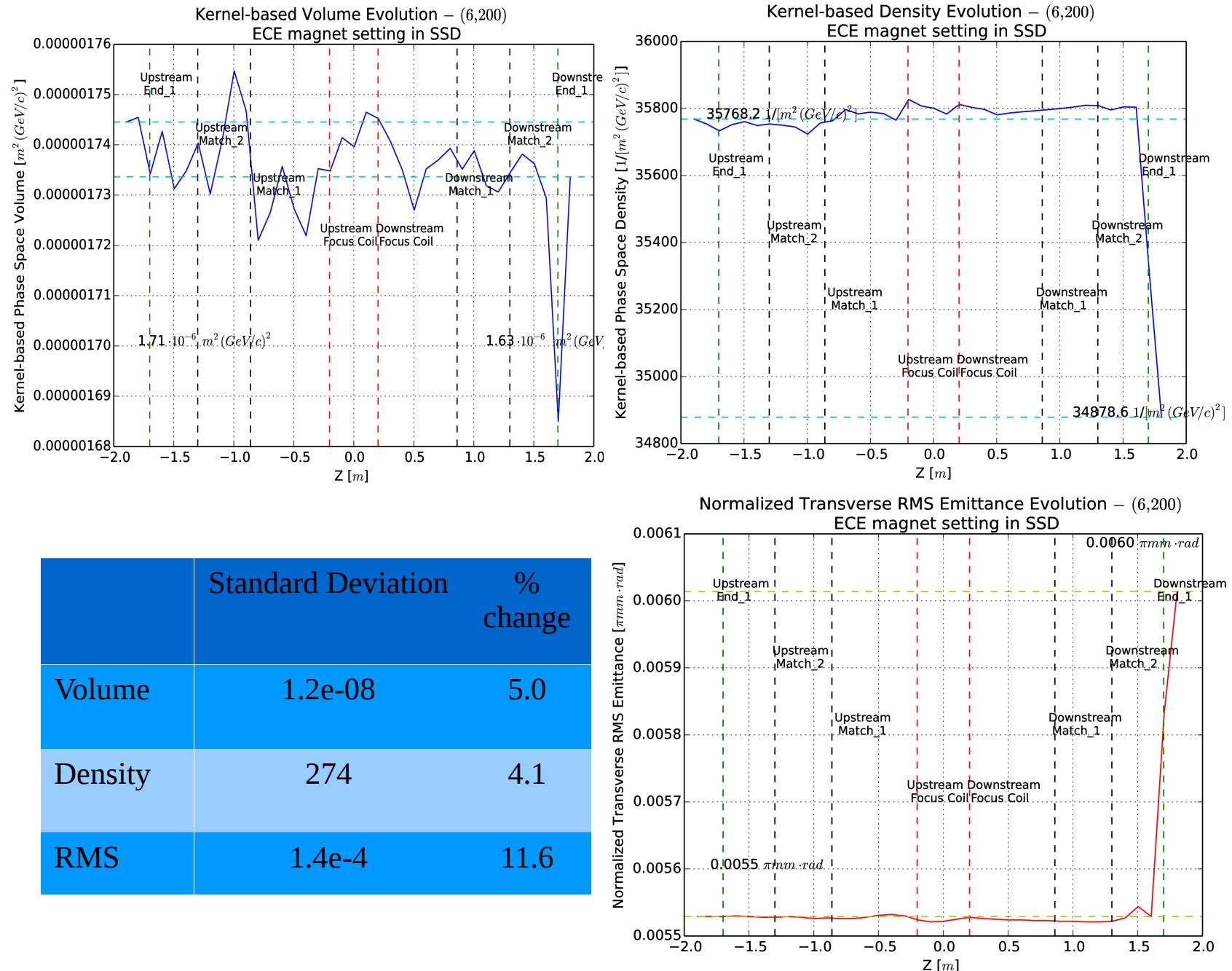
# Bandwidth's Effect on Density Cont. – Quantitative Comparison

- Left: ratio of downstream density to maximum upstream density vs. radius of four dimensional phase space with the larger bandwidth factor, Scott's factor.
- Right: ratio of downstream density to maximum upstream density vs. radius of four dimensional phase space with 0.423 times the Scott's factor. This bandwidth factor is a better representation of the conservation of density inside the solenoid.
- Results:
  - ✓ Choice of bandwidth factor has strong effects on the data interpretation.
  - ✓ KDE is sensitive to tails.



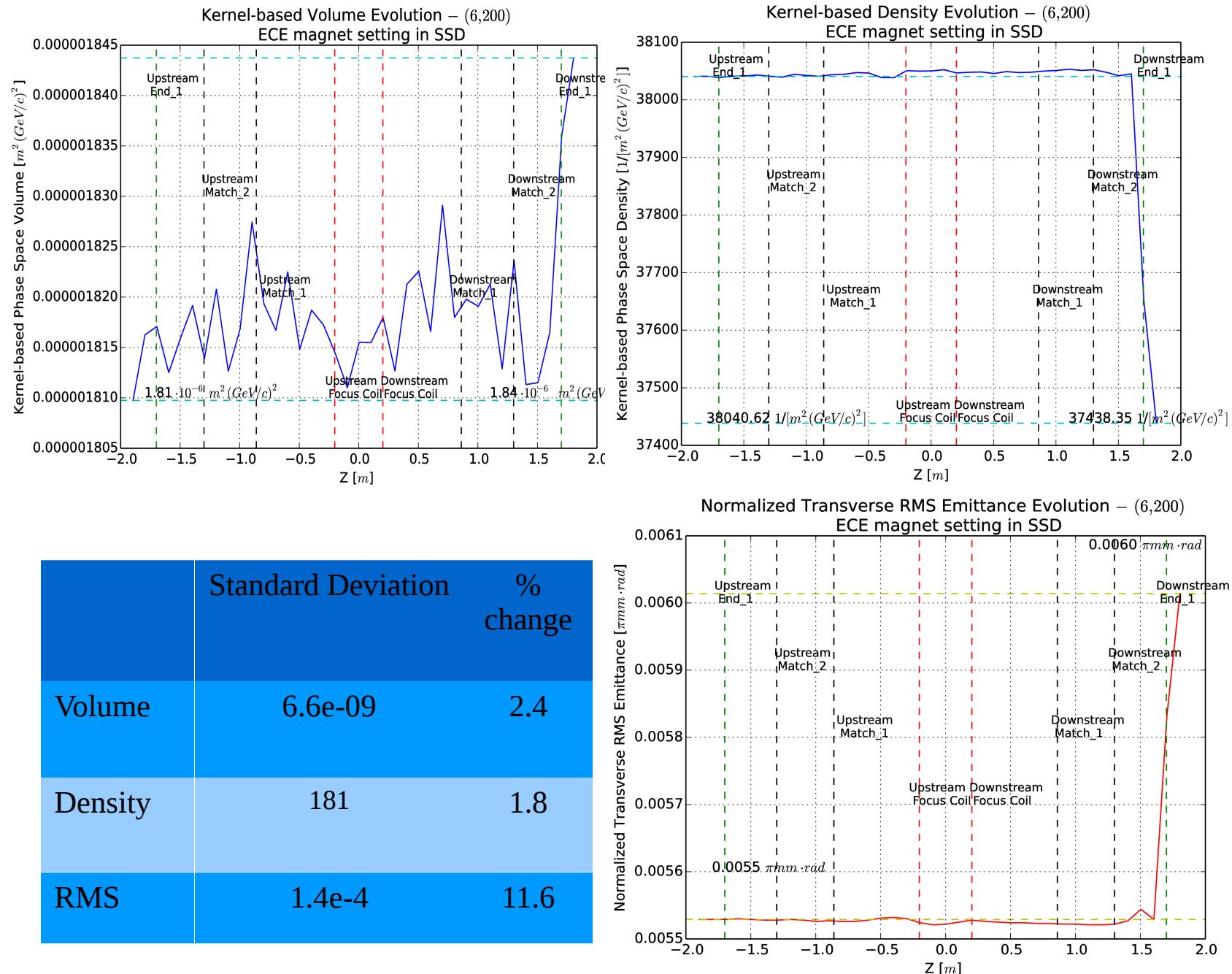
# MICE Optics with KDE

- Evolutions of emittance, beam-core volume and density along an empty-absorber channel:
  - ✓ Downstream match coils turned off.
  - ✓ Initial beam optics: (6, 200).
  - ✓ Trackers removed to ensure minimized energy loss effects.
  - ✓ 10,000 input with transmission of ~95%.
  - Result: spikes start to show up at  $z = 1.5$  m (close to the location of the downstream match coils).



# MICE Optics with KDE Cont. – Larger Sample Size

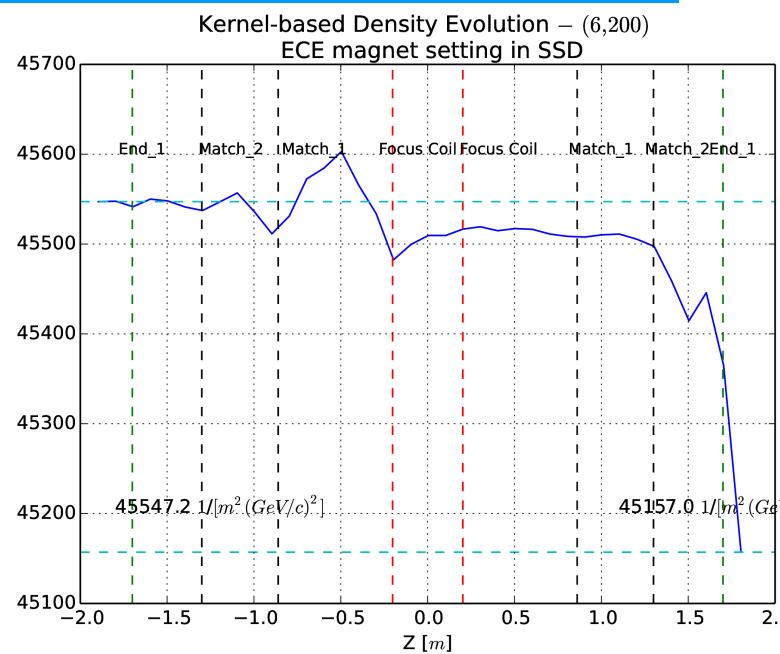
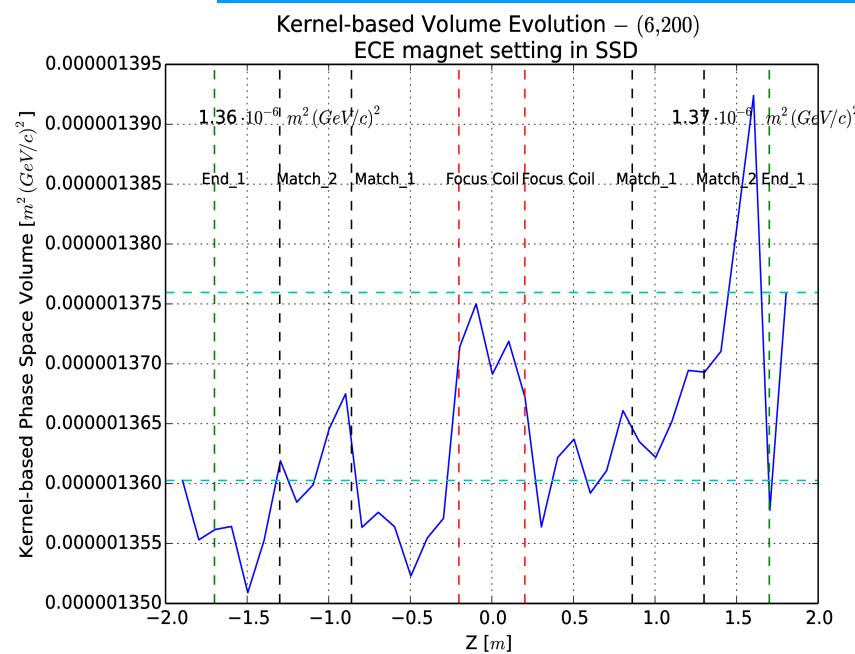
- Evolutions of emittance, beam-core volume and density along an empty-absorber channel:
  - ✓ Downstream match coils turned off.
  - ✓ Initial beam optics: (6, 200).
  - ✓ Trackers removed to ensure minimized energy loss effects.
  - ✓ 100,000 input with transmission of ~95%.
- Results: shown on the table.



# MICE Optics with KDE Cont. – $5\sigma$ (High Amplitude) Cut

- Used KDE to apply a high amplitude cut,
- $5\sigma$  of the beam roughly contain 85% of the total muons in four dimensions (S.Y. Lee) → apply KDE on the distribution, do a binary search to isolate contour with 85% of all muons, and record  $(x, p_x, y, p_y)$  of the contained muons.
- After the cut, redo the “old routine” (described in the previous slide) on this subset of muons. Emittance same as slide 2 (did not apply this to emittance). Results are below,

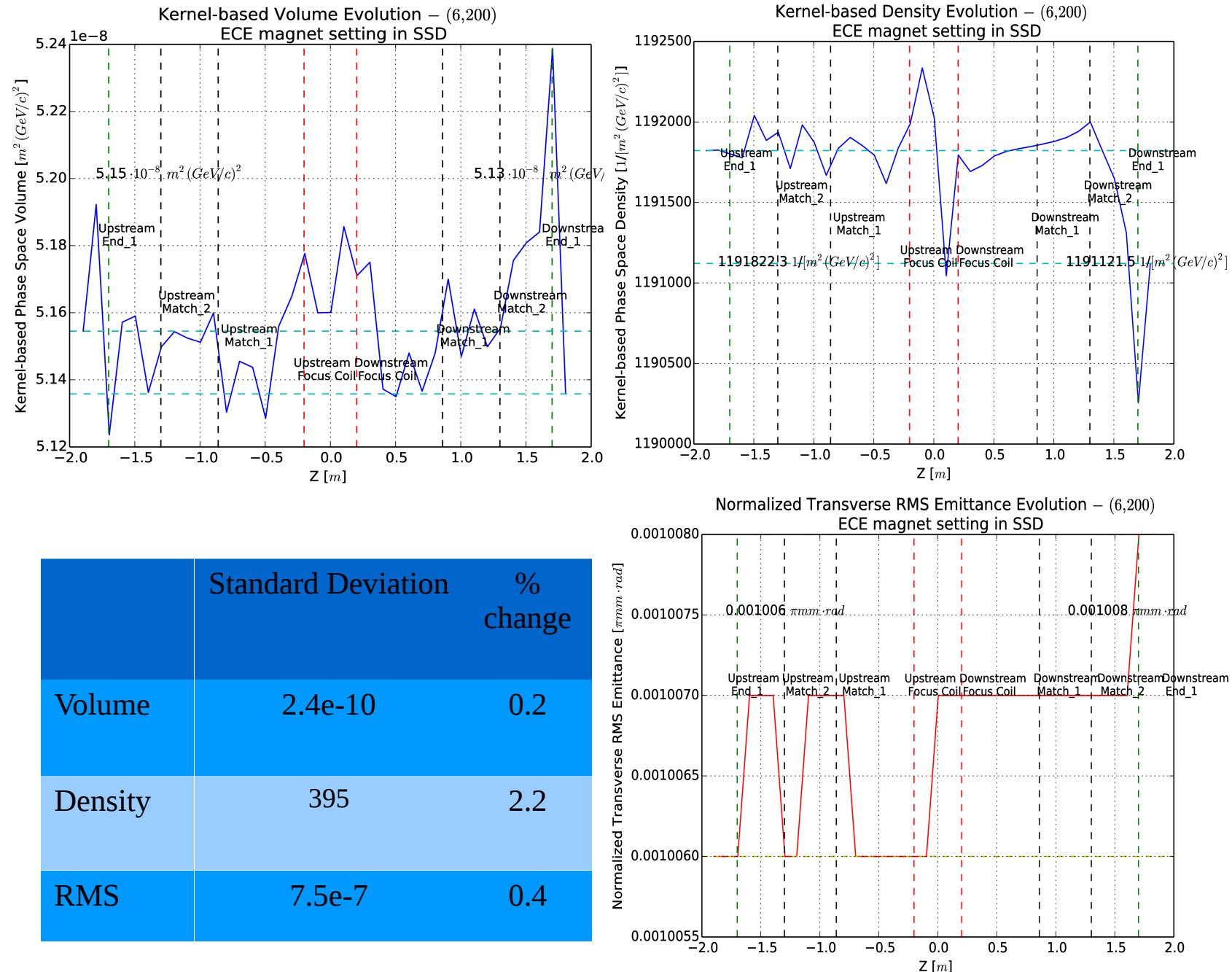
	Standard Deviation	% change
Volume	1.0e-08	5.9
Density	108.6	1.3
RMS	0.00014	11.56



- Evolutions of emittance, beam-core volume and density along an empty-absorber channel:
- Downstream match coils turned off.
- Initial beam optics: (6, 200). **But with a  $5\sigma$  cut.**
- Trackers removed to ensure minimized energy loss effects.
- 10,000 input with transmission of ~95%.
- Results: shown on the table.

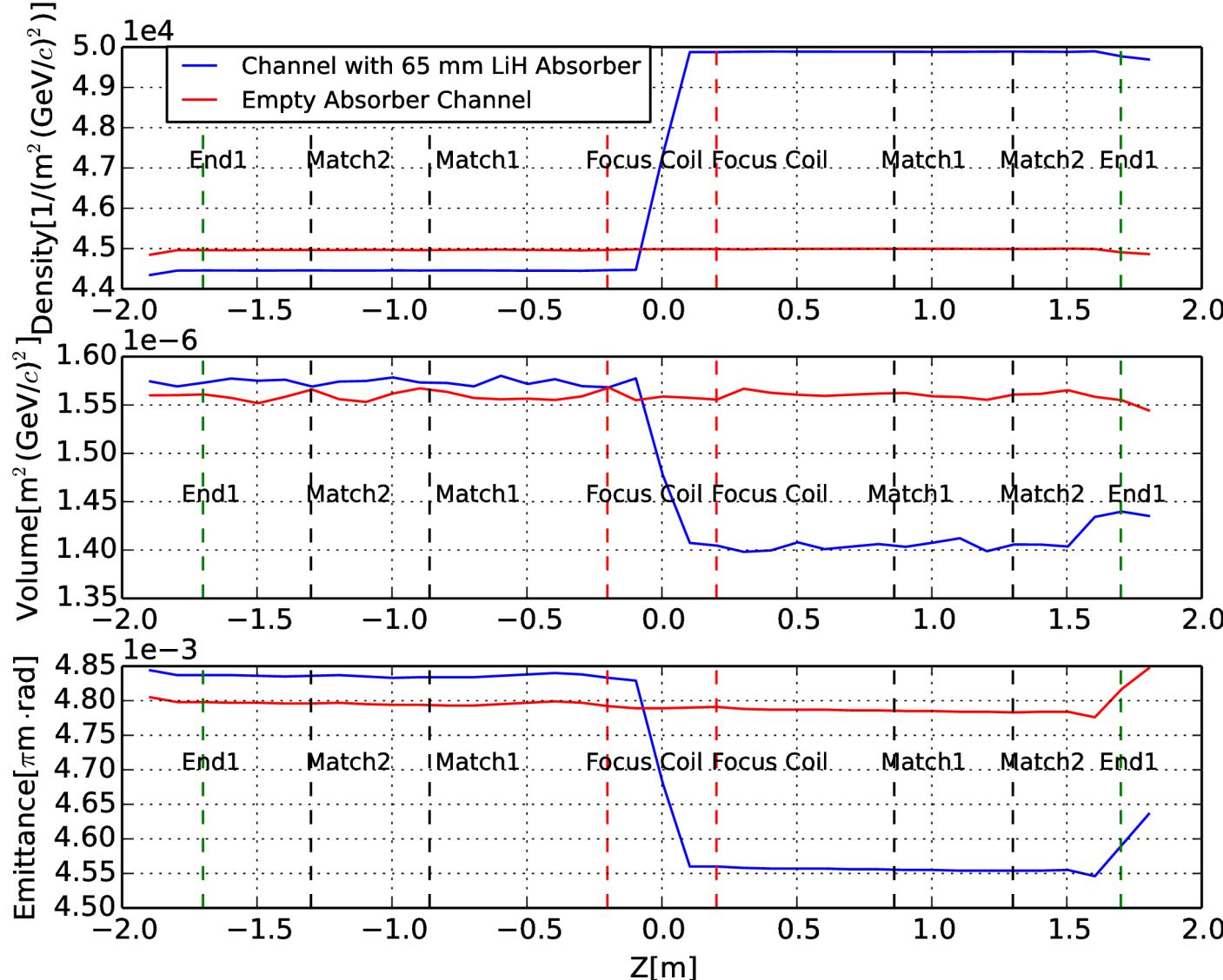
# MICE Optics with KDE Cont. – Smaller Initial Beam

- Evolutions of emittance, beam-core volume and density along an empty-absorber channel:
  - ✓ Downstream match coils turned off.
  - ✓ Initial beam optics: (1, 200).
  - ✓ Trackers removed to ensure minimized energy loss effects.
  - ✓ 10,000 muons input with transmission of ~95%.
  - Results: shown on the table.



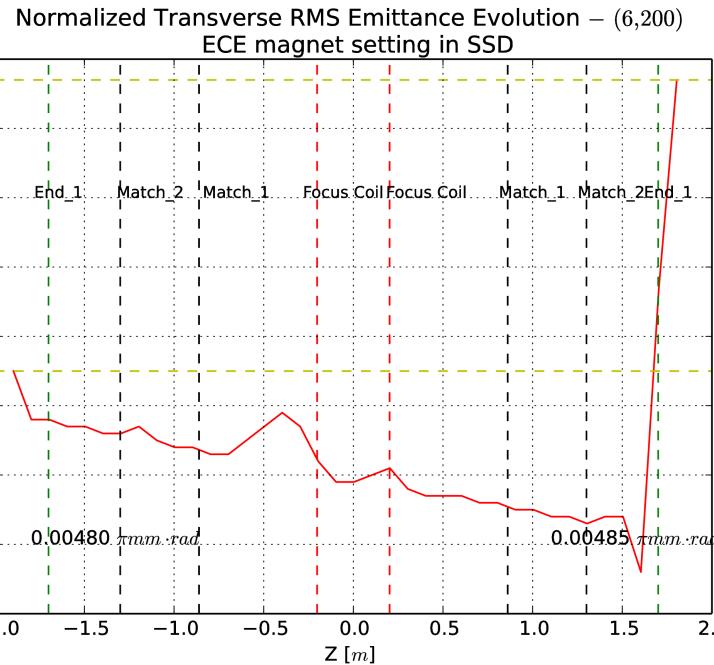
# MICE Optics with KDE Cont. – Cooling Channel

- Spikes still present in empty absorber channel but in the scale of cooling effect, they have smaller percentages.

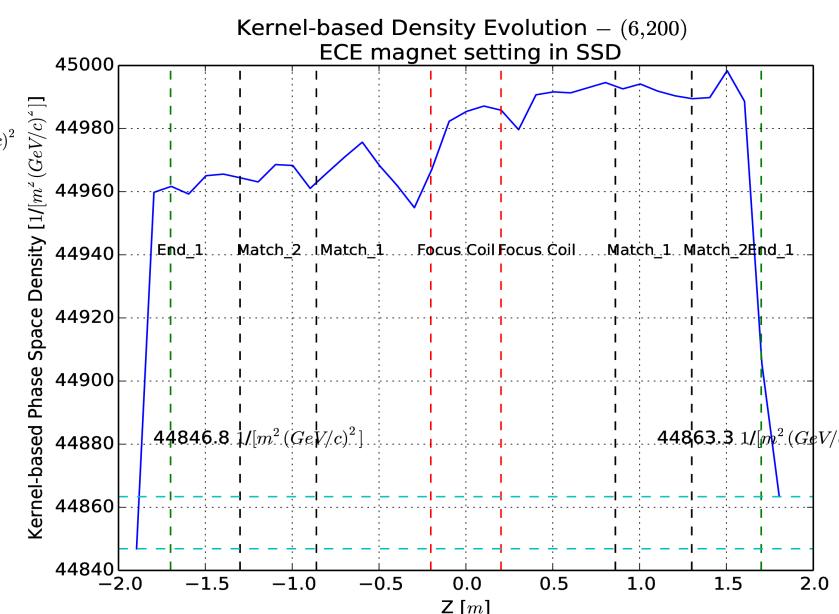
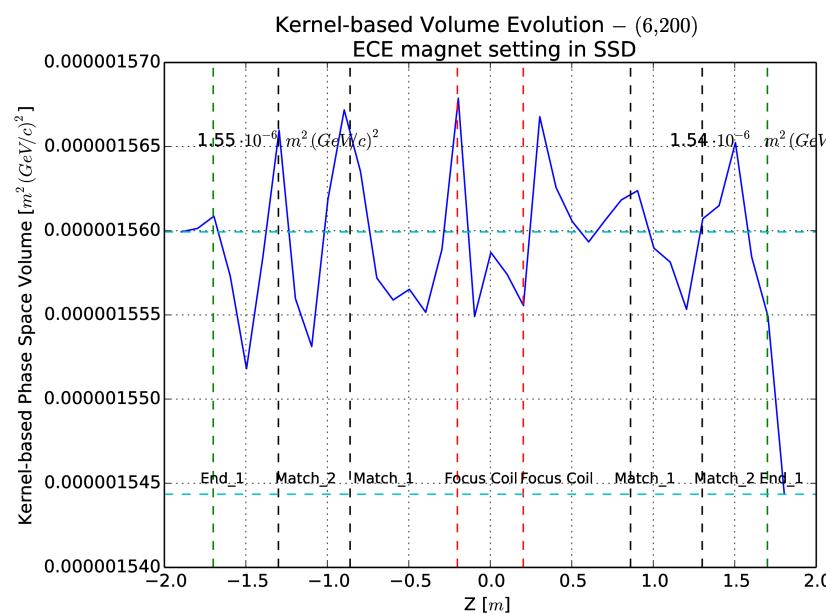


- Evolutions of emittance, beam-core volume and density along the empty-absorber **and the cooling channels**:
- Downstream match coils turned off.
- Initial beam optics: (6, 200).
- Trackers **placed back in the geometry** to ensure realistic energy loss processes.
- 100,000** input muons with a transmission of  $\sim 85\%$ .
- Results: 4% and 9% reductions in emittance and volume with 12% increase in density.

# MICE Optics with KDE Cont. – Cooling Channel Cont.



	Standard Deviation	% change
Volume	7.0e-09	2.8
Density	31.6	0.33
RMS	1.8e-05	2.28



The evolutions of emittance along with beam-core volume and density in the empty-absorber lattice, zoomed in.

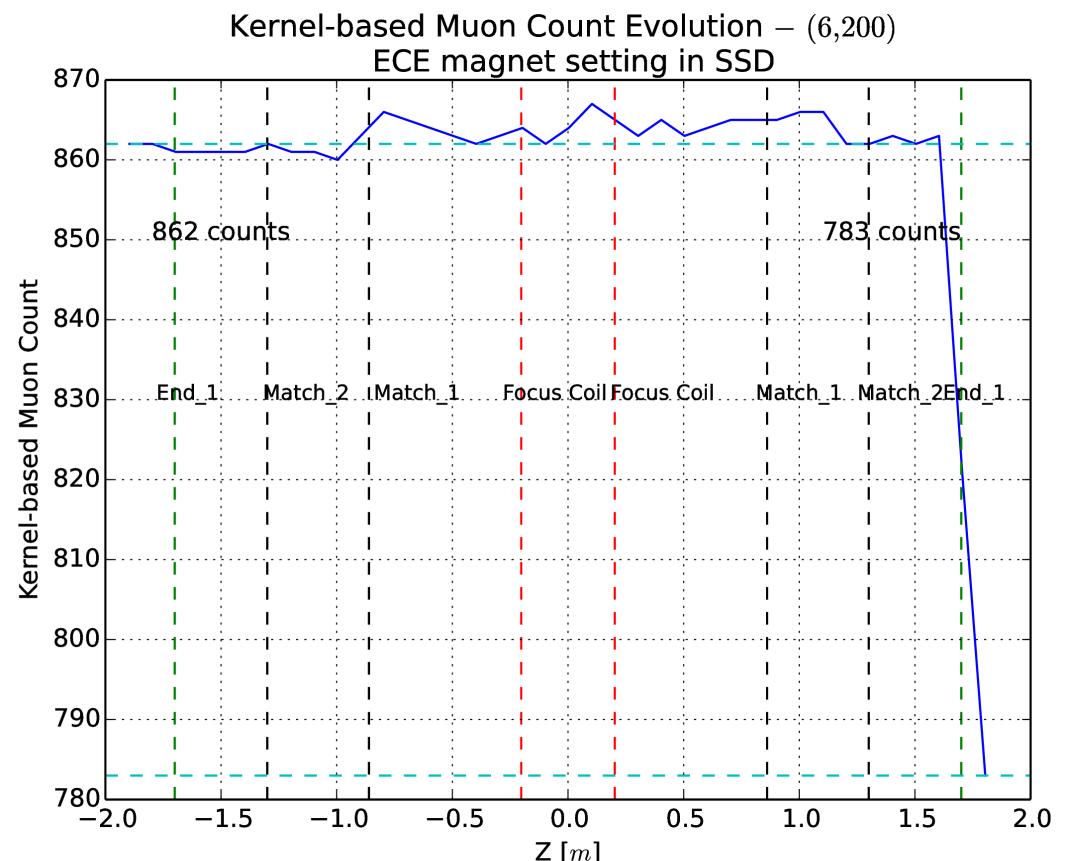
Results: shown on the table.

# Number of Muons as Figure of Merit

- My old routine:
  - ✓ Search for phase space contour which contains 9% of the total number of muons (roughly 1 sigma of the beam) and record the (constant) density corresponding to that contour and compute volume using MC technique.
- My new routine:
  - ✓ Rely on density information from old routine (need to improve this part) to choose a fixed density: take the averages of the 9% contour densities at every region in the channel. The contour corresponding to this average density will then be tracked.
  - ✓ Count the muons inside this constant density and obtain “number of muons” evolution.

Muon #	Standard Deviation	% change
	23	14

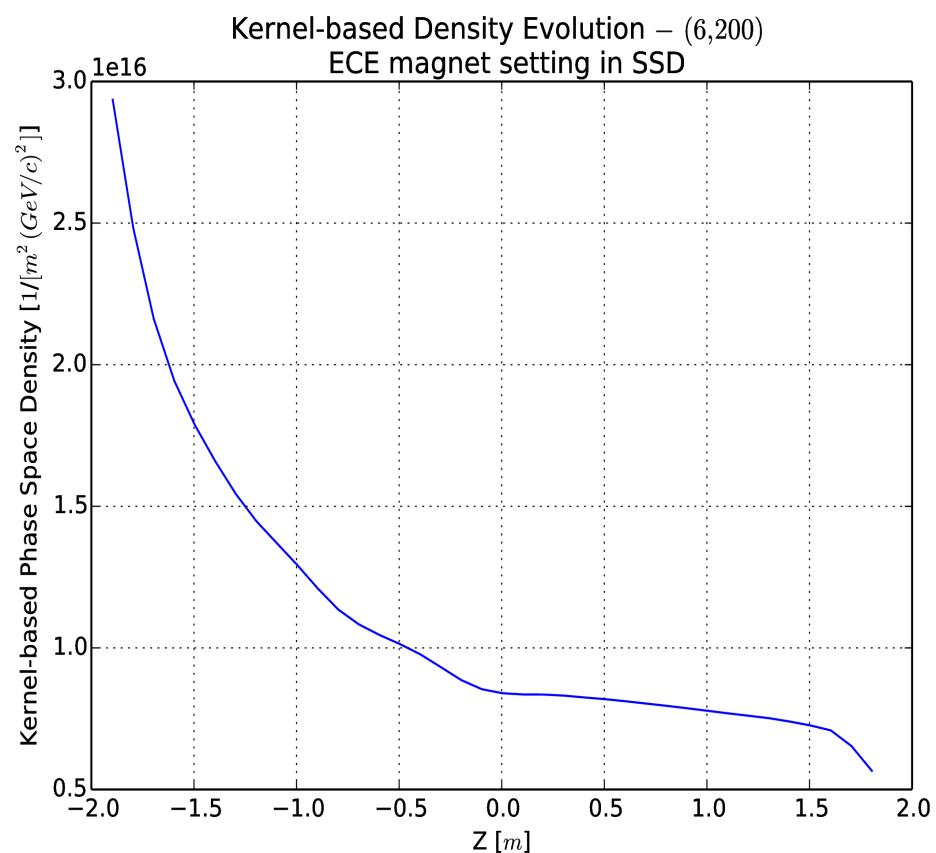
- Same lattice and beam optics as slide 13:
  - ✓ Downstream match coils turned off.
  - ✓ Initial beam optics: (6, 200).
  - ✓ Trackers removed to ensure minimized energy loss effects.
- 10,000 input muons with a transmission of ~95%.



# Preliminary Six Dimensional KDE

- Chris recommended that spikes may be caused by transverse and longitudinal correlations.
- Longitudinal initial beam optics:  $\sigma_t$  and  $\sigma_E = 0.3e-9$  s and 10 MeV.
- Same lattice and beam optics as slide 13:
  - ✓ Downstream match coils turned off.
  - ✓ Initial beam optics: (6, 200).
  - ✓ Trackers removed to ensure minimized energy loss effects.
  - ✓ 10,000 input muons with transmission of ~95%.
- Only analyzed density for now.
- Need to investigate this further.

	Standard Deviation	% change
Muon #	5.3e15	83



# Conclusions and To Do's

- My hypothesis on spikes: the distributions at disabled match coils are long-tailed and KDE (as shown with the stand-alone solenoidal study) in its standard form is sensitive to tails. Methods exist for a systematic adaptation of the level of smoothing to the localization of data: nearest neighbor or variable kernel methods. Could help remove the spikes. Currently working on this.
- With and without tracker lattices result in substantially different evolution plots. This has to do with the change in initial emittance. Beam starts about 0.4 m upstream of the center of the center coil.
- Currently running a toy MC with a sample size of 100,000 for comparison with Chris's analytical relation between volume and bandwidth parameter.
- Investigate the problem with the six-dimensional KDE.
- Verify “the increase in mean density with increasing data” against actual PDF.