Current matrix elements in HAL QCD method of lattice hadron potentials

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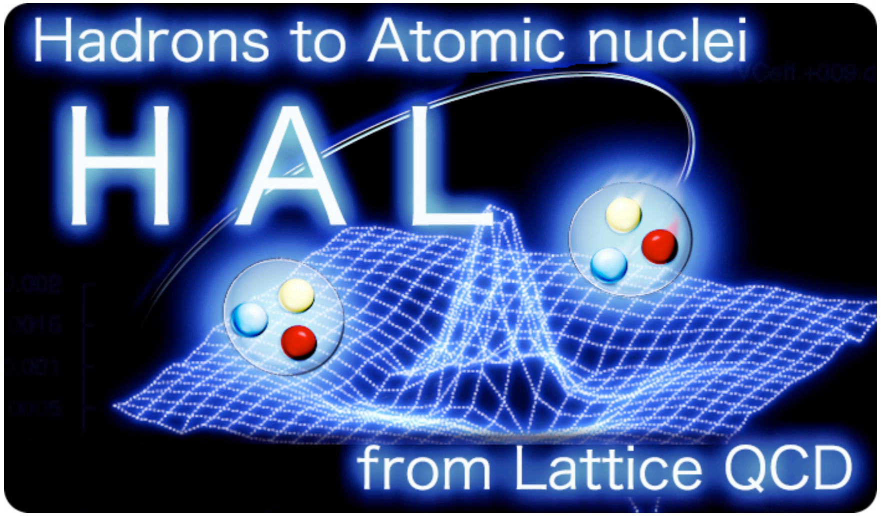
and



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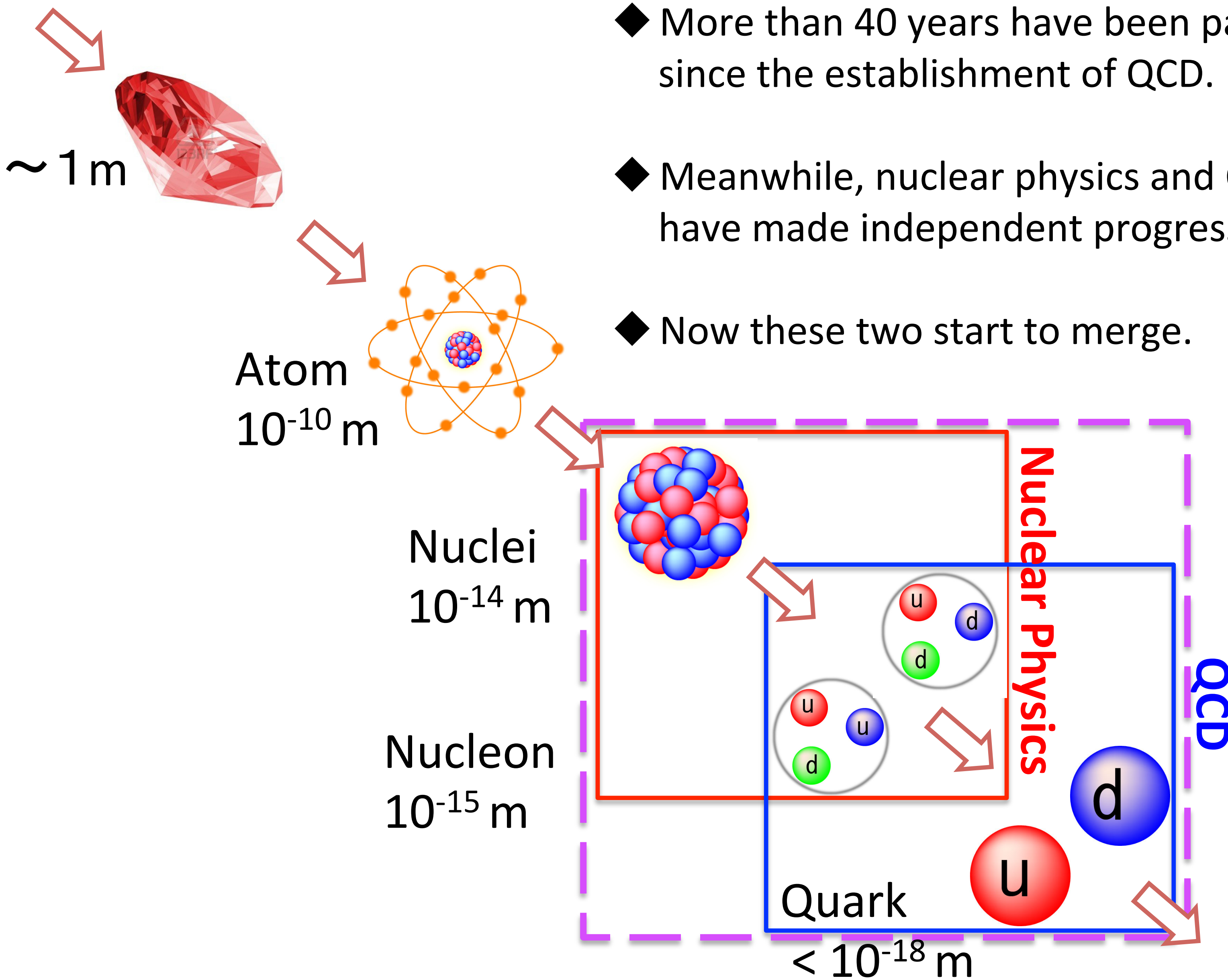
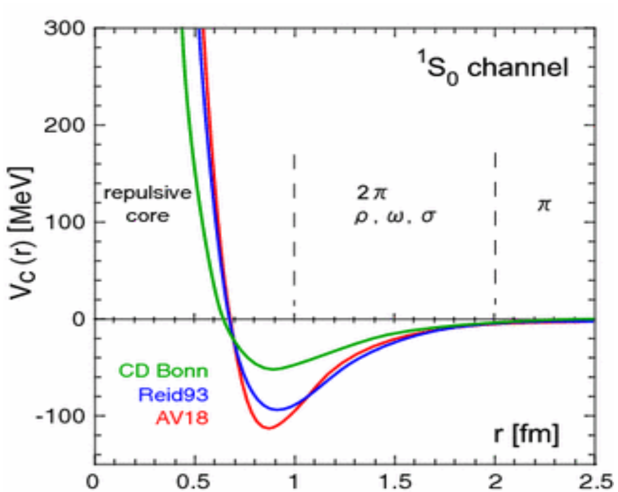
Nuclear physics and Quantum Choromodynamics(QCD)

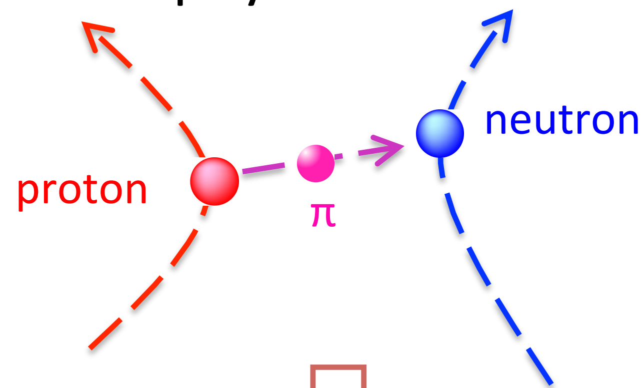
- More than 40 years have been passed since the establishment of QCD.
- Meanwhile, nuclear physics and QCD have made independent progress.
- Now these two start to merge.



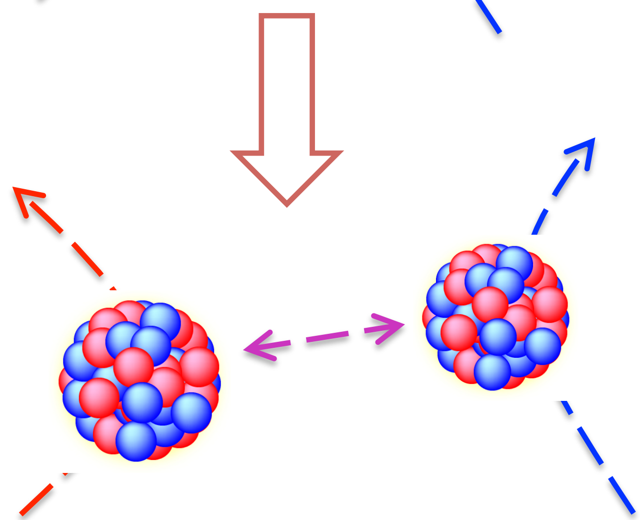
Background

◆The nuclear force is important for nuclear / astro phys.





◆Structures and reactions of atomic nuclei



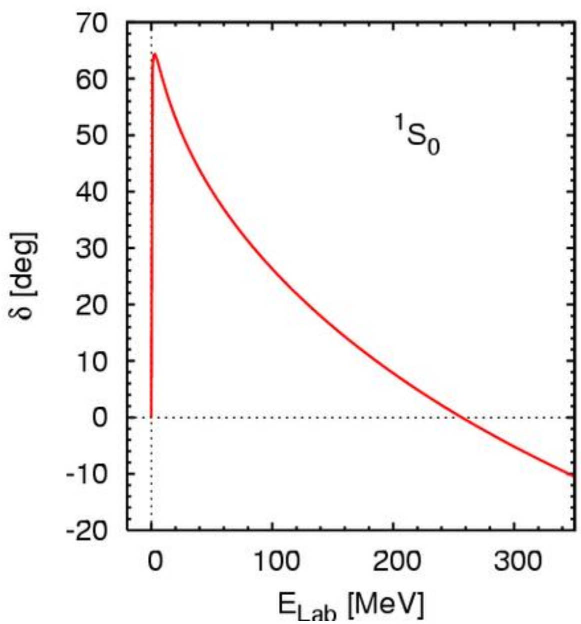
◆Supernova explosions and neutron stars

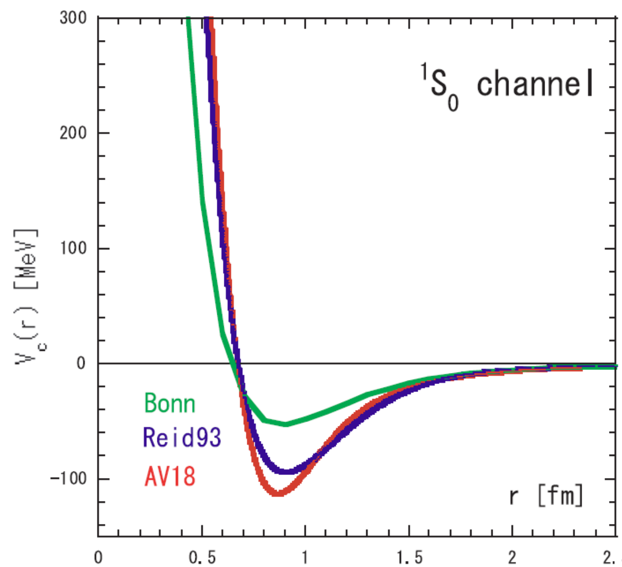
Background

◆Experimental determination of nuclear force.

NN scattering data (～4000 data)

High Precision Nuclear Force





◆The same method does not work for

- ☐ Hyperon-Hyperon interactions
- ☐ Three nucleon interactions

➔ **We need a lattice QCD method !**

Background

HAL QCD method is a LQCD method
to calculate nuclear/hyperon forces and hadronic potentials,
which are faithful to scattering phase shift of QCD.

- ☐ Brief review of HAL QCD method
- ☐ A new attempt:
  Matrix element of current for HAL QCD method

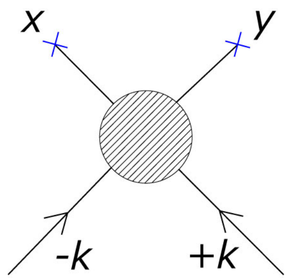
Brief review of HAL QCD method for lattice hadron potentials

HALQCD method

[Aoki,Hatsuda,Ishii,PTP123(2010)89]

◆Nambu-Bethe-Salpeter (NBS) wave func.

$$\left\langle 0 \middle| T\left[N(x)N(y)\right] \middle| N(+k)N(-k),in \right\rangle$$



◆**Relation to S-matrix** by LSZ reduction formula

Bosonic notation
to avoid lengthy notations.

$$\left\langle N(p_1)N(p_2),out \middle| N(+k)N(-k),in \right\rangle_{\text{connected}}$$
$$=\left(iZ_N^{-1/2}\right)^2 \int d^4x_1 d^4x_2\, e^{ip_1x_1}\left(\Box_1+m_N^2\right)e^{ip_2x_2}\left(\Box_2+m_N^2\right)\left\langle 0 \middle| T\left[N(x_1)N(x_2)\right] \middle| N(+k)N(-k),in \right\rangle$$

◆**Equal-time** restriction of **NBS wave func.**

[C.-J.D.Lin et al., NPB619,467(2001).]

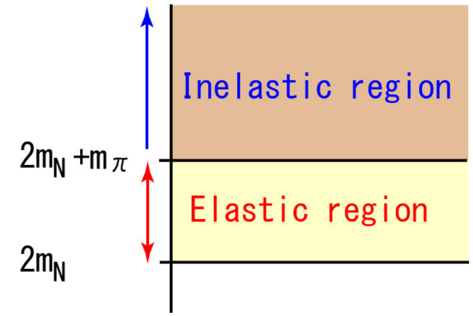
$$\psi_k(\vec{x}-\vec{y}) \equiv \lim_{x_0\to+0} Z_N^{-1}\left\langle 0 \middle| T\left[N(\vec{x},x_0)N(\vec{y},0)\right] \middle| N(+k)N(-k),in \right\rangle$$
$$= Z_N^{-1}\left\langle 0 \middle| N(\vec{x},0)N(\vec{y},0) \middle| N(+k)N(-k),in \right\rangle$$
$$\simeq e^{i\delta(k)}\frac{\sin\left(kr+\delta(k)\right)}{kr}+\cdots \quad \text{as } r\equiv|\vec{x}-\vec{y}|\to \text{large}$$

(for S-wave)

❖ Exactly the same func. form as scat. wave func's in Q.M.

Definition of the potential

◆**Def.** of potential from **equal-time NBS wave func's**:



$$\left(k^2/m_N - H_0\right)\psi_k(\vec{r}) = \int d^3r' U(\vec{r},\vec{r}')\psi_k(\vec{r}')$$

$$\text{for } 2\sqrt{m_N^2 + k^2} < E_{\text{th}} \equiv 2m_N + m_\pi$$

◆**U(r,r') is E-indep.** (One can prove its existence.)

$$H_0 \equiv -\frac{\nabla^2}{m_N}$$

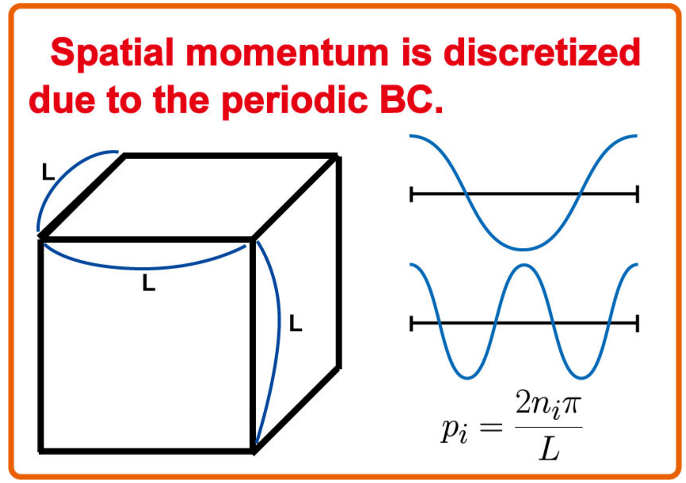
◆**U(r,r') reproduces the scattering phase δ(k),**

(together with equal-time NBS wave func's)

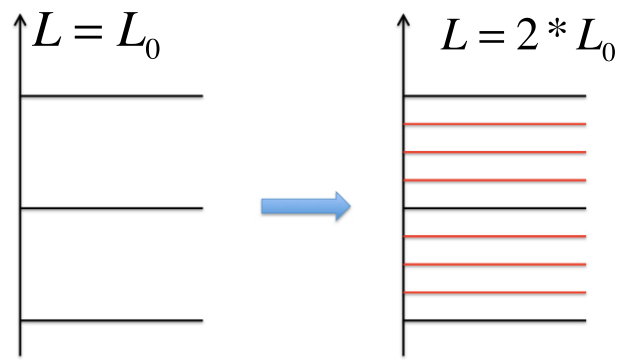
$$\psi_k(\vec{x}-\vec{y}) \simeq e^{i\delta(k)}\frac{\sin\left(kr+\delta(k)\right)}{kr} + \cdots \quad \text{as } r \equiv |\vec{x}-\vec{y}| \to \text{large}$$

Ground state saturation becomes difficult for large spatial vol.

$$\Delta E = E_{n+1} - E_n \sim \frac{1}{m_N}\left(\frac{2\pi}{L}\right)^2$$
$$= O\left(1/L^2\right)$$



	L=3 fm	L=6 fm	L=9 fm	L=12 fm
Δ E	181.5 MeV	45.3 MeV	20.2 MeV	11.3 MeV



Determination of potentials

We have a special strategy against the ground state saturation.

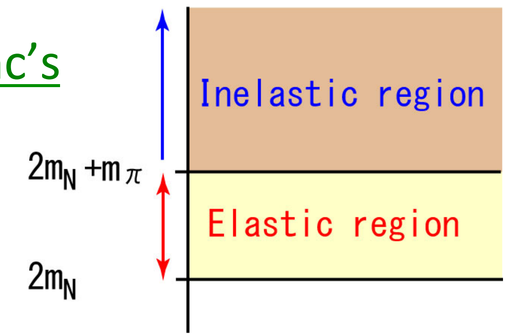
◆ Def. **R-correlator** [Ishii et al.,PLB712(2012)437]

$$R(\vec{x}-\vec{y},t) \equiv e^{2mt} \left\langle 0 \middle| T\left[B(\vec{x},t)B(\vec{y},t)\cdot \overline{BB}(t=0)\right] \middle| 0 \right\rangle$$

$$= \sum_n \psi_{k_n}(\vec{x}-\vec{y}) \cdot \exp\left(-(E_n - 2m)t\right) \cdot a_n$$

◆ Schroedinger eq. satisfied by HAL QCD pot. and NBS wave func's

$$\left(-H_0 + k_n^2/m\right)\psi_{k_n}(\vec{r}) = \int d^3r' V(\vec{r},\vec{r}')\, \psi_{k_n}(\vec{r}')$$



➔ R-corr. satisfies **time-dependent Schroedinger-like eq.**

$$\boxed{\left(-H_0 + \frac{1}{4m}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right) R(\vec{r},t) = \int d^3r' V(\vec{r},\vec{r}') R(\vec{r}',t)}$$

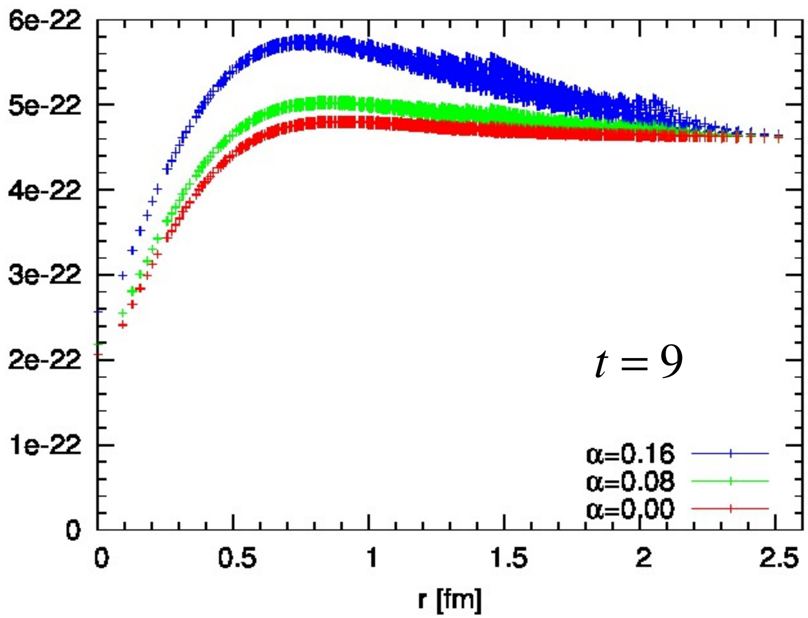
☐ Only **Elastic saturation** is needed.
Ground state saturation is not needed.

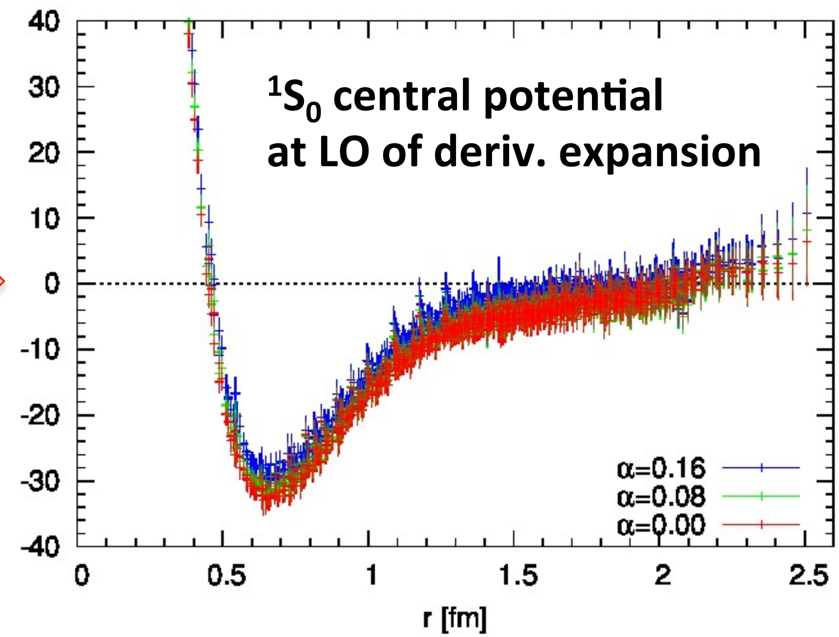
☐ **Elastic saturation** is much easier than **ground state saturation**.

Ground state saturation is not needed in HAL QCD method.

$$\langle 0|T[N(\vec{x},t)N(\vec{y},t)\cdot\overline{NN}(t=0;\alpha)]|0\rangle = \sum_n \psi_n(\vec{x}-\vec{y})\cdot a_n(\alpha)\cdot\exp(-E_n t)$$

$$V_C(\vec{x}) = -\frac{H_0 R(t,\vec{x})}{R(t,\vec{x})} - \frac{(\partial/\partial t)R(t,\vec{x})}{R(t,\vec{x})} + \frac{1}{4m_N}\frac{(\partial/\partial t)^2 R(t,\vec{x})}{R(t,\vec{x})}$$



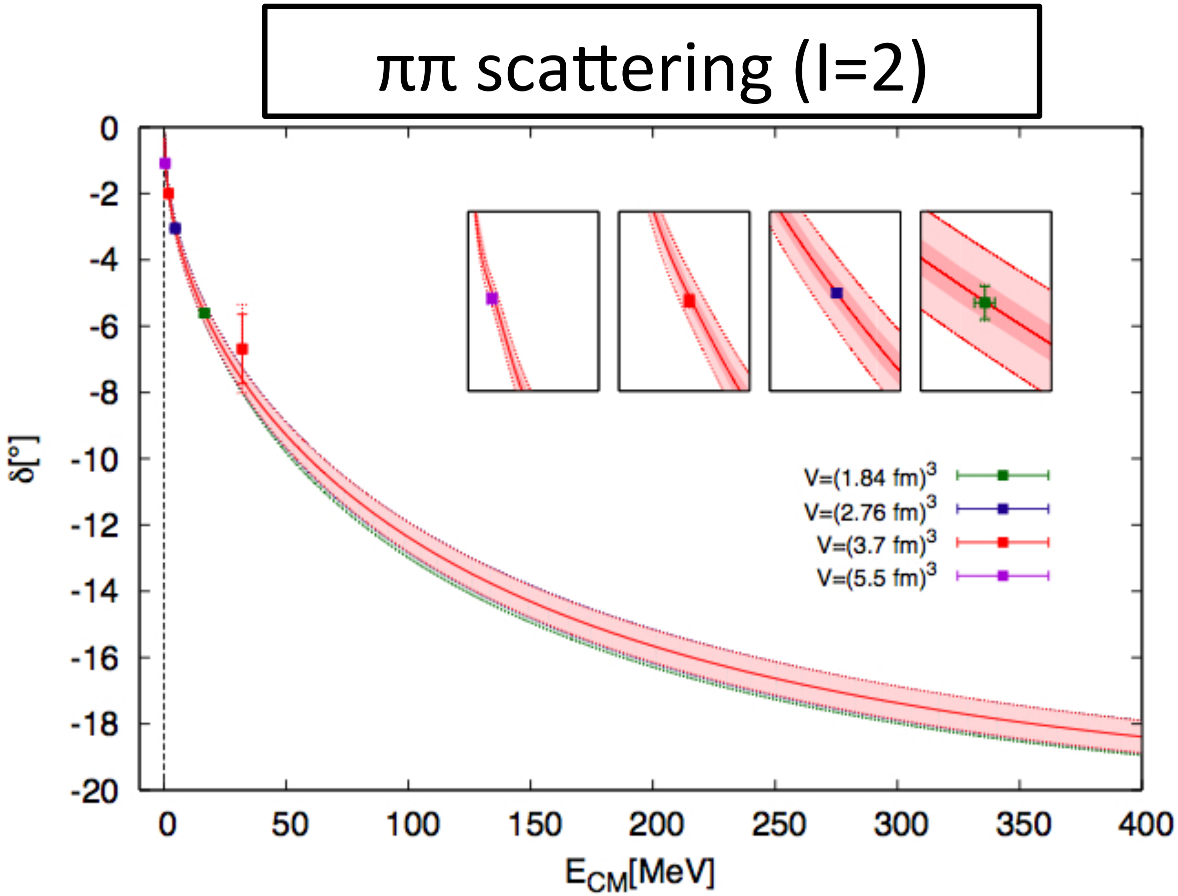


Different mixture of NBS waves are generated by different α

$$f(x,y,z) = 1 + \alpha\left(\cos(2\pi x/L) + \cos(2\pi y/L) + \cos(2\pi z/L)\right)$$

Good agreement !
➔ Our method works !

Comparison: HAL QCD method and Luescher's finite vol. method



Ns=16,24,32,48, Nt=128, a=0.115 fm.

m_{pi} = 940 MeV by Quenched QCD [Kurth et al., JHEP **1312**(2013)015.]

Good agreement !

➔ HAL QCD pot. is faithful to the scat. phase.

Lattice QCD at (almost) physical point

- Multi-baryon systems at physical point requires huge spatial volume

- Such gauge configurations have been generated by K computer.

96^4 lattice, $a \sim 0.085$ fm, $L \sim 8.2$ fm, $m_\pi \sim 145$ MeV

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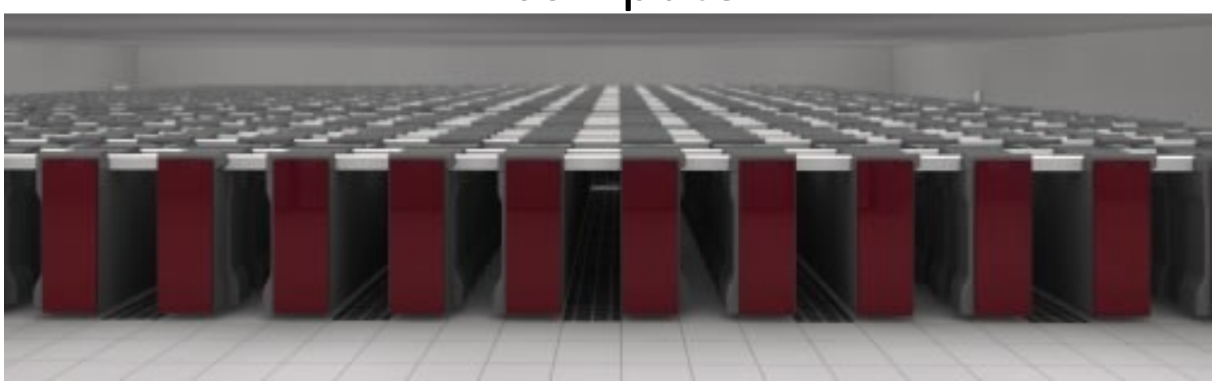
- Determination of baryon-baryon potentials at "phys. point".

K computer



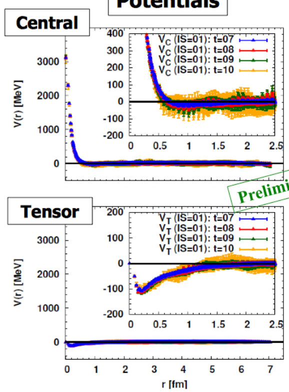
11.28 PFLOPS

"Physical point" calc. for nuclear/hyperon forces is currently going on.

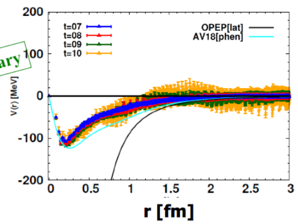
◆ S=0: NN

NN system ($^3S_1$-$^3D_1$)

Potentials



- **Vc: repulsive core + long-range attraction**
- **Vt: strong tensor force !**

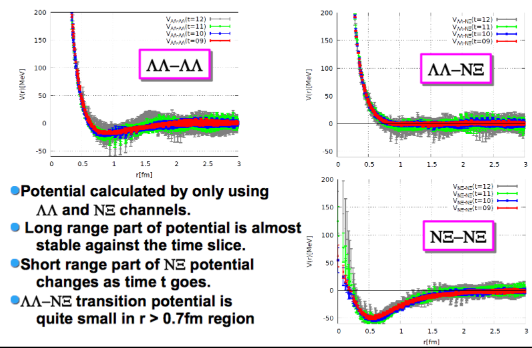


m(eff) for single N: ~2-4% sys err for t = 8-10

◆ S=-2: ΛΛ-NΞ

ΛΛ, NΞ (I=0) $^1S_0$ potential (2ch calc.)

▶ **N_f = 2+1 full QCD with L = 8fm, m_π = 146 MeV** **Preliminary!**

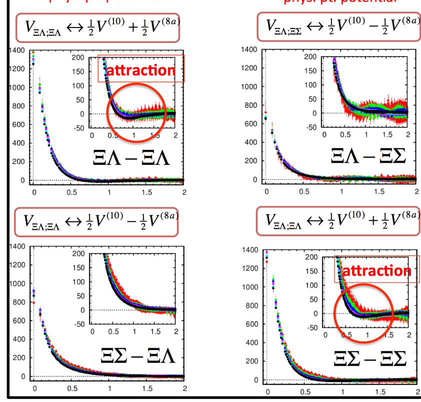


- **Potential calculated by only using ΛΛ and NΞ channels.**
- **Long range part of potential is almost stable against the time slice.**
- **Short range part of NΞ potential changes as time t goes.**
- **ΛΛ–NΞ transition potential is quite small in r > 0.7fm region**

Kenji Sasaki (YITP, Kyoto University) for HAL QCD Collaboration

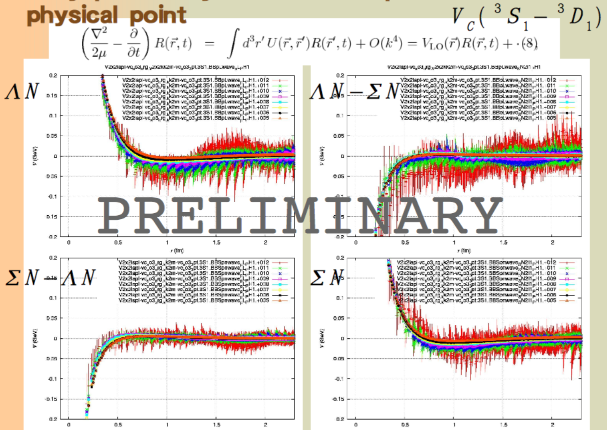
◆ S=-3: ΞΛ-ΞΣ

ΞΛ-ΞΣ potential (I=1/2, spin triplet) central force

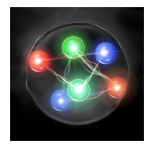


◆ S=-1: NΛ-NΣ

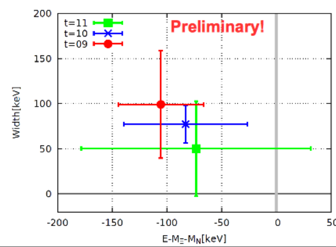
Very preliminary result of LN potential at the physical point



◆ H dibaryon



- **Sharp resonance is found just below the NΞ threshold.**
- **Resonance position and width from Breit-Wigner type fit**

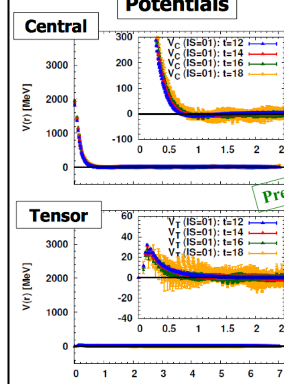


➔ **Next speaker**

◆ S=-4: ΞΞ

ΞΞ system ($^3S_1$-$^3D_1$)

Potentials



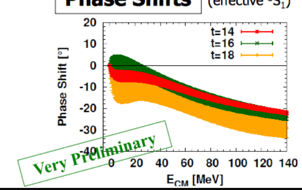
10plet ⇔ unique w/ hyperon DoF

Flavor SU(3)-partner of Σ^- n

⇨ • Σ^- in neutron star ?

Central: Strong Repulsion
Tensor: Weak

Phase Shifts (effective $^3S_1$)



Matrix element of current for HAL QCD method

Matrix elements in HAL QCD method

◆HAL QCD method

➔ potentials which are faithful to the S-matrix of QCD.

different Interpolating field

different wave func.

different potential

but

unique S-matrix

different probability density (charge density) by naïve formula

$$\psi^*(x)\psi(x)$$

Exchange currents

In the theory of nuclear force,

◆ current/charge has additional term (exchange current/exch. charge)

$$\rho(x) = \rho^{(1)}(x) + \rho^{(2)}(x)$$

$$\vec{J}(x) = \vec{J}^{(1)}(x) + \vec{J}^{(2)}(x)$$

◆ The current conservation (gauge invariance) implies

$$\vec{\nabla}\cdot\vec{J}^{(2)}(x) = -i\left[V, \rho^{(1)}(x)\right] \qquad \left(\rho^{(2)}(x) \sim 0 \text{ is assumed}\right)$$

➔ The exch. current is closely related to the potential.

◆ For phenomenologically determined nuclear forces,
all one can do is to impose **a constraint.**

◆ For HAL QCD potentials,
we may use QCD to obtain explicit (exchange) currents and charges. [←our aim]
(Depending on the interpolating field, these contributions can be large)

◆ We proceed step by step:
Non-rela. Q.M. → Lorentz cov. system → composite particles → Lattice QCD

A toy model (non-rela. two-channel coupling model)

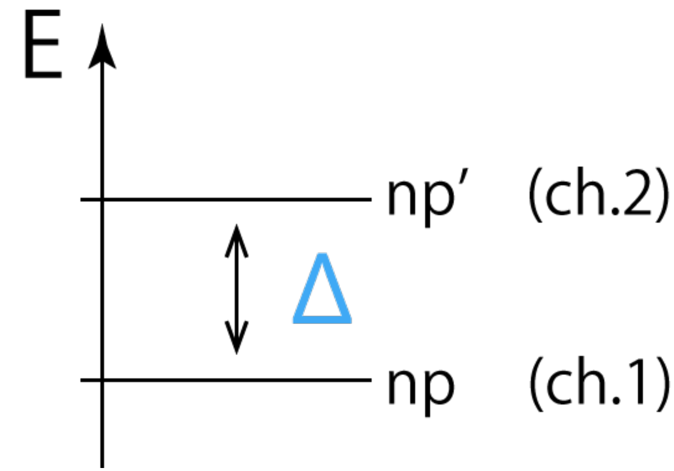
$$H \equiv H_1 + H_2 + V_{11} + V_{12} + V_{21} + V_{22}$$

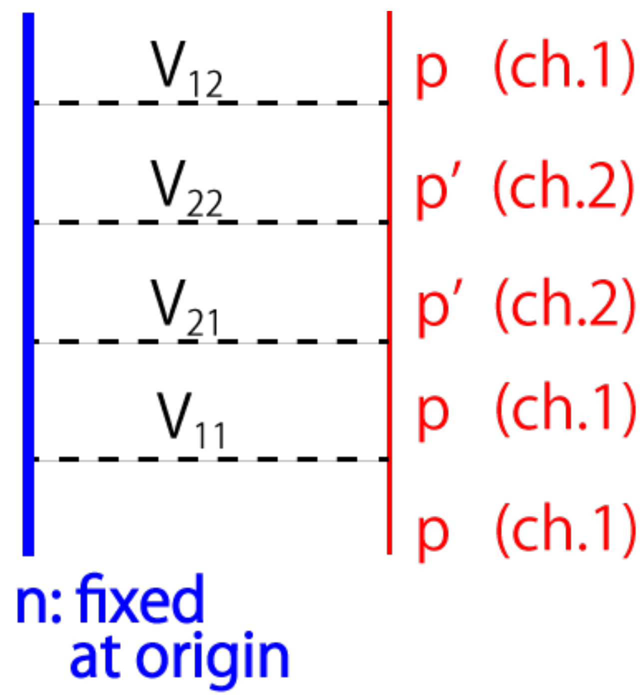
$$H_1 \equiv \int d^3x\, \phi_1^\dagger(\vec{x}) \left(-\frac{\nabla^2}{2m} \right) \phi_1(\vec{x})$$

$$H_2 \equiv \int d^3x\, \phi_2^\dagger(\vec{x}) \left(-\frac{\nabla^2}{2m} + \Delta \right) \phi_2(\vec{x})$$

$$V_{\alpha\beta} \equiv \int d^3x\, \phi_\alpha^\dagger(\vec{x}) V_{\alpha\beta}(\vec{x}) \phi_\beta(\vec{x})$$

$$(\alpha, \beta = 1,2)$$





- ◆ np-np′ coupling system is mimicked
- ◆ n is fixed at the origin
- ◆ n, p, p′ are scalar bosons for simplicity.

"Nambu-Bethe-Salpeter (NBS)" wave function

◆ Def ("NBS wave function")

$$\psi_\alpha(x) \equiv \langle 0 | \phi_\alpha(x) | \psi \rangle \qquad (\alpha = 1,2)$$

☐ Single particle state

$$|\psi\rangle = \int d^3x \left(\phi_1^\dagger(\vec{x})|0\rangle \psi_1(\vec{x}) + \phi_2^\dagger(\vec{x})|0\rangle \psi_2(\vec{x}) \right)$$

◆ NBS wave functions satisfy a coupled channel eq.

$$\left(i\partial_0 + \frac{1}{2m}\partial_i^2 \right)\psi_1(x) = V_{11}(\vec{x})\psi_1(x) + V_{12}(\vec{x})\psi_2(x)$$

$$\left(i\partial_0 + \frac{1}{2m}\partial_i^2 - \Delta \right)\psi_2(x) = V_{21}(\vec{x})\psi_1(x) + V_{22}(\vec{x})\psi_2(x)$$

HAL QCD potential of this toy model

(Elimination of ψ_2 to obtain an effective theory of ψ_1 for E < Δ)

We define the E-indep. non-local potential [HAL QCD pot.] as

$$v(\vec{x},\vec{x}') \equiv \sum_{\substack{m \\ E_m<\Delta}} \left(V_{11}(\vec{x})\psi_{1,m}(\vec{x}) + V_{12}(\vec{x})\psi_{2,m}(\vec{x}) \right) \tilde{\psi}_{1,m}(\vec{x}')$$

by using the dual vector which satisfies the orthogonality relation

$$\int d^3x \, \tilde{\psi}_{1,n}(\vec{x}) \psi_{1,m}(\vec{x}) = \delta_{nm}$$

➔

NBS wave func. $\psi_{1,n}(x) \equiv \langle 0 | \phi_1(x) | n \rangle$ for $E_n < \Delta$ satisfy the Schrödinger eq:

$$\left(i\partial_0 + \frac{1}{2m}\nabla^2 \right) \psi_{1,n}(x) = \int d^3x' \, v(\vec{x},\vec{x}') \psi_{1,n}(x')$$

(Proof) $\int d^3x' \, v(\vec{x},\vec{x}') \psi_{1,n}(x')$

$$= \sum_m \left(V_{11}(\vec{x})\psi_{1,m}(\vec{x}) + V_{12}(\vec{x})\psi_{2,m}(\vec{x}) \right) \underbrace{\int d^3x' \, \tilde{\psi}_{1,m}(\vec{x}') \cdot \psi_{1,n}(\vec{x}')}_{\Rightarrow \delta_{nm}} e^{-iE_n x_0}$$

$$= V_{11}(\vec{x})\psi_{1,n}(x) + V_{12}(\vec{x})\psi_{2,n}(x)$$

Strategy for the current matrix element

To obtain the explicit expression for the exch. current/charges,

- We embed the system in an external field A.
- The current/charge is obtained from a response of the system to the ext. field A.

External field

$$H[A_t] \equiv H_1[A_t] + H_2[A_t] + V_{11} + V_{12} + V_{21} + V_{22}$$

◆ We introduce an ext. field in the unperturbed one-body Hamiltonians

$$H_1 \equiv \int d^3x\, \phi_1^\dagger(\vec{x}) \left(-\frac{1}{2m}\left(\partial_i - iA_i(\vec{x},t)\right)^2 - A_0(\vec{x},t) \right) \phi_1(\vec{x})$$

$$H_2 \equiv \int d^3x\, \phi_2^\dagger(\vec{x}) \left(-\frac{1}{2m}\left(\partial_i - iA_i(\vec{x},t)\right)^2 - A_0(\vec{x},t) + \Delta \right) \phi_2(\vec{x})$$

◆ We do not introduce an ext. field in the potential for simplicity.

$$V_{\alpha\beta} \equiv \int d^3x\, \phi_\alpha^\dagger(\vec{x}) V_{\alpha\beta}(\vec{x}) \phi_\beta(\vec{x})$$

Time evolution in an ext. field with a cut off

◆ Hamiltonian with Cut off (E < Λ ≡ Δ)

$$H_\Lambda[A_t] \equiv \mathbb{P}_\Lambda H[A_t]\mathbb{P}_\Lambda \quad \text{with} \quad \mathbb{P}_\Lambda \equiv \sum_{E<\Lambda} |E\rangle\langle E|$$

◆ NBS wave func. in an ext. field with a cutoff

$$\psi_\alpha(x;A) \equiv \langle 0|\phi_\alpha(x;A)|\psi\rangle \quad \text{with} \quad \phi_\alpha(x;A) \equiv U_\Lambda(0,x_0;A)\phi_\alpha(\vec{x})U_\Lambda(x_0,0;A)$$

$$U_\Lambda(t_1,t_0;A) \equiv T\exp\left(i\int_{t_0}^{t_1} dt\, H_\Lambda[A_t]\right)$$

◆ NBS wave func. satisfy the coupled channel eq.

$$\boxed{\left(iD_0 + \frac{1}{2m}D_i^2\right)\psi_\alpha(x;A) = \sum_{\beta=1,2}\int d^3x'\, V_{\alpha,\beta;\Lambda}(\vec{x},\vec{x}';A_{x_0})\psi_\beta(x';A)}$$

$$D_\mu \equiv \partial_\mu - iA_\mu(x)$$

$$V_{\alpha\beta;\Lambda}(\vec{x},\vec{x}';A_{x_0}) \equiv V_{\alpha\beta}(\vec{x})\delta^3(x-x') + \delta V_{\alpha\beta;\Lambda}(\vec{x},\vec{x}',A_{x_0})$$

$$\delta V_{\alpha\beta;\Lambda}(\vec{x},\vec{x}';A_{x_0}) \equiv \langle 0|\phi_\alpha(\vec{x})\left(\mathbb{P}_\Lambda - 1\right)H[A_{x_0}]\mathbb{P}_\Lambda\phi_\beta^\dagger(\vec{x}')|\rangle$$

HAL QCD potential in an ext. field

(Elimination of ψ_2 to obtain an effective theory of ψ_1 for E < Δ)

We define state-indep. non-local potential [HAL QCD pot. in ext. field] as

$$v_\Lambda(\vec{x},\vec{x}';A_{x_0})$$
$$\equiv \sum_{\substack{m \\ E_m<\Lambda}} \int d^3x'' \left(V_{11,\Lambda}(\vec{x},\vec{x}'';A_{x_0})\psi_{1,m}(\vec{x}'') + V_{12\Lambda}(\vec{x},\vec{x}'';A_{x_0})\psi_{2,m}(\vec{x}'') \right) \tilde{\psi}_{1,m}(\vec{x}')$$

by using the dual vector which satisfies the orthogonality relation

$$\int d^3x \tilde{\psi}_{1,n}(\vec{x})\psi_{1,m}(\vec{x}) = \delta_{nm}$$

➔

NBS wave func $\psi_{1,n}(x;A) \equiv \langle 0 | \phi_1(x;A) | n \rangle$ in ext. field satisfy

the Schrödinger eq in ext. field:

$$\left(iD_0 + \frac{1}{2m} D_i^2 \right) \psi_{1,n}(x;A) = \int d^3x' \, v_\Lambda(\vec{x},\vec{x}';A_{x_0}) \psi_{1,n}(x;A) \quad \text{for} \quad E_n < \Lambda$$

$$D_\mu \equiv \partial_\mu - iA_\mu(x)$$

(proof is similar)

Current matrix element

Schrödinger eq. in an ext. field

$$\left(iD_0 + \frac{1}{2m} D_i^2 \right) \psi_{1,n}(x;A) = \int d^3x' \, v_\Lambda(\vec{x},\vec{x}';A_{x_0}) \psi_{1,n}(x;A) \quad \text{for} \quad E_n < \Lambda$$

$$\frac{\delta}{\delta A_\mu(z)} (\text{both side}) \Big|_{A\equiv 0}$$

$A \equiv 0$

$$\left(i\partial_0 + \frac{1}{2m}\partial_i^2 - v_\Lambda \right) \frac{\delta \psi_{1,n}(x;A)}{\delta A_\mu(z)} = \underbrace{\frac{\delta}{\delta A_\mu(z)} \left(-iD_0 - \frac{1}{2m} D_i^2 + v_\Lambda \right)}_{\equiv K_\mu(z)} \cdot \psi_{1,n}(x)$$

use Green's func.

$A \equiv 0$

$$\frac{\delta \psi_{1,n}(x;A)}{\delta A_\mu(z)} = \left(i\partial_0 + \frac{1}{2m}\partial_i^2 - v_\Lambda \right)^{-1} K_\mu(z) \cdot \psi_{1,n}(x)$$

Current matrix element formula

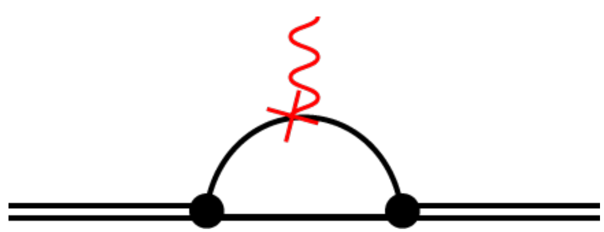
$$\boxed{\langle m | j_\mu(\vec z) | n\rangle = -\int d^3x \int d^3x' \, \psi_m^L(\vec x)\, K_\mu(\vec x,\vec x';\vec z)\, \psi_n^R(\vec x')}$$

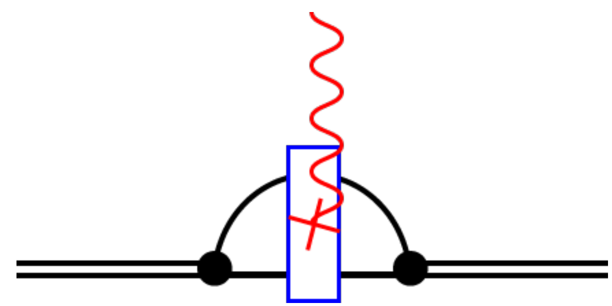
$$K_\mu(\vec x,\vec x';\vec z) \equiv \frac{\delta}{\delta A_\mu(z)}\left[\left(-iD_0+\frac{1}{2m}D_i^2\right)\delta^3(x-x') + v_\Lambda(\vec x,\vec x';A_{x_0})\right]_{A\equiv 0}$$

$\psi_n^R(\vec x),\ \psi_n^L(\vec x)$ right and left eig. vectors of $h \equiv -\frac{1}{2m}\partial_i^2 + v_\Lambda$

- μ=0 component:

$$\langle m|j_0(\vec z)|n\rangle = \underbrace{\psi_m^L(\vec z)\psi_n^R(\vec z)}_{\text{direct charge}} - \underbrace{\int d^3x\int d^3x' \, \psi_m^L(\vec x)\frac{\delta v_\Lambda(\vec x,\vec x';A)}{\delta A_0(\vec z)}\psi_n^R(\vec x')}_{\text{exch. charge}}$$





Procedure which fits HAL QCD method for LQCD

- Generate NBS wave func. in an ext. field with cut off

$$\psi_{1,n}(x;A) \equiv \langle 0 | \phi_1(x;A) | n \rangle \quad \text{for} \quad E_n < \Lambda \equiv \Delta$$

- Determine HAL QCD potential in ext. field
  so that $\psi_{1,n}(x;A)$ can be reproduced by Schrodinger eq.

$$\left(iD_0 + \frac{1}{2m} D_i^2 \right) \psi_{1,n}(x;A) = \int d^3x' \, v_\Lambda(\vec{x},\vec{x}';A_{x_0}) \psi_{1,n}(x;A) \quad \text{for} \quad E_n < \Lambda$$

- Formula for the current matrix element

$$\langle m | j_\mu(\vec{z}) | n \rangle = -\int d^3x \int d^3x' \, \psi_m^L(\vec{x}) K_\mu(\vec{x},\vec{x}';\vec{z}) \psi_n^R(\vec{x}') \qquad \text{for} \;\; E_m, E_n < \Lambda$$

$$K_\mu(\vec{x},\vec{x}';\vec{z}) \equiv \frac{\delta}{\delta A_\mu(z)} \left[\left(-iD_0 + \frac{1}{2m} D_i^2 \right) \delta^3(x-x') + v_\Lambda(\vec{x},\vec{x}';A_{x_0}) \right]_{A\equiv 0}$$

$\psi_n^R(\vec{x}),\ \psi_n^L(\vec{x})$ right and left eig. vectors of $h \equiv -\dfrac{1}{2m}\partial_i^2 + v_\Lambda$

Summary

- HAL QCD method for the lattice hadron potentials
  - It is faithful to the scattering phase shift.
  - The potential is obtained without achieving the ground state saturation.
  - Physical point calculations of nuclear/hyperon forces are currently going on.

- Current matrix element in HAL QCD method
  - Interpolating field dependence of the potential/NBS wave func.
    ➔ in the calculation of current matrix element in HAL QCD method, exchange currents and exchange charges may be important.

  - We derived a formula in a non-rela. 2 channel coupling model.
    - ✓ The formula is inconvenient.
      NBS wave in an ext. field with cutoff may not easily be obtained.

    - ✓ The formula should be extended to
      → Lorentz covariant system.
      → Composite particle system
      → LQCD.