

Kaon-Nucleon scattering states in the Skyrme model

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1. Introduction

Introduction

Kaon nucleon systems are very attractive

- Strong attraction between the anti-kaon(\bar{K}) and the nucleon(N)
Y. Akaishi and T. Yamazaki, Phys. Rev. C **65** (2002)
- $\bar{K}N$ bound state = $\Lambda(1405)$
- Few body nuclear system with $\bar{K} \rightarrow$ under debate

$\bar{K}N$ interaction is important
to investigate the few body systems with \bar{K}

Theoretical studies of $\bar{K}N$ interaction

- Phenomenological approach
Y. Akaishi and T. Yamazaki, Phys. Rev. C **65** (2002) etc
- Chiral theory: based on a 4-point local interaction
T. Hyodo and W. Weise, Phys. Rev. C **77** (2008)
K. Miyahara and T. Hyodo, Phys. Rev. C **93** (2016) etc

Investigate the KN system in the Skyrme model
where the nucleon is described as a soliton.

2. Method

The Skyrme model and our ansatz

- SKyrme model T.H.R. Skyrme, Nucl. Phys. **31** (1962); Proc. Roy. Soc. A **260** (1961)

- Describe the meson-baryon interaction by mesons
- Baryon emerges as a soliton of meson fields.

$$L = \frac{F_\pi^2}{16} \text{tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{tr} [(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2 + L_{SB} + L_{WZ}$$

F_π, e : parameter m_π : mass less, m_K : massive

• Ansatz

$$U = (3 \times 3 \text{ matrix}) \rightarrow \sqrt{U_\pi} U_K \sqrt{U_\pi}$$

C.G. Callan and I. Klebanov, Nucl. Phys. **B 262** (1985)

C .G.Callan, K .Hornbostel and I. Klebanov, Phys. Lett. **B 202** (1988)

$$U_\pi = \begin{pmatrix} U_H & 0 \\ 0 & 1 \end{pmatrix}$$

Hedgehog soliton

$$\begin{cases} U_\pi \rightarrow A(t) U_\pi A^\dagger(t) & A(t): \text{isospin rotation matrix} \\ U_K = U_K & (2 \times 2 \text{ matrix}) \end{cases}$$

$$U_K = \exp \left[i \frac{2}{F_\pi} \lambda_a K_a \right], \quad a = 3, 4, 5, 6$$

$$U = A(t) \sqrt{U_\pi} A^\dagger(t) U_K A(t) \sqrt{U_\pi} A^\dagger(t)$$

$$\lambda_a K_a = \sqrt{2} \begin{pmatrix} 0_{2 \times 2} & K \\ K^\dagger & 0 \end{pmatrix} \quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad K^\dagger = \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}$$

T. Ezoe. and A. Hosaka Phys. Rev. D **94**, 034022 (2016)

Obtaining Lagrangian

$$L = L_{SU(2)} + L_{KN}$$

$$\begin{aligned}
 L_{SU(2)} &= \frac{1}{16} F_\pi^2 \text{tr} \left[\partial_\mu \tilde{U}^\dagger \partial^\mu \tilde{U} \right] + \frac{1}{32e^2} \text{tr} \left[\partial_\mu \tilde{U} \tilde{U}^\dagger, \partial_\nu \tilde{U} \tilde{U}^\dagger \right]^2 \\
 L_{KN} &= (D_\mu \mathbf{K})^\dagger D^\mu \mathbf{K} - \mathbf{K}^\dagger a_\mu^\dagger a^\mu \mathbf{K} - m_K^2 \mathbf{K}^\dagger \mathbf{K} \\
 &\quad + \frac{1}{(eF_\pi)^2} \left\{ -\mathbf{K}^\dagger \mathbf{K} \text{tr} \left[\partial_\mu \tilde{U} \tilde{U}^\dagger, \partial_\nu \tilde{U} \tilde{U}^\dagger \right]^2 - 2 (D_\mu \mathbf{K})^\dagger D_\nu \mathbf{K} \text{tr} (a^\mu a^\nu) \right. \\
 &\quad \left. - \frac{1}{2} (D_\mu \mathbf{K})^\dagger D^\mu \mathbf{K} \text{tr} (\partial_\nu \tilde{U}^\dagger \partial^\nu \tilde{U}) + 6 (D_\nu \mathbf{K})^\dagger [a^\nu, a^\mu] D_\mu \mathbf{K} \right\} \\
 &\quad + \frac{3i}{F_\pi^2} B^\mu \left[(D_\mu \mathbf{K})^\dagger \mathbf{K} - \mathbf{K}^\dagger (D_\mu \mathbf{K}) \right]
 \end{aligned}$$

- Decompose the kaon field in partial waves

$$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \psi_I K(t, \mathbf{r}) \rightarrow \underbrace{\psi_I}_{\text{Isospin wave function}} \underbrace{K(\mathbf{r})}_{\text{Spatial wave function}} e^{-iEt}$$

$$K(\mathbf{r}) = \sum_{l,m} C_{lm\alpha} Y_{lm}(\theta, \phi) k_l^\alpha(r)$$

$Y_{lm}(\theta, \varphi)$: Spherical harmonics
 l : orbital angular momentum
 m : the 3rd component of l
 α : the other quantum numbers

Equation of motion and potential

• Equation of motion(E.o.M)

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 h(r) \frac{dk_l^\alpha(r)}{dr} \right) - E^2 f(r) k_l^\alpha(r) + (m_K^2 + V(r)) k_l^\alpha(r) = 0 : \text{Klein-Gordon like}$$

→
$$-\frac{1}{m_K + E} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dk_l^\alpha(r)}{dr} \right) + U(r) k_l^\alpha(r) = \varepsilon k_l^\alpha(r) : \text{Schrödinger like}$$

$$U(r) = -\frac{1}{m_K + E} \left[\frac{h(r) - 1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{dh(r)}{dr} \frac{d}{dr} \right] - \frac{(f(r) - 1) E^2}{m_K + E} + \frac{V(r)}{m_K + E}$$

Equivalent local potential: $\tilde{U}(r) = \frac{U(r) k_l^\alpha(r)}{k_l^\alpha(r)}$

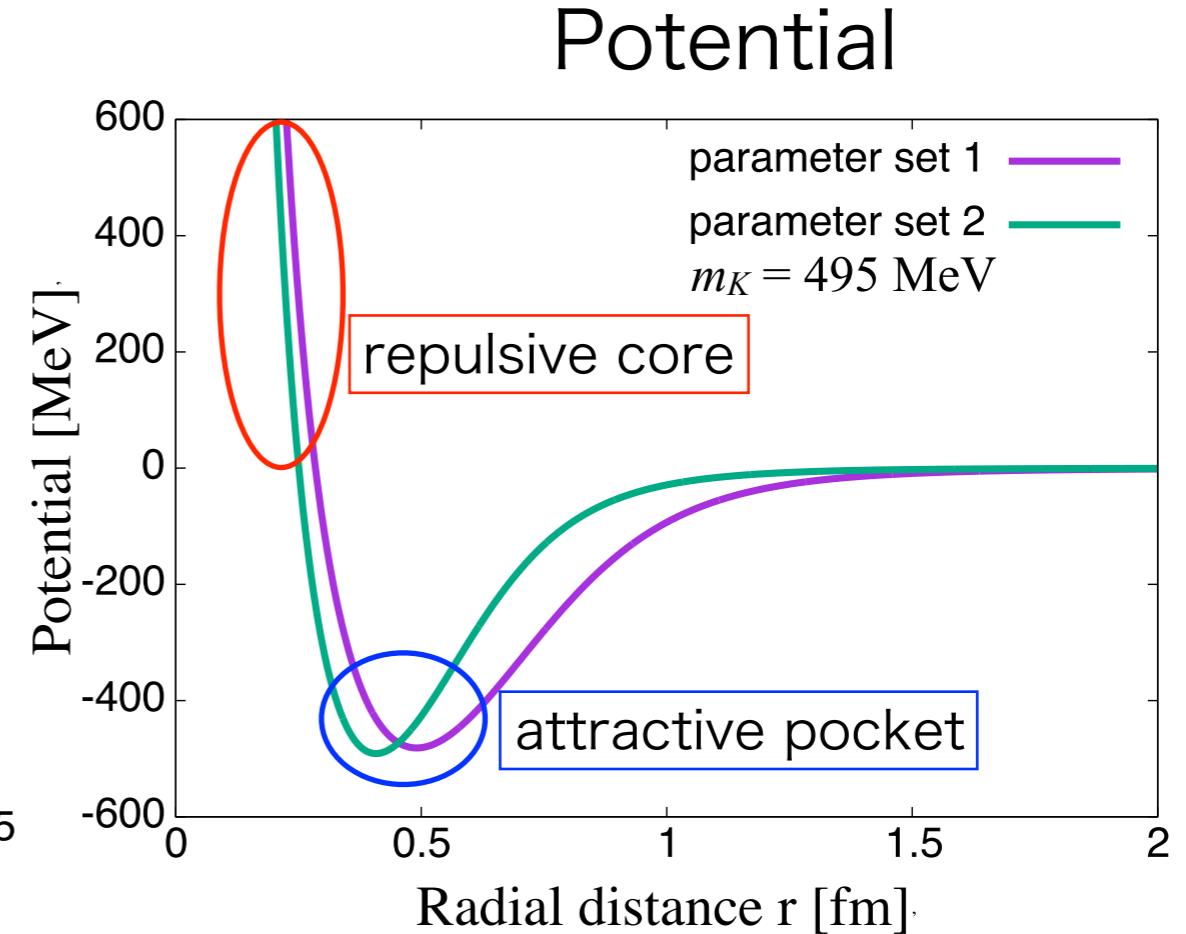
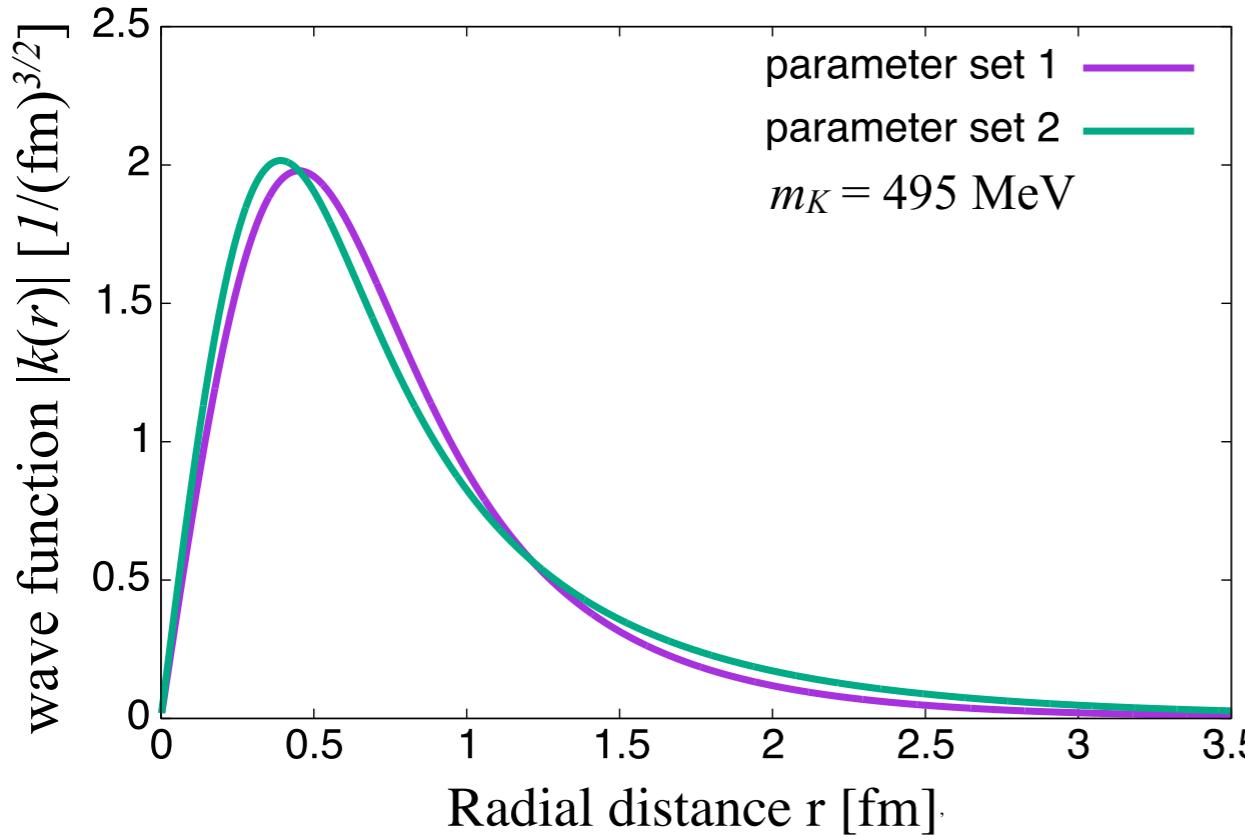
• Properties of resulting potential U

1. Nonlocal and depend on the kaon energy
2. Contain isospin dependent and independent central forces and the similar **spin-orbit(LS) forces**
3. A repulsive component is proportional to $1/r^2$ at short distances

3. Results and discussions

Result 1: $\bar{K}N(I = 0, L = 0)$ Bound state

Kaon wave function



• Model parameters and physical properties

	F_π [MeV]	e	B.E. [MeV]	$\langle r_N^2 \rangle^{1/2}$ [fm]	$\langle r_K^2 \rangle^{1/2}$ [fm]
parameter set 1	129	5.45	82.9	0.59	0.99
parameter set 2	186	4.82	32.9	0.46	1.18

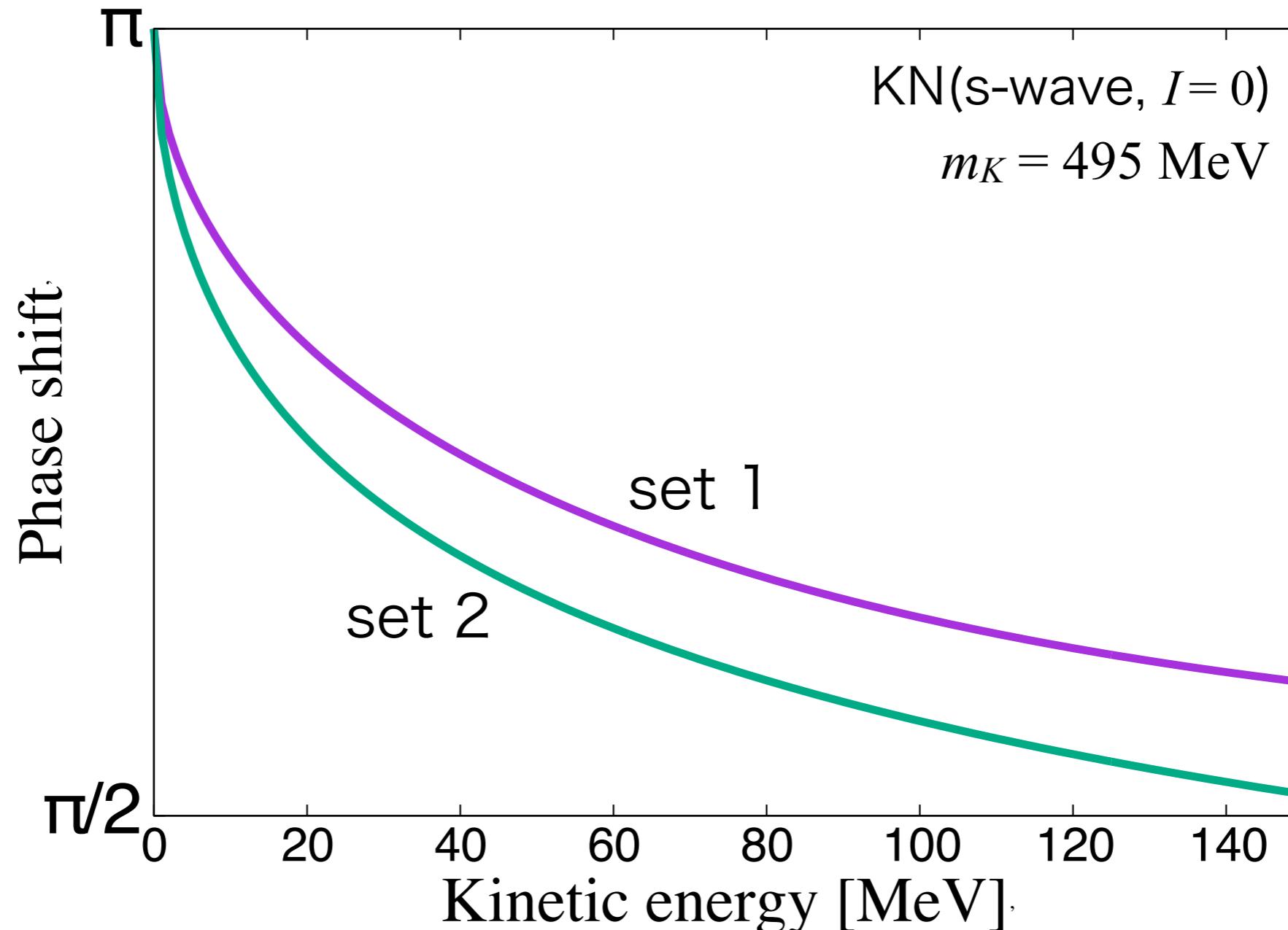
T. Ezoe. and A. Hosaka Phys. Rev. D **94**, 034022 (2016)

$$\langle r_N^2 \rangle = \int_0^\infty dr \ r^2 \rho_B(r), \quad \rho_B(r) = -\frac{2}{\pi} \sin^2 F F' \quad \text{G. S. Adkins, C. R. Nappi and E. Witten,}$$

Nucl. Phys. B **228** (1983)

$$\langle r_K^2 \rangle = \int dV \ r^2 [Y_{00}(\hat{r}) k_0^0(r)]^2 = \int_0^\infty dr \ r^4 k^2(r) \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

Result 2: $\bar{K}N(I = 0, L = 0)$ phase shift



	F_π [MeV]	e	B.E. [MeV]
parameter set 1	129	5.45	82.9
parameter set 2	186	4.82	32.9

Result 3: Fitting the potentials

$$\tilde{U}(r) = \tilde{U}_0^c(r) + \tilde{U}_\tau^c(r)(\mathbf{I}^K \cdot \mathbf{I}^N) + \tilde{U}_0^{LS}(r)(\mathbf{L}^K \cdot \mathbf{J}^N) + \tilde{U}_\tau^{LS}(r)(\mathbf{L}^K \cdot \mathbf{J}^N)(\mathbf{I}^K \cdot \mathbf{I}^N)$$

	Isospin	Normal term	Wess-Zumino term
Central	indep.	$u_0^c(N, r) + v_0^c(N, r)E_{kin}$ $G_1(r) + G_2(r) + G_2(r)$	$u_0^c(WZ, r) + v_0^c(WZ, r)E_{kin}$ $G_2(r) + G_2(r)$
	dep.	$u_\tau^c(N, r) + v_\tau^c(N, r)E_{kin}$ $G_2(r) + G_3(r)$	— —
LS	indep.	$u_0^{LS}(N, r) + v_0^{LS}(N, r)E_{kin}$ $G_2(r) + G_2(r)$	$u_0^{LS}(WZ, r) + v_0^{LS}(WZ, r)E_{kin}$ $G_2(r) + G_2(r)$
	dep.	$u_\tau^{LS}(N, r) + v_\tau^{LS}(N, r)E_{kin}$ $G_1(r) + G_1(r)$	— —

$$\begin{aligned}\tilde{U}(r) &\simeq \tilde{U}(r) + \frac{\partial \tilde{U}(r)}{\partial E_{kin}} E_{kin} \\ &\equiv u(r) + v(r) E_{kin}\end{aligned}$$

Gaussian-fit

$$\begin{aligned}G_1(r) &= \textcolor{red}{C_{-2}} \frac{1}{r^2/\textcolor{blue}{R}^2} \exp\left(-\frac{r^2}{\textcolor{blue}{R}^2}\right) \\ G_2(r) &= \textcolor{red}{C_0} \exp\left(-\frac{r^2}{\textcolor{blue}{R}^2}\right) \\ G_3(r) &= \textcolor{red}{C_2} \frac{r^2}{\textcolor{blue}{R}^2} \exp\left(-\frac{r^2}{\textcolor{blue}{R}^2}\right)\end{aligned}$$

Result 3: Fitting the potentials

$$\tilde{U}(r) = \tilde{U}_0^c(r) + \tilde{U}_\tau^c(r)(\mathbf{I}^K \cdot \mathbf{I}^N) + \tilde{U}_0^{LS}(r)(\mathbf{L}^K \cdot \mathbf{J}^N) + \tilde{U}_\tau^{LS}(r)(\mathbf{L}^K \cdot \mathbf{J}^N)(\mathbf{I}^K \cdot \mathbf{I}^N)$$

	Isospin	Normal term	Wess-Zumino term
Central	indep.	$u_0^c(N, r) + v_0^c(N, r)E_{kin}$ $G_1(r) + G_2(r) + G_2(r)$	$u_0^c(WZ, r) + v_0^c(WZ, r)E_{kin}$ $G_2(r) + G_2(r)$
	dep.	$u_\tau^c(N, r) + v_\tau^c(N, r)E_{kin}$ $G_2(r) + G_3(r)$	— —
LS	indep.	$u_0^{LS}(N, r) + v_0^{LS}(N, r)E_{kin}$ $G_2(r) + G_2(r)$	$u_0^{LS}(WZ, r) + v_0^{LS}(WZ, r)E_{kin}$ $G_2(r) + G_2(r)$
	dep.	$u_\tau^{LS}(N, r) + v_\tau^{LS}(N, r)E_{kin}$ $G_1(r) + G_1(r)$	— —

$$\begin{aligned}\tilde{U}(r) &\simeq \tilde{U}(r) + \frac{\partial \tilde{U}(r)}{\partial E_{kin}} E_{kin} \\ &\equiv u(r) + v(r) E_{kin}\end{aligned}$$

Fit by the Gaussian

$$\begin{aligned}G_1(r) &= C_{-2} \frac{1}{r^2 / R_{-2}^2} \exp\left(-\frac{r^2}{R_{-2}^2}\right) \\ G_2(r) &= C_0 \exp\left(-\frac{r^2}{R_0^2}\right) \\ G_3(r) &= C_2 \frac{r^2}{R_2^2} \exp\left(-\frac{r^2}{R_2^2}\right)\end{aligned}$$

Result 3: fitting parameters

parameter set 2: $F_\pi = 186 \text{ MeV}$, $e = 4.82$

- **Isospin independent normal term**

$$C_{-2} \frac{1}{r^2/R_{-2}^2} \exp\left(-\frac{r^2}{R_{-2}^2}\right) + C_0 \exp\left(-\frac{r^2}{R_0^2}\right) + C'_0 \exp\left(-\frac{r^2}{R'_0^2}\right)$$

R_{-2}	0.156 fm
R_0	0.280 fm
R'_0	0.427 fm

	C_{-2}	C_0	C'_0
$u(r)$	3883.32 MeV	3877.77 MeV	-983.591 MeV
$v(r)$	-3.75119	-4.42205	-0.712553

- **Isospin dependent normal term**

$$C_0 \exp\left(-\frac{r^2}{R_0^2}\right) + C_2 \frac{r^2}{R_2^2} \exp\left(-\frac{r^2}{R_2^2}\right)$$

R_0	0.265 fm
R_2	0.524 fm

	C_0	C_2
$u(r)$	401.337 MeV	290.964 MeV
$v(r)$	1.07039	0.293903

- **Wess-Zumino term**

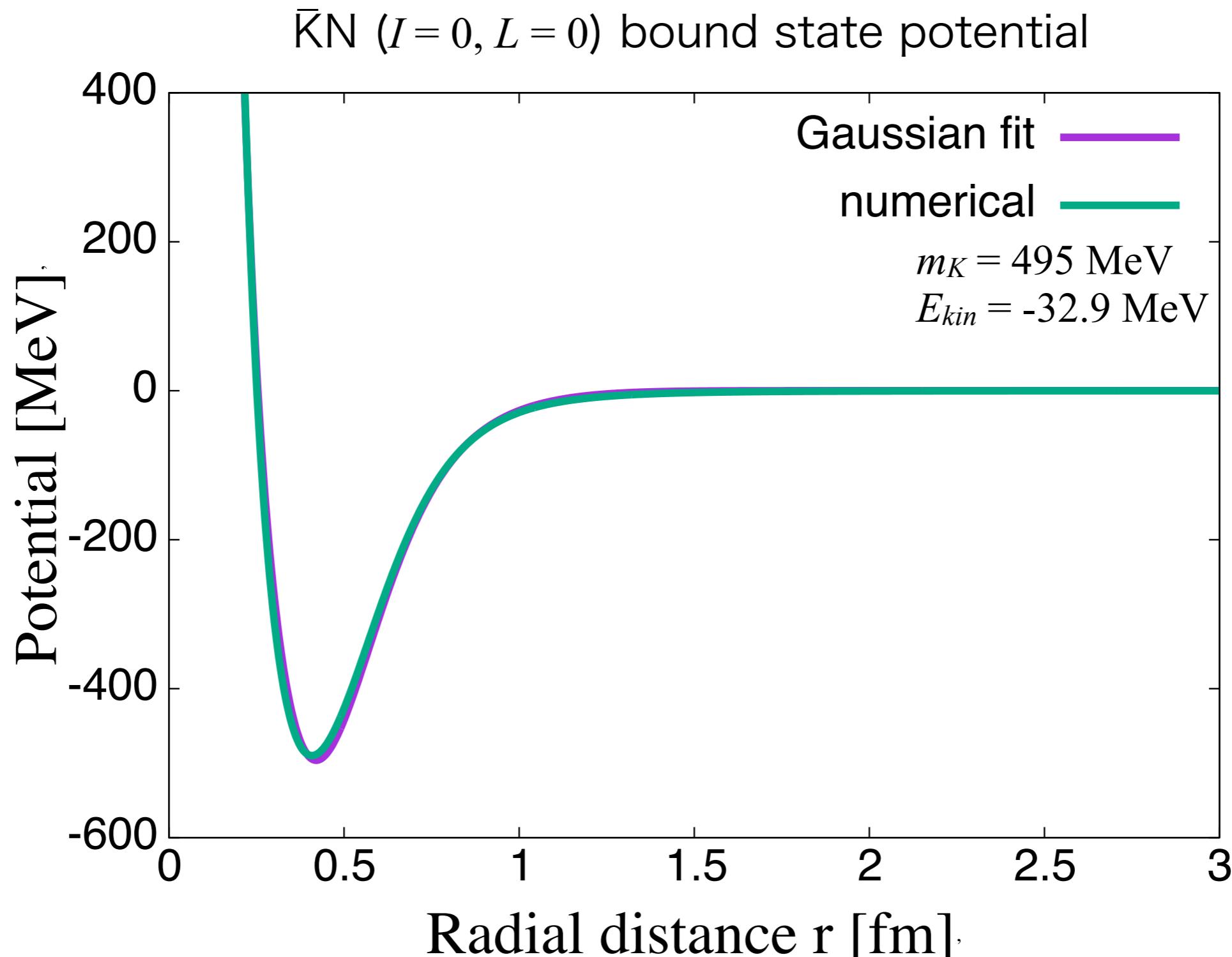
$$C_0 \exp\left(-\frac{r^2}{R^2}\right) + C'_0 \exp\left(-\frac{r^2}{R'^2}\right)$$

R_0	0.282 fm
R'_0	0.404 fm

	C_0	C'_0
$u(r)$	-676.51 MeV	-1207.07 MeV
$v(r)$	-3.483	-0.995

Result 3: Comparing

parameter set 2: $F_\pi = 186 \text{ MeV}$, $e = 4.82$



5. Summary

Summaries

Investigate the kaon-nucleon systems by a modified bound state approach in the Skyrme model and apply to a channel

- **Results**

1. Properties of the obtaining potential
 - a. nonlocal and depends on the kaon energy
 - b. contain **central and LS terms**
with and without isospin dependence
 - c. repulsion proportional to $1/r^2$ for small r
2. $\bar{K}N(I=0)$ bound states exist with B.E. of order ten MeV
3. The phase shift diagram reflects the property of the bound state
4. Fit the potential with the Gaussians

- **Future works**

1. The $\pi \Sigma$ system
2. The properties of $\Lambda(1405)$
3. few body nuclear system with kaon

Thank you for
your attention

back-up

1. The Skyrme model

The Skyrme model 1

T.H.R. Skyrme, Nucl. Phys. **31** (1962);
Proc. Roy. Soc. A **260** (1961)

- Describe the interaction between mesons and baryons by mesons
- Baryon emerges as a soliton of meson fields.

$$\phi = \frac{1}{\sqrt{2}} \lambda_a \phi_a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$
$$U = \exp \left[i \frac{2}{F_\pi} \lambda_a \phi_a \right] \quad \lambda_a: \text{Gell-Mann matrices } (a = 1, 2, \dots, 8)$$

The Skyrme model 1

T.H.R. Skyrme, Nucl. Phys. **31** (1962);
Proc. Roy. Soc. A **260** (1961)

- Describe the interaction between mesons and baryons by mesons
- Baryon emerges as a soliton of meson fields.

$$\phi = \frac{1}{\sqrt{2}} \lambda_a \phi_a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & & K^+ \\ \frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\eta & \pi^- & \bar{K}^0 & K^0 \\ \pi^- & K^- & -\frac{2}{\sqrt{6}}\eta & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

$U = \exp \left[i \frac{2}{F_\pi} \lambda_a \phi_a \right]$

λ_a : Gell-Mann matrices ($a = 1, 2, \dots, 8$)

- For SU(2)

$$L = \frac{F_\pi^2}{16} \text{tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{tr} [(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2$$

The kinetic term

F_π, e : parameters

The interaction term
(the Skyrme term)

The Skyrme model 2

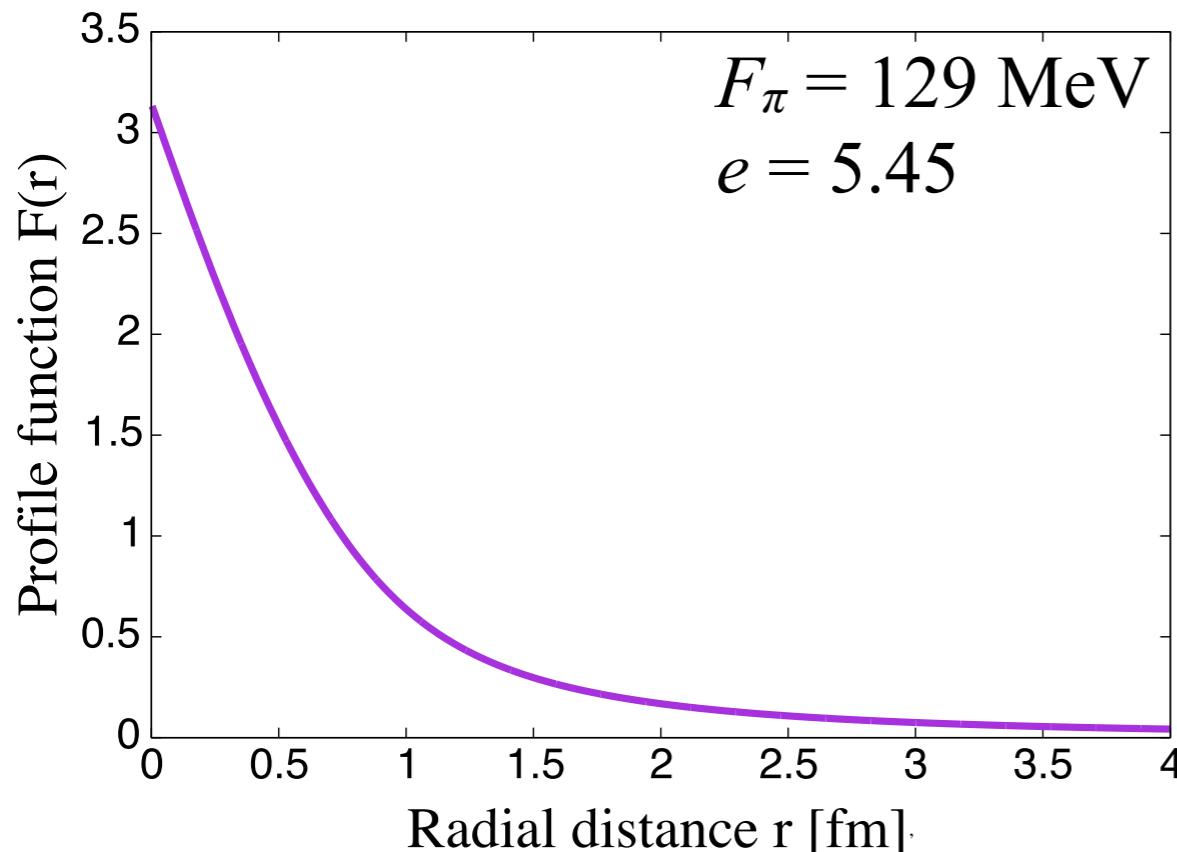
- Hedgehog ansatz

π has three degrees of freedom(π^0, π^+, π^-)

- two of these: the angles of the radial vector, θ, φ
- the rest: a function depending on r

⇒ a special configuration called the hedgehog ansatz

Hedgehog ansatz: $U_H = \exp [i\boldsymbol{\tau} \cdot \hat{r} F(r)]$



minimize the mass of the soliton
with B.C. for $F(r)$: $F(\infty) = 0, F(0) = \pi$

G. S. Adkins, C. R. Nappi and E. Witten,
Nucl. Phys. B **228** (1983)

The Skyrme model 3

• Quantization

The hedgehog ansatz is classical

→ without spin or isospin

→ become a physical state by quantization

$$U_H(\mathbf{x}) \rightarrow U_H(t, \mathbf{x}) = A(t) \exp [i\tau_a R_{ab}(t) \hat{r}_b F(r)] A^\dagger(t)$$

$A(t)$: 2×2 isospin rotation matrix

$R_{ab}(t)$: 3×3 spatial rotation matrix

The baryon with $I=J$ from the symmetry

which the hedgehog ansatz has

• Quantized Hamiltonian

$$H = M_{sol} + \frac{J(J+1)}{2\Lambda}$$

the rotation energy

M_{sol} : soliton mass

J : spin or isospin value

Λ : moment of inertia

2. The CK and Our approaches

Method

SU(3) symmetry is broken $\rightarrow m_u = m_d = 0, m_s \neq 0$

Callan-Klebanov approach (CK approach)

- Introduce the kaon as fluctuations **around the hedgehog soliton**
- Form a bound state of the kaon and the hedgehog soliton
- **rotate the system** to generate hyperons
- Follow the $1/N_c$ counting rule
- Projection after variation, The strong coupling

C.G. Callan and I. Klebanov, Nucl. Phys. **B 262** (1985)

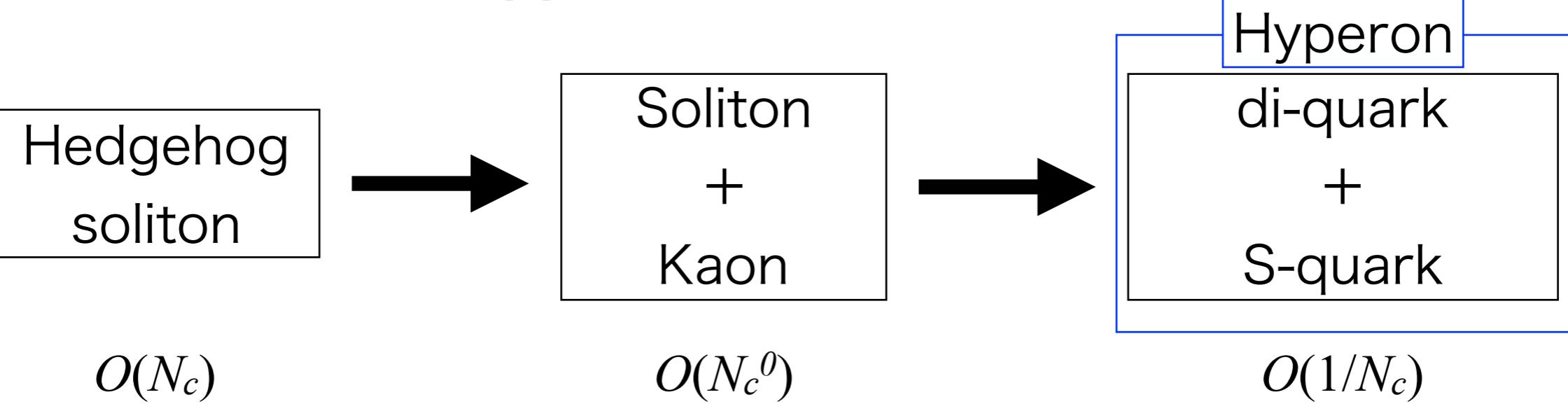
C .G.Callan, K .Hornbostel and I. Klebanov, Phys. Lett. **B 202** (1988)

Our approach

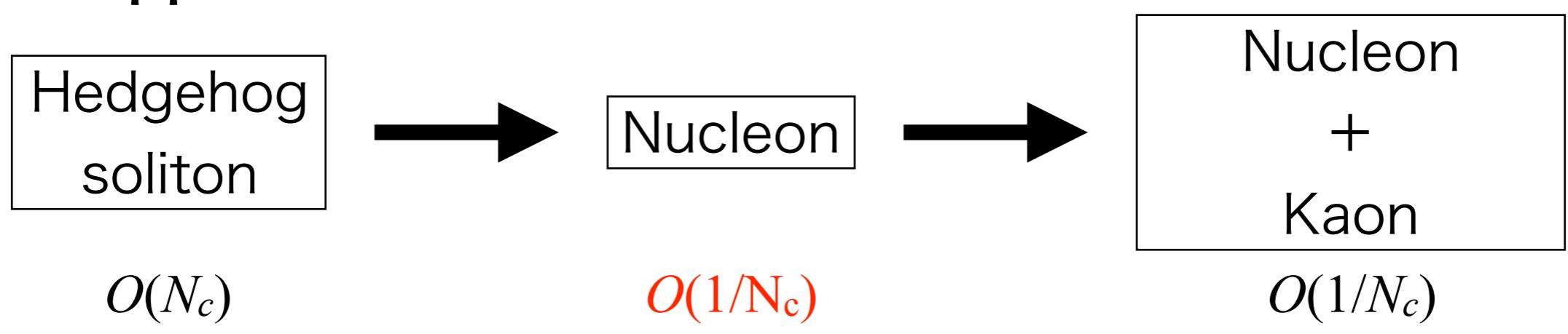
- **Rotate the hedgehog soliton** to generate the nucleon
- Introduce the kaon as fluctuations **around the nucleon**
- describe kaon-nucleon systems
- Violate the $1/N_c$ counting rule
- Variation after projection, The weak coupling

$1/N_c$ expansion

Callan-Klebanov approach



Our approach



$O(N_c^0)$ is missing

3. Our method and results (Bound state)

Lagrangian and ansatz

- Expand to the SU(3) Skyrme model

$$L = \frac{F_\pi^2}{16} \text{tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{tr} [(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2 + L_{SB} + L_{WZ}$$

- Ansatz

$$U = \begin{cases} A(t) \sqrt{U_\pi} U_K \sqrt{U_\pi} A^\dagger(t) : \text{Callan-Klebanov ansatz} \\ A(t) \sqrt{U_\pi} A^\dagger(t) U_K A(t) \sqrt{U_\pi} A^\dagger(t) : \text{Our ansatz} \end{cases}$$

$$U_\pi = \begin{pmatrix} U_H & 0 \\ 0 & 1 \end{pmatrix}$$

Hedgehog ansatz
(2×2 matrix)

$$U_K = \exp \left[i \frac{2}{F_\pi} \lambda_a K_a \right], \quad a = 3, 4, 5, 6$$

$$\lambda_a K_a = \sqrt{2} \begin{pmatrix} 0_{2 \times 2} & K \\ K^\dagger & 0 \end{pmatrix} \quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad K^\dagger = \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}$$

Lagrangian and ansatz

- Expand to the SU(3) Skyrme model

$$L = \frac{F_\pi^2}{16} \text{tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{tr} [(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2 + L_{SB} + L_{WZ}$$

- Ansatz

$$U = \begin{cases} A(t) \sqrt{U_\pi} U_K \sqrt{U_\pi} A^\dagger(t) & : \text{Callan-Klebanov ansatz} \\ [A(t) \sqrt{U_\pi} A^\dagger(t)] U_K [A(t) \sqrt{U_\pi} A^\dagger(t)] & : \text{Our ansatz} \end{cases}$$

the kaon around the hedgehog soliton

the kaon around the **rotating** hedgehog soliton

Derivation 1

- Substitute our ansatz for the Lagrangian

Ansatz

$$U = A(t) \sqrt{U_\pi} A^\dagger(t) U_K A(t) \sqrt{U_\pi} A^\dagger(t)$$

$$U_K = \exp \left[i \frac{2}{F_\pi} \lambda_a K_a \right], \quad a = 3, 4, 5, 6 \qquad \qquad U_\pi = \begin{pmatrix} U_H & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_a K_a = \sqrt{2} \begin{pmatrix} 0_{2 \times 2} & K \\ K^\dagger & 0 \end{pmatrix} \quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad K^\dagger = \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}$$

Lagrangian

$$\begin{aligned} L = & \frac{F_\pi^2}{16} \text{tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{tr} [(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2 \\ & + L_{SB} + L_{WZ} \end{aligned}$$

Obtaining Lagrangian

$$L = L_{SU(2)} + L_{KN}$$

$$\begin{aligned}
L_{SU(2)} &= \frac{1}{16} F_\pi^2 \text{tr} \left[\partial_\mu \tilde{U}^\dagger \partial^\mu \tilde{U} \right] + \frac{1}{32e^2} \text{tr} \left[\partial_\mu \tilde{U} \tilde{U}^\dagger, \partial_\nu \tilde{U} \tilde{U}^\dagger \right]^2 \\
L_{KN} &= (D_\mu K)^\dagger D^\mu K - K^\dagger a_\mu^\dagger a^\mu K - m_K^2 K^\dagger K \\
&\quad + \frac{1}{(eF_\pi)^2} \left\{ -K^\dagger K \text{tr} \left[\partial_\mu U_H U_H^\dagger, \partial_\nu U_H U_H^\dagger \right]^2 - 2 (D_\mu K)^\dagger D_\nu K \text{tr} (a^\mu a^\nu) \right. \\
&\quad \left. - \frac{1}{2} (D_\mu K)^\dagger D^\mu K \text{tr} \left(\partial_\nu U_H^\dagger \partial^\nu U_H \right) + 6 (D_\nu K)^\dagger [a^\nu, a^\mu] D_\mu K \right\} \\
&\quad + \frac{3i}{F_\pi^2} B^\mu \left[(D_\mu K)^\dagger K - K^\dagger (D_\mu K) \right]
\end{aligned}$$

$$\tilde{U} = A(t) U_H A^\dagger(t), \quad \tilde{\xi} = A(t) \sqrt{U_H} A^\dagger(t) \quad D_\mu K = \partial_\mu K + v_\mu K$$

$$v_\mu = \frac{1}{2} \left(\tilde{\xi}^\dagger \partial_\mu \tilde{\xi} + \tilde{\xi} \partial_\mu \tilde{\xi}^\dagger \right)$$

$$a_\mu = \frac{1}{2} \left(\tilde{\xi}^\dagger \partial_\mu \tilde{\xi} - \tilde{\xi} \partial_\mu \tilde{\xi}^\dagger \right)$$

$$B^\mu = -\frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2} \text{tr} \left[\left(U_H^\dagger \partial_\nu U_H \right) \left(U_H^\dagger \partial_\alpha U_H \right) \left(U_H^\dagger \partial_\beta U_H \right) \right]$$

G. S. Adkins, C. R. Nappi and E. Witten,
Nucl. Phys. B **228** (1983)

Derivation 2

- Decompose the kaon field

$$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \psi_I K(t, \mathbf{r}) \rightarrow \underbrace{\psi_I}_{\text{Isospin wave function}} \underbrace{K(\mathbf{r}) e^{-iEt}}_{\text{Spatial wave function}}$$

- Expand the $K(r)$ by the spherical harmonics

$$K(\mathbf{r}) = \sum_{l,m} C_{lm\alpha} Y_{lm}(\theta, \phi) k_l^\alpha(r)$$

$Y_{lm}(\theta, \phi)$: Spherical harmonics
 l : orbital angular momentum
 m : the 3rd component of l
 α : the other quantum numbers

- Take a variation with respect to the kaon radial function

⇒ Obtain the equation of motion for the kaon around the nucleon

Results (Equation of motion)

- **Equation of motion(E.o.M)**

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 h(r) \frac{dk_l^\alpha(r)}{dr} \right) - E^2 f(r) k_l^\alpha(r) + (m_K^2 + V(r)) k_l^\alpha(r) = 0$$

$h(r), f(r)$: functions depending on r

E : kaon energy, m_K : kaon mass

$V(r)$: the kaon nucleon interaction term

Investigate KN systems by solving this E.o.M



Concentrate on $\bar{K}N(I=0)$ system

Interaction term

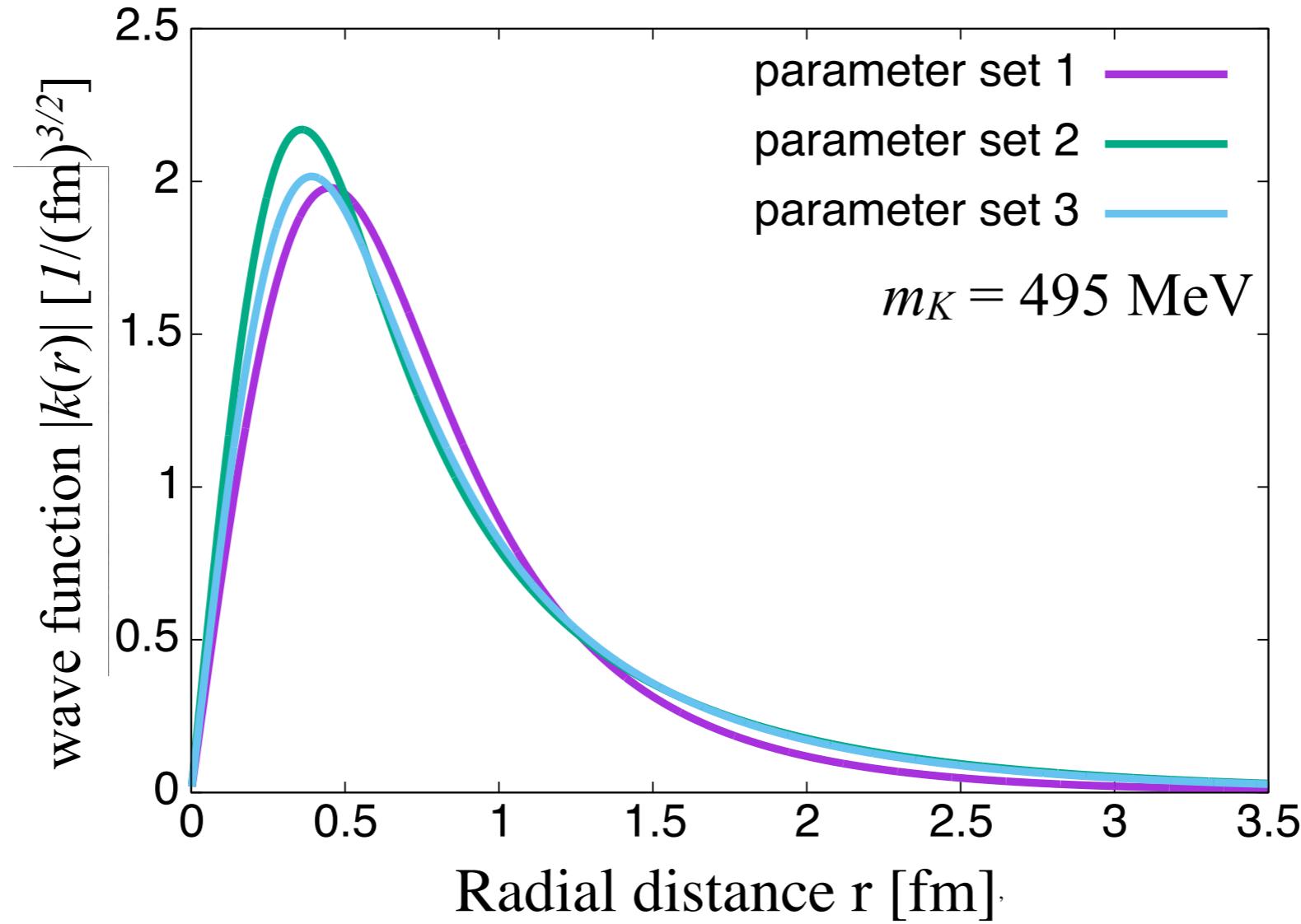
$$V(r) = V_{nor}(r) + V_{WZ}(r)$$

$$\begin{aligned}
V_{nor}(r) &= -\frac{1}{4} \left(2 \frac{\sin^2 F}{r^2} + F'^2 \right) + 2 \frac{s^4}{r^2} - \frac{1}{(eF_\pi)^2} \left[2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F'^2 \right) - 2 \frac{s^4}{r^2} \left(F'^2 + \frac{\sin^2 F}{r^2} \right) \right] \\
&\quad + \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[\frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{ s^2 \sin F F' \} \right] \\
&\quad + \frac{2E}{\Lambda} s^2 \left[1 + \frac{1}{(eF_\pi)^2} \left(F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 I_{KN} \boxed{I_{KN}} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[F'^2 + \frac{4}{r^2} \sin^2 F \right] \boxed{I_{KN}} \\
&\quad + \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{4}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \boxed{I_{KN}} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right] \\
&\quad + \left[1 + \frac{1}{(eF_\pi)^2} \left(\frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{l(l+1)}{r^2} - \left[1 + \frac{1}{(eF_\pi)^2} \left(4 \frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{16s^2}{3r^2} \boxed{J_{KN}} \boxed{I_{KN}} \\
&\quad + \frac{1}{(eF_\pi)^2} \frac{2E \sin^2 F}{\Lambda r^2} \boxed{J_{KN}} - \frac{1}{(eF_\pi)^2} \frac{8}{r^2} \frac{d}{dr} (\sin F F') \boxed{J_{KN}} \boxed{I_{KN}}
\end{aligned}$$

$$V_{WZ}(r) = \frac{3E}{(\pi F_\pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F' + \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F}{\Lambda r^2} F' \boxed{J_{KN}}$$

$$s = \sin(F/2) \quad \boxed{I_{KN} = \mathbf{I}^K \cdot \mathbf{I}^N}, \quad \boxed{J_{KN} = \mathbf{L}^K \cdot \mathbf{J}^N}$$

$\bar{K}N(L = 0, I_{tot} = 0)$ Bound state



- $\bar{K}N$ bound states with $I_{tot} = 0, l = 0$ (Binding Energy: B.E.)

	F_π [MeV]	e	B.E. [MeV]
parameter set 1	129	5.45	82.9
parameter set 2	186	5.45	27.2
parameter set 3	186	4.82	32.9

when F_π is the exp. value,
B.E. is of order ten MeV

Bound state properties

- Root mean square radii for N and K

$$\langle r_N^2 \rangle = \int_0^\infty dr \ r^2 \rho_B(r), \quad \rho_B(r) = -\frac{2}{\pi} \sin^2 F F'$$

G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B **228** (1983)

$$\langle r_K^2 \rangle = \int dV \ r^2 [Y_{00}(\hat{r}) k_0^0(r)]^2 = \int_0^\infty dr \ r^4 k^2(r) \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

- properties of the $\bar{K}N(I=0)$ bound states

	F_π [MeV]	e	B.E. [MeV]	$\langle r_N^2 \rangle^{1/2}$ [fm]	$\langle r_K^2 \rangle^{1/2}$ [fm]
parameter set 1	129	5.45	82.9	0.59	0.99
parameter set 2	186	5.45	27.2	0.41	1.19
parameter set 3	186	4.82	32.9	0.46	1.18

The anti-kaon is **weakly binding** to the nucleon

Potential

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 h(r) \frac{dk_l^\alpha(r)}{dr} \right) - E^2 f(r) k_l^\alpha(r) + (m_K^2 + V(r)) k_l^\alpha(r) = 0$$

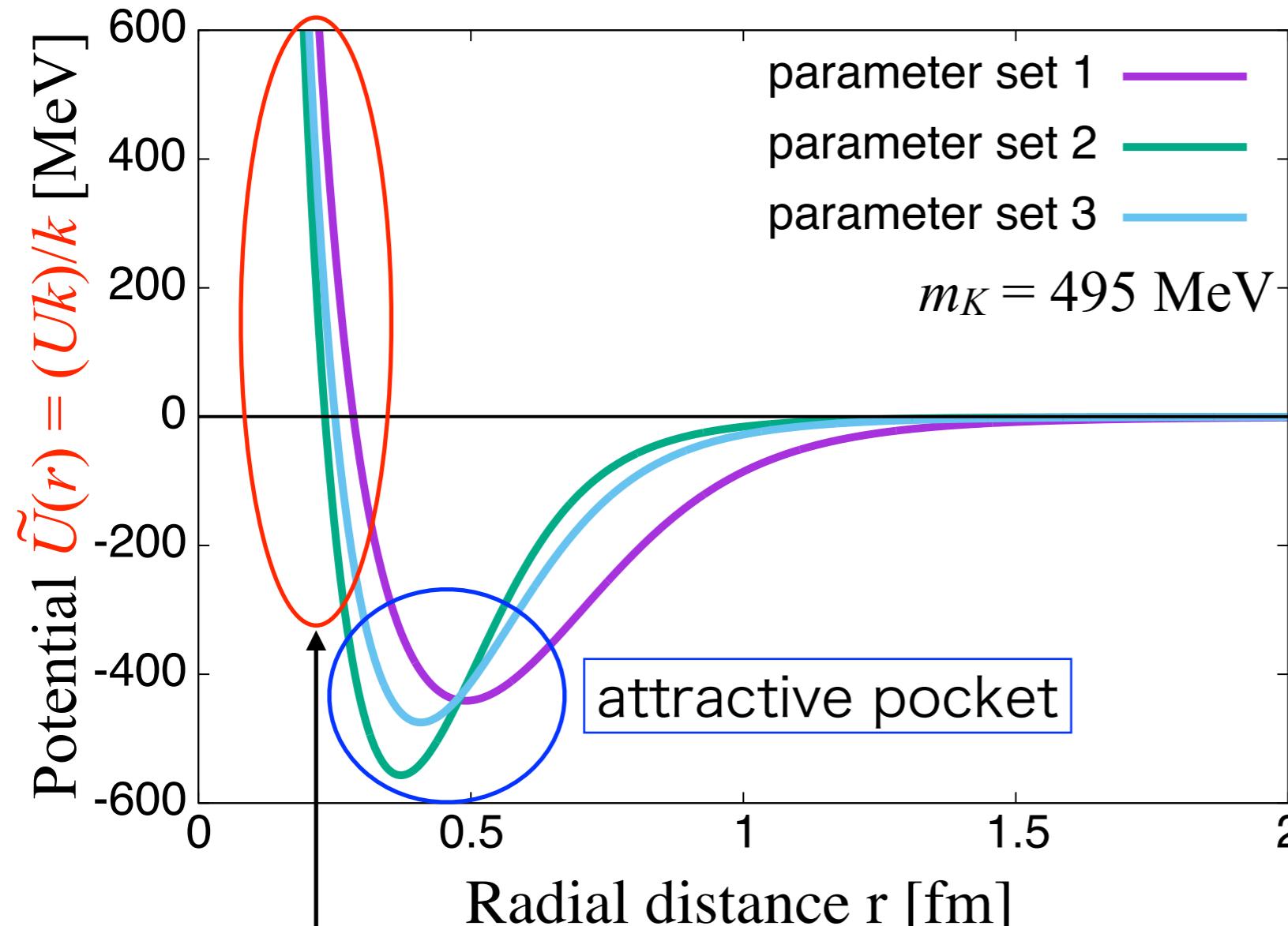
→ $-\frac{1}{m_K + E} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dk_l^\alpha(r)}{dr} \right) + U(r) k_l^\alpha(r) = \varepsilon k_l^\alpha(r)$

$$\begin{aligned} U(r) &= -\frac{1}{m_K + E} \left[\frac{h(r) - 1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{dh(r)}{dr} \frac{d}{dr} \right] - \frac{(f(r) - 1) E^2}{m_K + E} \\ &\quad + \frac{V(r)}{m_K + E} \end{aligned}$$

• properties of resulting potential U

1. Nonlocal and depend on the kaon energy
2. Contain isospin dependent and independent central forces and the similar **spin-orbit(LS) forces**
3. In the short range, behave as a repulsive force proportional to $1/r^2$

$\bar{K}N(L = 0, I_{tot} = 0)$ potential



- **Equivalent local potential**

$$\tilde{U}(r) = \frac{U(r)k(r)}{k(r)}$$

	F_π [MeV]	e
parameter set 1	129	5.45
parameter set 2	186	5.45
parameter set 3	186	4.82

T. Ezoe and A. Hosaka
Phys. Rev. D 94, 034022 (2016)

Comparisons with the chiral theory 1

• Weinberg-Tomozawa interaction

$$L_{WT} = \frac{2}{F_\pi^2} \{ \bar{N} \mathbf{I}^N \gamma^\mu N \cdot (\partial_\mu K^\dagger \mathbf{I}^K K - K^\dagger \mathbf{I}^K \partial_\mu K) \} \propto \frac{1}{F_\pi^2}$$



S. Weinberg, Phys. Rev. Lett. **17** (1966)
Y. Tomozawa, Nuovo Cim. A **46** (1966)

The strength of $L_{WT} \propto 1/F_\pi^2$

→ For $F_\pi = 129$ MeV and 186 MeV,

$$1/129^2 : 1/186^2 \sim \boxed{15 : 7}$$

• The interaction for KN($I=0$) bound state

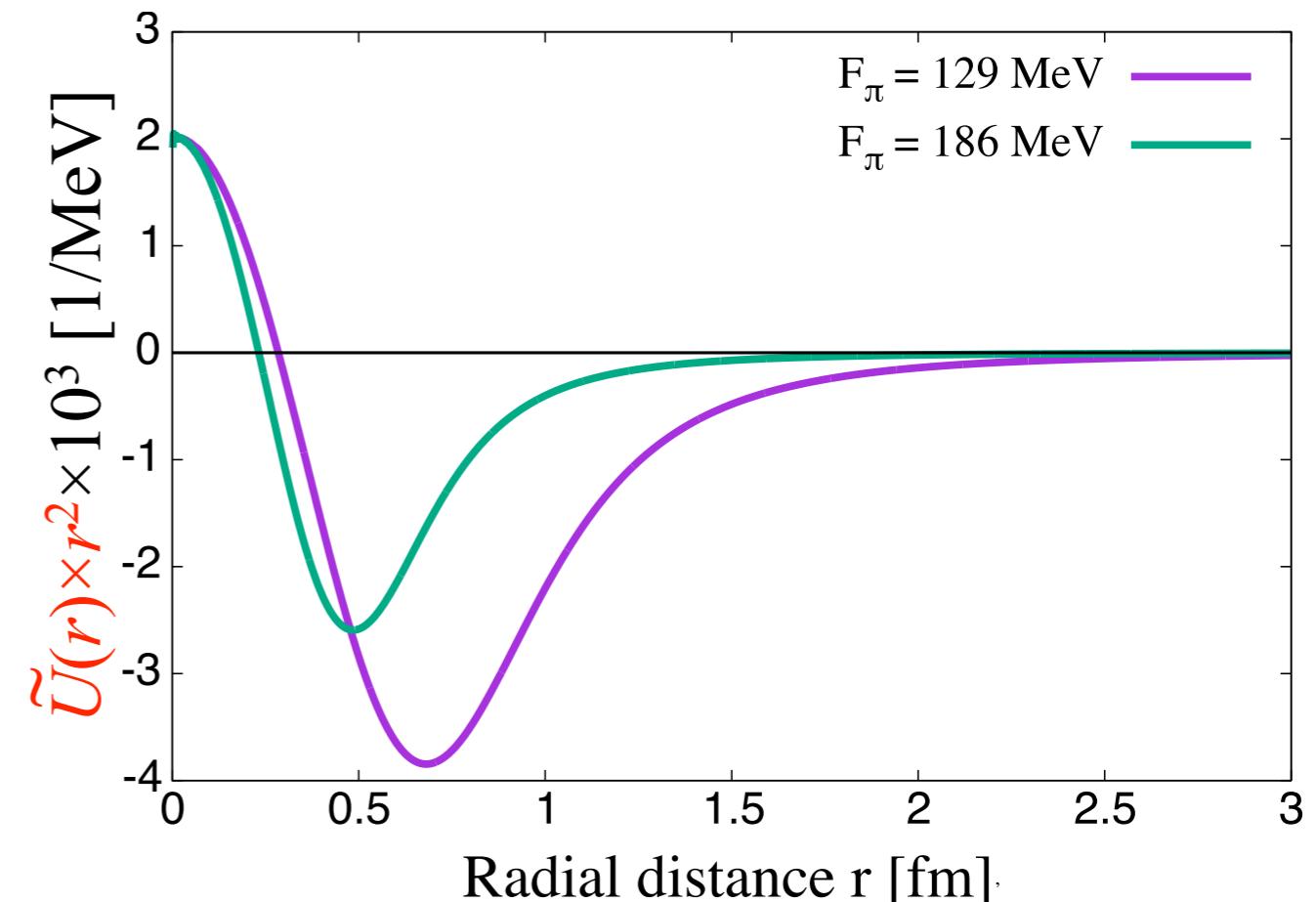
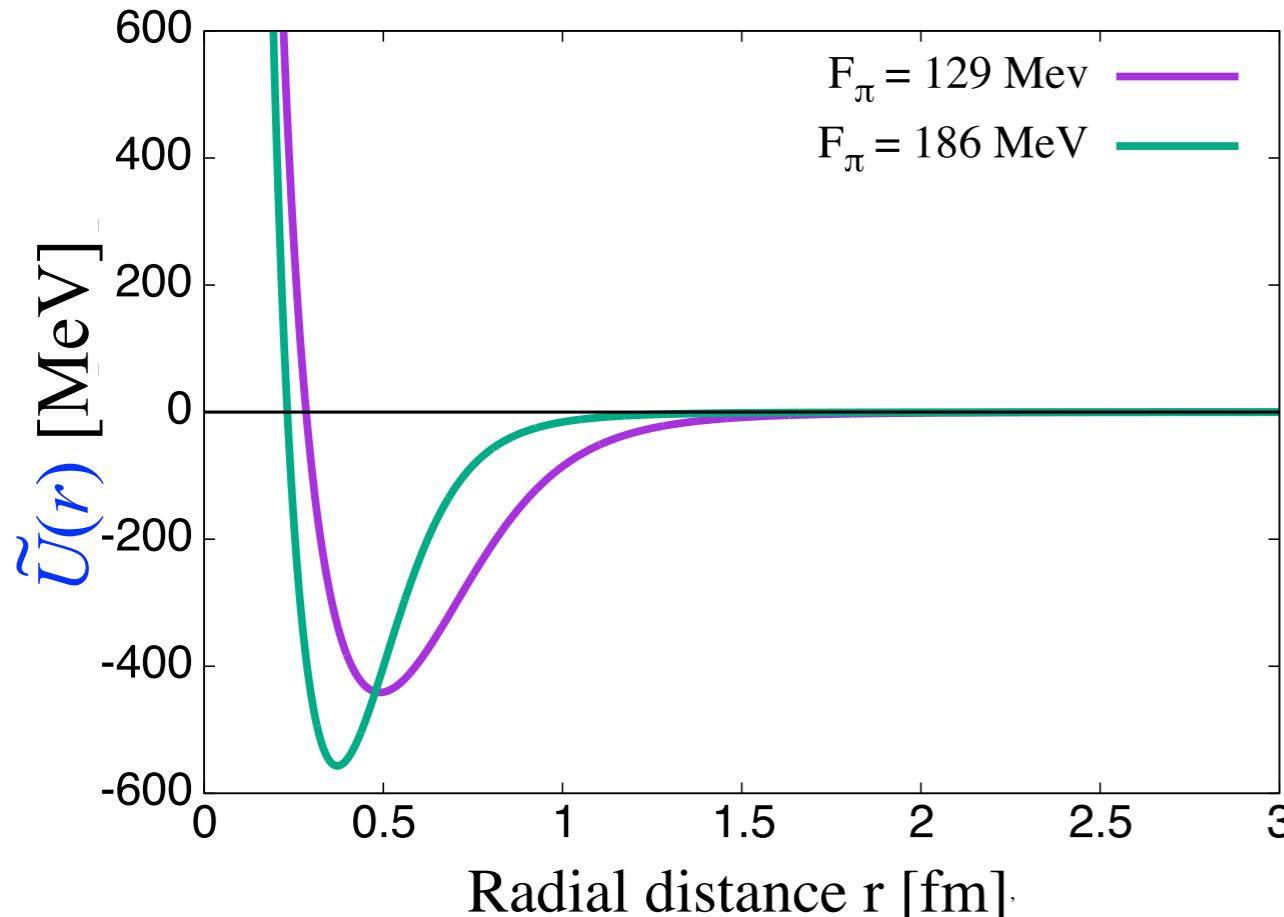
$$W \equiv 4\pi \int r^2 dr \tilde{U}(r)$$

F_π [MeV]	e	$-W \times 10^5$ [1/MeV ²]
129	5.45	$1.2 \times 4\pi$
186	5.45	$0.48 \times 4\pi$



$$\boxed{5:2}$$

Comparisons with the chiral theory 2



- The interaction for KN($I=0$) bound state

$$W \equiv 4\pi \int r^2 dr \tilde{U}(r)$$

F_π [MeV]	e	$-W \times 10^5$ [$1/\text{MeV}^2$]
129	5.45	$1.2 \times 4\pi$
186	5.45	$0.48 \times 4\pi$

Comparisons with the CK approach

- Comparisons between the CK and our approach**

Callan-Klebanov approach				Our approach			Physical state
l	l_{eff}	B.E. [MeV]	$\langle r_K^2 \rangle^{1/2}$ [fm]	l	B.E. [MeV]	$\langle r_K^2 \rangle^{1/2}$ [fm]	
0	1	61.7	0.93	0	32.9	1.18	$\Lambda(1405)$
1	0	326.6	0.54	—	—	—	$\Lambda(1116)$

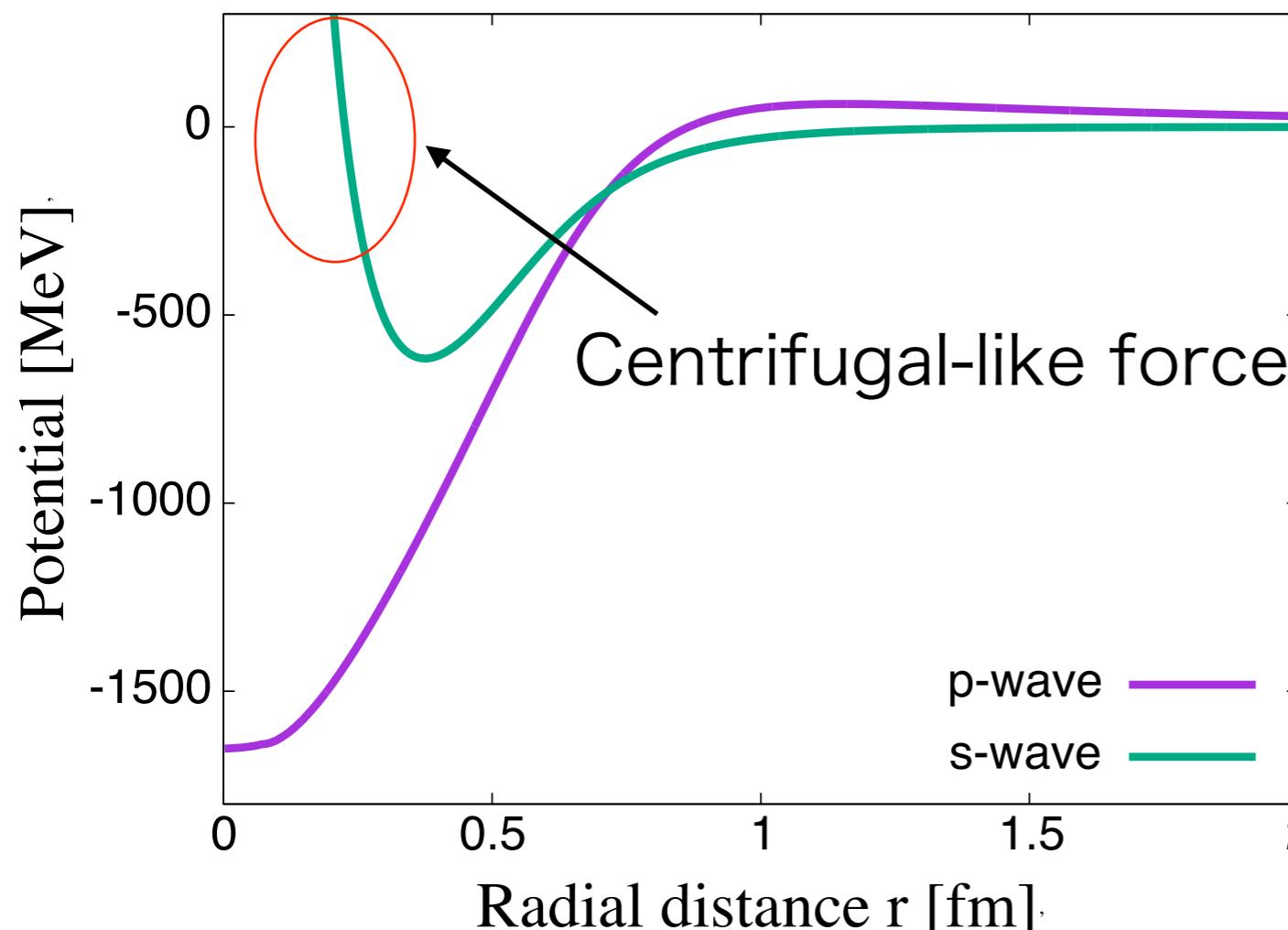
- Anti-kaon hedgehog potential**

$$l_{eff} (l_{eff} + 1) = l(l + 1) + 4\mathbf{I} \cdot \mathbf{L} + 2$$

C.G. Callan and I. Klebanov,
Nucl. Phys. B **262** (1985)

Parameters:

$$F_\pi = 186 \text{ MeV}, e = 4.82$$

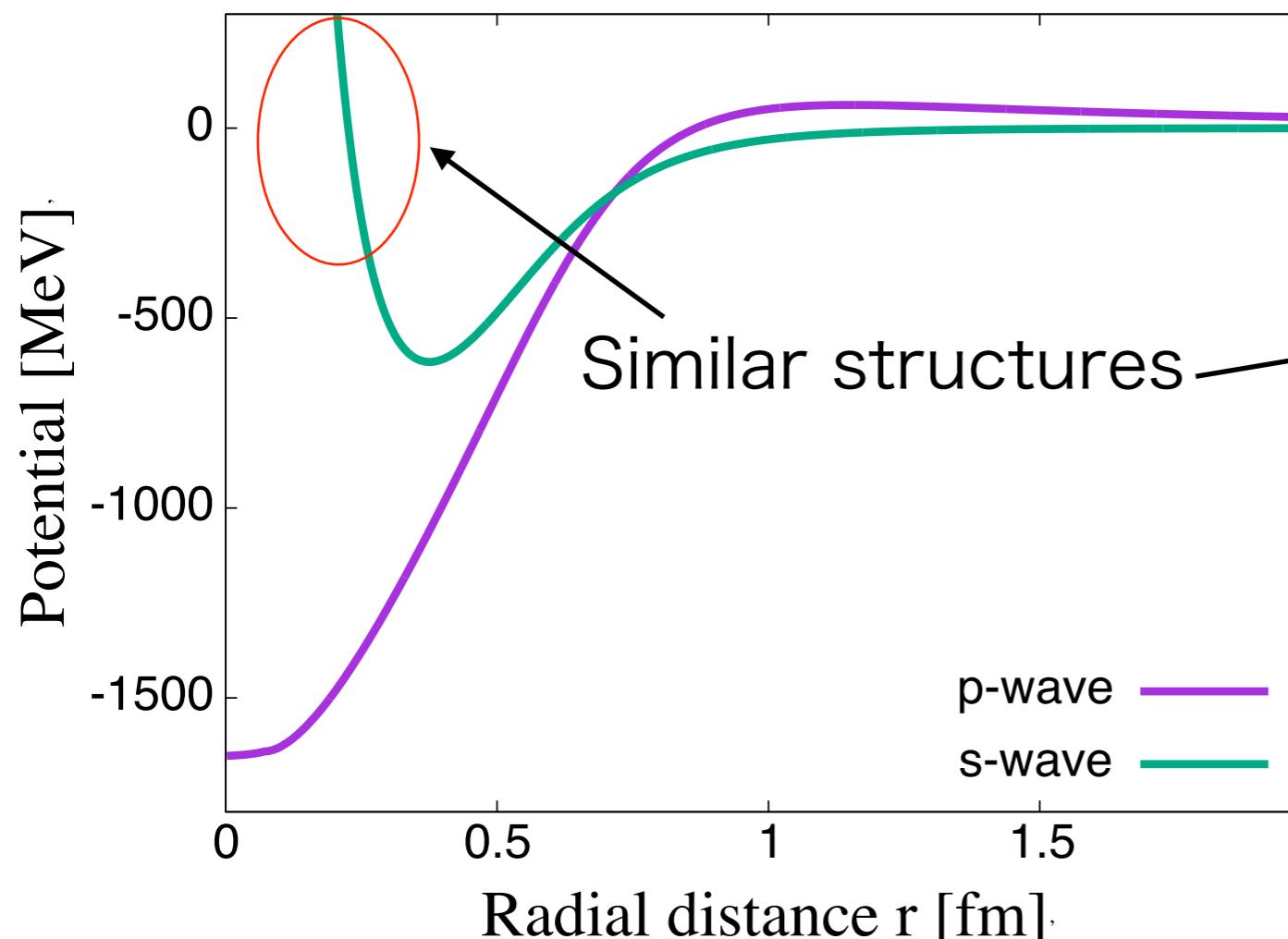


Comparisons with the CK approach

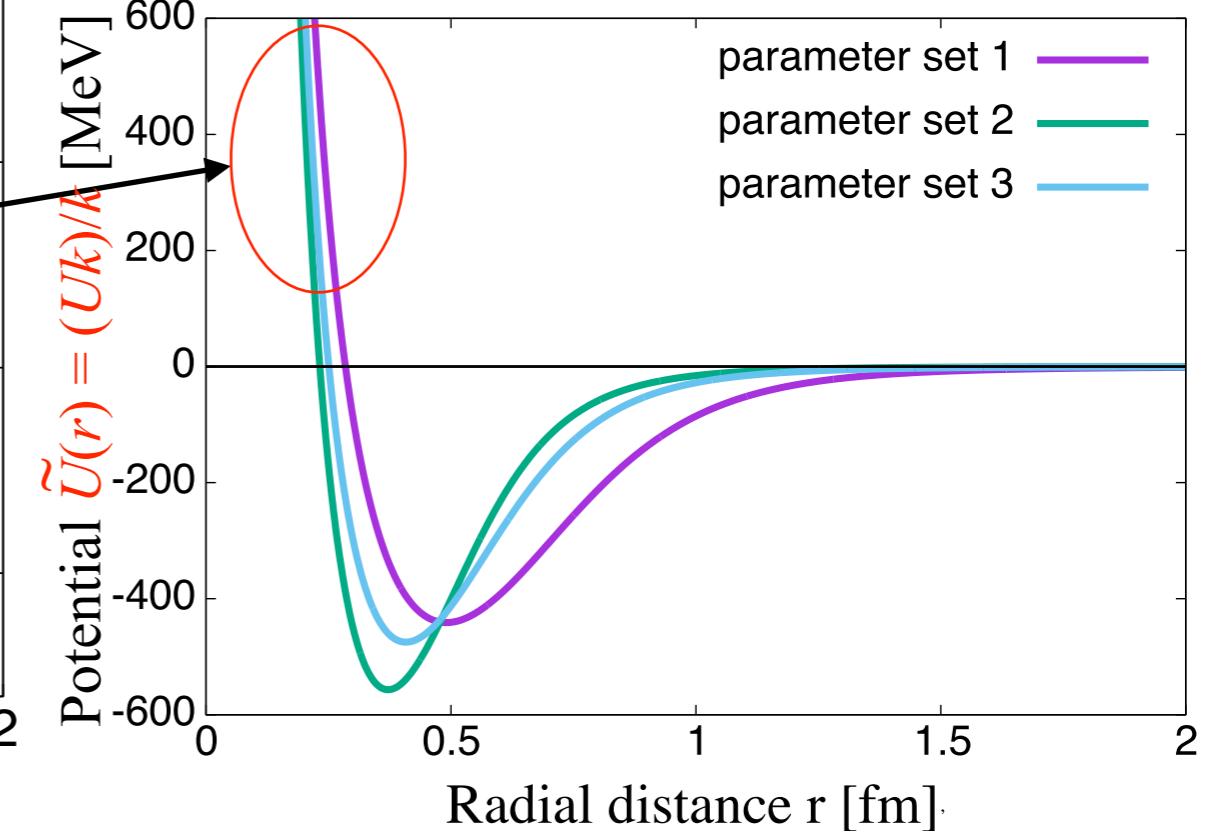
- Comparisons between the CK and our approach

Callan-Klebanov approach				Our approach			Physical state
l	l_{eff}	B.E. [MeV]	$\langle r_K^2 \rangle^{1/2}$ [fm]	l	B.E. [MeV]	$\langle r_K^2 \rangle^{1/2}$ [fm]	
0	1	61.7	0.93	0	32.9	1.18	$\Lambda(1405)$
1	0	326.6	0.54	—	—	—	$\Lambda(1116)$

- The CK approach



- The present approach



Results 1 E.o.M with $l = 0$

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 h(r) \frac{dk_l^\alpha(r)}{dr} \right) - E^2 f(r) k_l^\alpha(r) + (m_K^2 + V(r)) k_l^\alpha(r) = 0$$

$$-\frac{1}{m_K + E} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dk}{dr} \right) + \frac{2}{m_K + E} \frac{1}{r^2} k \simeq 0, \quad (r \sim 0)$$

$$h(r) = 1 + \frac{1}{(eF_\pi)^2} \frac{2}{r^2} \sin^2 F \quad f(r) = 1 + \frac{1}{(eF_\pi)^2} \left(\frac{2}{r^2} \sin^2 F + F'^2 \right) \quad V(r) = V_{eff}(r) \pm V_{WZ}(r)$$

$s = \sin(F/2)$

$$\begin{aligned} V_{nor}(r) &= -\frac{1}{4} \left(2 \frac{\sin^2 F}{r^2} + F'^2 \right) + \boxed{2 \frac{s^4}{r^2}} + \frac{1}{(eF_\pi)^2} \left[2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F'^2 \right) - \boxed{2 \frac{s^4}{r^2} \left(F'^2 + \frac{\sin^2 F}{r^2} \right)} \right] \\ &\quad + \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[\frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{ s^2 \sin F F' \} \right] \\ &\quad + \boxed{\frac{2E}{\Lambda} s^2 \left[1 + \frac{1}{(eF_\pi)^2} \left(F'^2 + \frac{5}{r^2} \sin^2 F \right) \right]} + \boxed{\frac{8E}{3\Lambda} s^2 I_{KN}} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[F'^2 + \frac{4}{r^2} \sin^2 F \right] I_{KN} \\ &\quad + \boxed{\frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{4}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} I_{KN} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right]} \end{aligned}$$

$$V_{WZ}(r) = \frac{3E}{(\pi F_\pi)^2} \frac{\sin^2 F}{r^2} F' \boxed{- \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F'}$$

$I_{KN} = \mathbf{I}^K \cdot \mathbf{I}^N$

Bule: dominant in $r \sim 0$

Red: $O(1/N_C)$ contributions

Results 1 potential for l = 0

$$U(r) = -\frac{1}{m_K + E} \left[\frac{h(r) - 1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{dh(r)}{dr} \frac{d}{dr} \right] - \frac{(f(r) - 1) E^2}{m_K + E}$$

$$+ \frac{V(r)}{m_K + E}$$

$$h(r) = 1 + \frac{1}{(eF_\pi)^2} \frac{2}{r^2} \sin^2 F \quad f(r) = 1 + \frac{1}{(eF_\pi)^2} \left(\frac{2}{r^2} \sin^2 F + F'^2 \right) \quad s = \sin(F/2)$$

$$V_{nor}(r) = -\frac{1}{4} \left(2 \frac{\sin^2 F}{r^2} + F'^2 \right) + 2 \frac{s^4}{r^2} - \frac{1}{(eF_\pi)^2} \left[2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F'^2 \right) - 2 \frac{s^4}{r^2} \left(F'^2 + \frac{\sin^2 F}{r^2} \right) \right]$$

$$+ \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[\frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{ s^2 \sin F F' \} \right]$$

$$+ \frac{2E}{\Lambda} s^2 \left[1 + \frac{1}{(eF_\pi)^2} \left(F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 I_{KN} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[F'^2 + \frac{4}{r^2} \sin^2 F \right] I_{KN}$$

$$+ \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{4}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} I_{KN} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right]$$

$$V_{WZ} = \frac{3E}{(\pi F_\pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F' \quad I_{KN} = \mathbf{I}^K \cdot \mathbf{I}^N$$

Results 2 potential for $\mathbf{l} \neq 0$

$$\begin{aligned}
V_{nor}(r) = & -\frac{1}{4} \left(2 \frac{\sin^2 F}{r^2} + F'^2 \right) + 2 \frac{s^4}{r^2} - \frac{1}{(eF_\pi)^2} \left[2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F'^2 \right) - 2 \frac{s^4}{r^2} \left(F'^2 + \frac{\sin^2 F}{r^2} \right) \right] \\
& + \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[\frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{ s^2 \sin FF' \} \right] \\
& + \frac{2E}{\Lambda} s^2 \left[1 + \frac{1}{(eF_\pi)^2} \left(F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 \underline{I_{KN}} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[F'^2 + \frac{4}{r^2} \sin^2 F \right] \underline{I_{KN}} \\
& + \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{4}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \underline{I_{KN}} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right] \\
& + \boxed{\left[1 + \frac{1}{(eF_\pi)^2} \left(\frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{l(l+1)}{r^2} - \left[1 + \frac{1}{(eF_\pi)^2} \left(4 \frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{16s^2}{3r^2} \underline{J_{KN} I_{KN}}} \\
& + \boxed{\frac{1}{(eF_\pi)^2} \frac{2E \sin^2 F}{\Lambda r^2} \underline{J_{KN}} - \frac{1}{(eF_\pi)^2} \frac{8}{r^2} \frac{d}{dr} (\sin FF') \underline{J_{KN} I_{KN}}}
\end{aligned}$$

$$V_{WZ}(r) = \frac{3E}{(\pi F_\pi)^2} \frac{\sin^2 F}{r^2} F' - \frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F s^2}{\Lambda r^2} F' + \boxed{\frac{3}{(\pi F_\pi)^2} \frac{\sin^2 F}{\Lambda r^2} F' \underline{J_{KN}}}$$

$$s = \sin(F/2) \quad I_{KN} = \mathbf{I}^K \cdot \mathbf{I}^N, \quad J_{KN} = \mathbf{L}^K \cdot \mathbf{J}^N$$

Red: New contributions for $\mathbf{l} \neq 0$

Results 2 potential for $|l| \neq 0$ ($r \sim 0$)

$$\begin{aligned}
V_{nor}(r) = & -\frac{1}{4} \left(2 \frac{\sin^2 F}{r^2} + F'^2 \right) + \boxed{\frac{2s^4}{r^2}} + \frac{1}{(eF_\pi)^2} \left[2 \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2F'^2 \right) \right] \\
& + \frac{1}{(eF_\pi)^2} \frac{6}{r^2} \left[\frac{s^4 \sin^2 F}{r^2} + \frac{d}{dr} \{ s^2 \sin FF' \} \right] \\
& + \frac{2E}{\Lambda} s^2 \left[1 + \frac{1}{(eF_\pi)^2} \left(F'^2 + \frac{5}{r^2} \sin^2 F \right) \right] + \frac{8E}{3\Lambda} s^2 I_{KN} + \frac{1}{(eF_\pi)^2} \frac{8Es^2}{3\Lambda} \left[F'^2 + \frac{4}{r^2} \sin^2 F \right] I_{KN} \\
& + \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{4}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} I_{KN} + \frac{3}{(eF_\pi)^2} \frac{EF' \sin F}{\Lambda} \right) \right] \\
& + \boxed{\left[1 + \frac{1}{(eF_\pi)^2} \left(\frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{l(l+1)}{r^2}} - \boxed{\left[1 + \frac{1}{(eF_\pi)^2} \left(4 \frac{\sin^2 F}{r^2} + F'^2 \right) \right] \frac{16s^2}{3r^2} J_{KN} I_{KN}} \\
& + \frac{1}{(eF_\pi)^2} \frac{2E \sin^2 F}{\Lambda r^2} J_{KN} - \frac{1}{(eF_\pi)^2} \frac{8}{r^2} \frac{d}{dr} (\sin FF') J_{KN} I_{KN}
\end{aligned}$$

Rule: dominant for $r \sim 0$

\downarrow
 $F(r \sim 0) \simeq \pi - ar$

$$\begin{aligned}
V(r) = & \boxed{\frac{2}{r^2} + \frac{a^2}{(eF_\pi)^2} \frac{4}{r^2} + \left[1 + \frac{2a^2}{(eF_\pi)^2} \right] \frac{l(l+1)}{r^2}} \xleftarrow{\text{Repulsion}} \\
& - \boxed{\left[1 + \frac{5a^2}{(eF_\pi)^2} \right] \frac{16}{3r^2} J_{KN} I_{KN} - \frac{a^2}{(eF_\pi)^2} \frac{8}{r^2} J_{KN} I_{KN}}
\end{aligned}$$

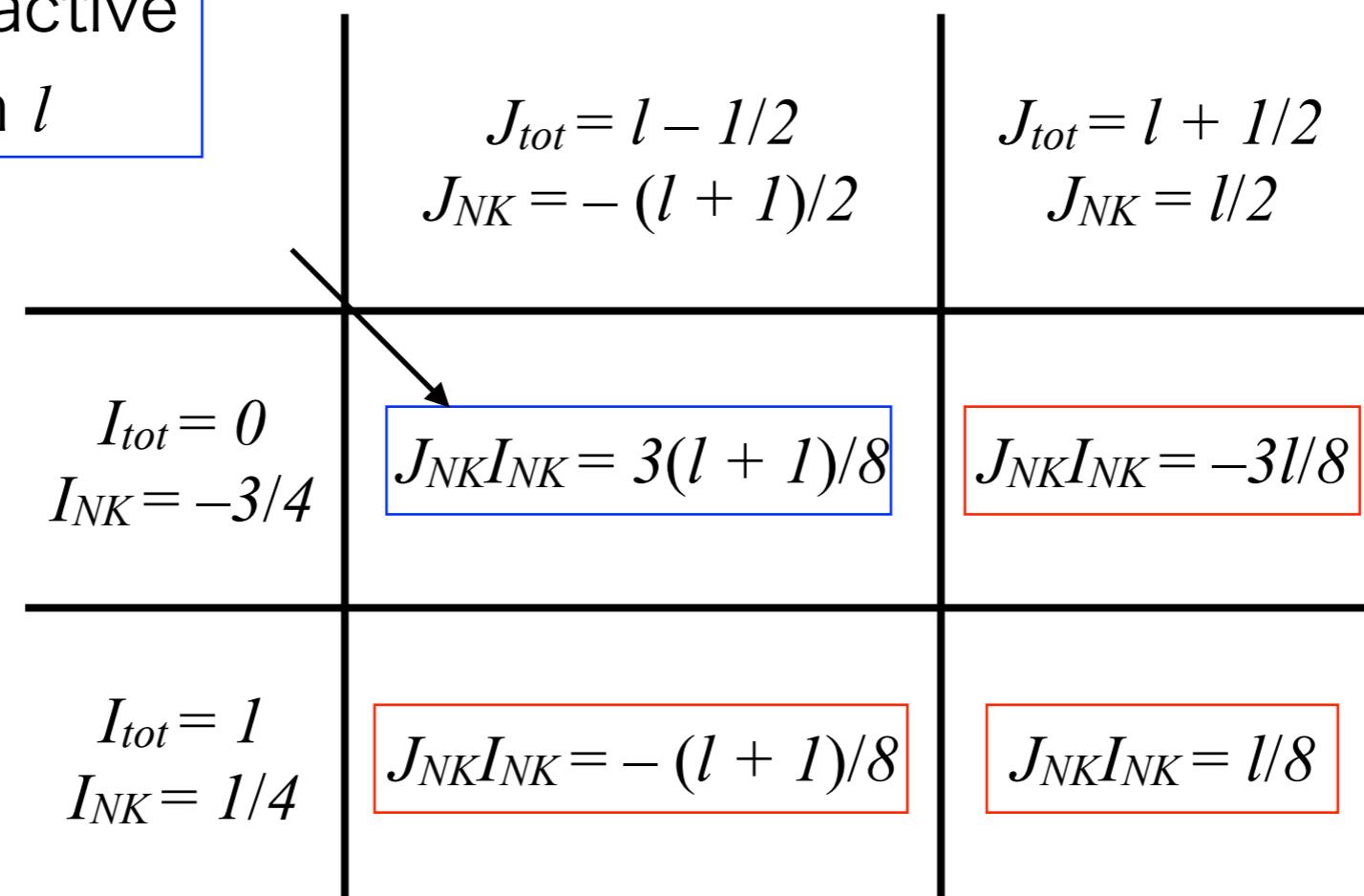
Repulsion
or
Attraction

(No contribution from WZ term)

Short range behavior of V_{eff}

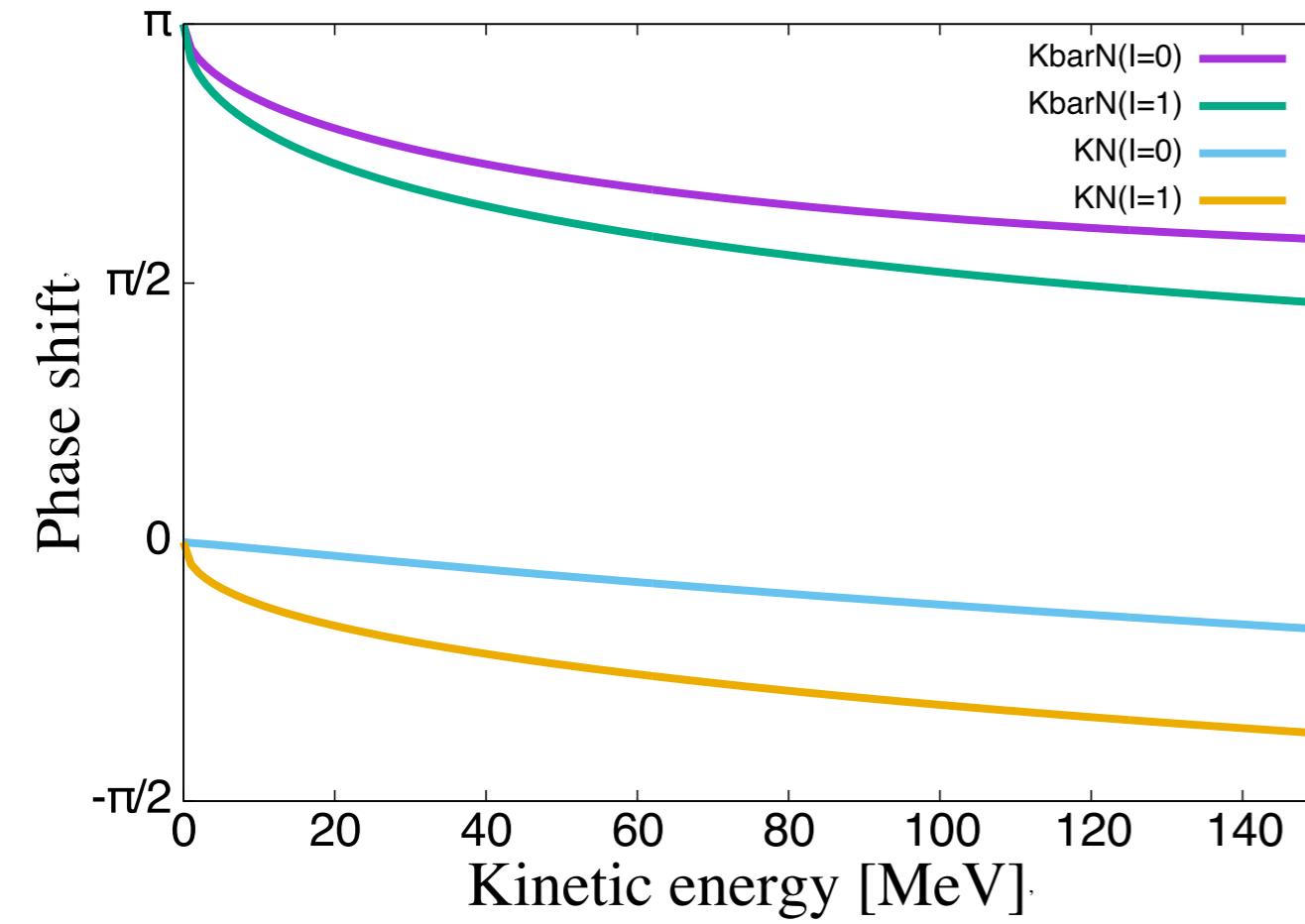
Repulsive or Attractive
depending on l

Analytically,
attraction ($1 \leq l \leq 3$)
repulsion ($l=0, 4 \leq l$)



Attractive for small r

Phase shift (parameter set 1)

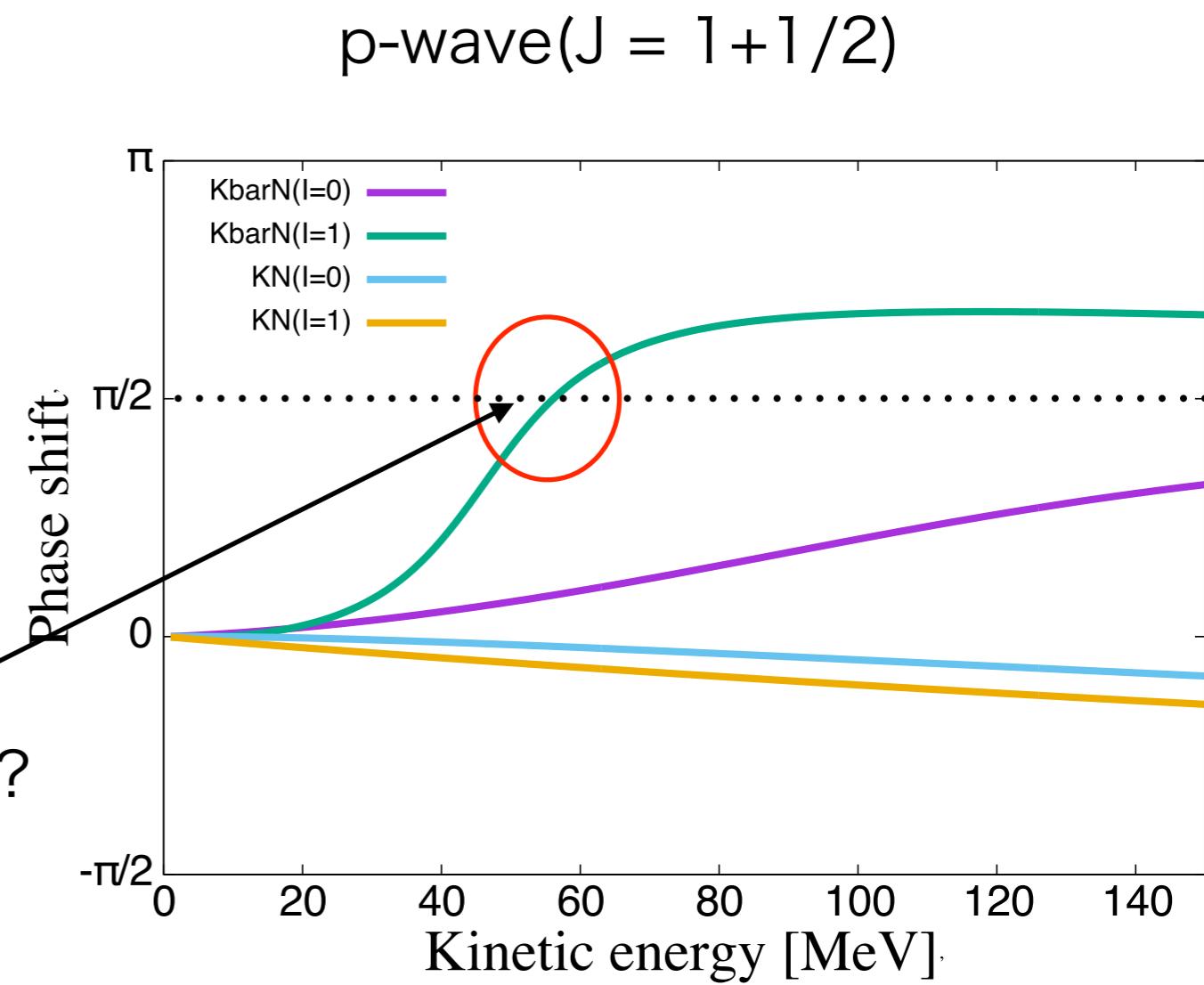


s-wave

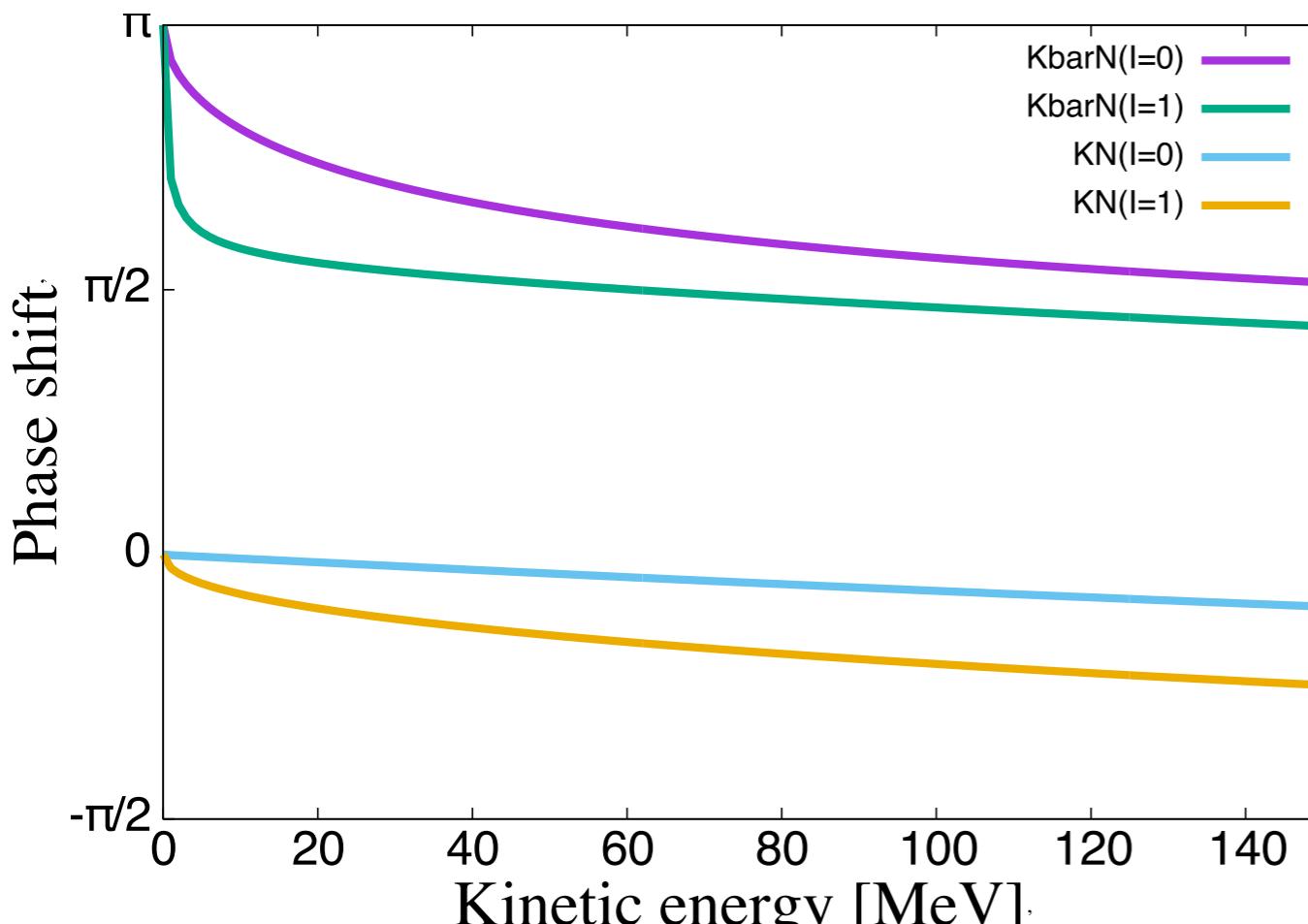
$\Sigma(1480)$ resonance?

$$I(J^P) = 1(3/2^+)$$

$$\Gamma = 46.5 \text{ MeV}$$



Phase shift (parameter set 2)



S-wave

