Two-body wave functions and compositeness from scattering amplitudes

Takayasu SEKIHARA

(Japan Atomic Energy Agency)

[1] T. S. , arXiv:1609.09496 [quant-ph].

[2] T. S. , T. Hyodo and D. Jido, *PTEP* 2015 063D04.

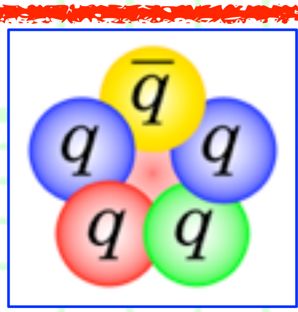
[3] T. S. , *PTEP* 2015 091D01 [Letters].

[4] T. S. , T. Arai, J. Yamagata-Sekihara and S. Yasui, *Phys. Rev.* C93 (2016) 035204.

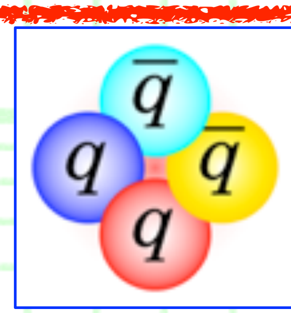
1. Introduction

++ Exotic hadrons and their structure ++

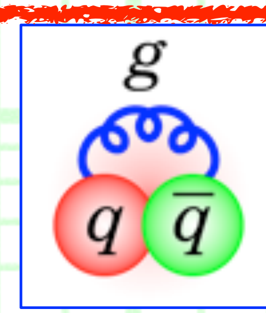
- Exotic hadrons --- not same quark component as ordinary hadrons
  = not qqq nor $q\bar{q}$.



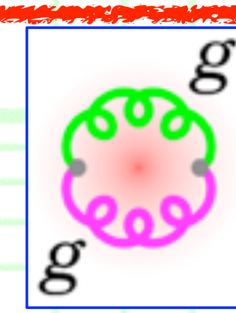
Penta-quarks



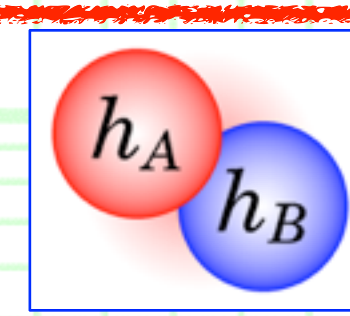
Tetra-quarks



Hybrids



Glueballs

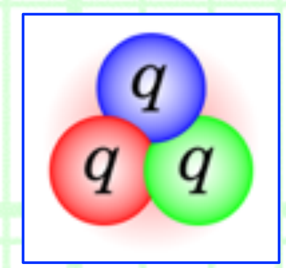


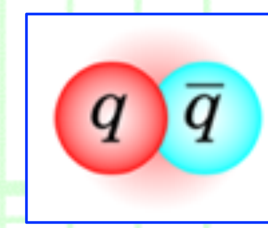
Hadronic molecules

...

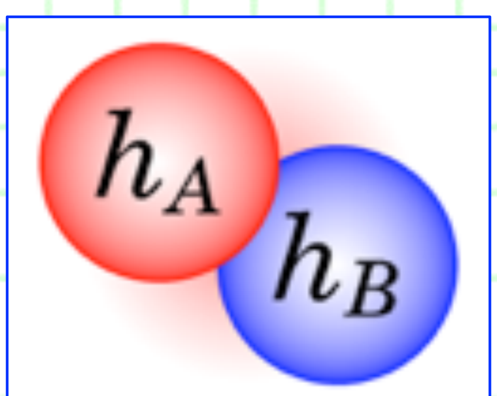
--- Actually some hadrons cannot be described by the quark model.

□ Do exotic hadrons really exist ?





Ordinary hadrons



(*cf.* deuteron)

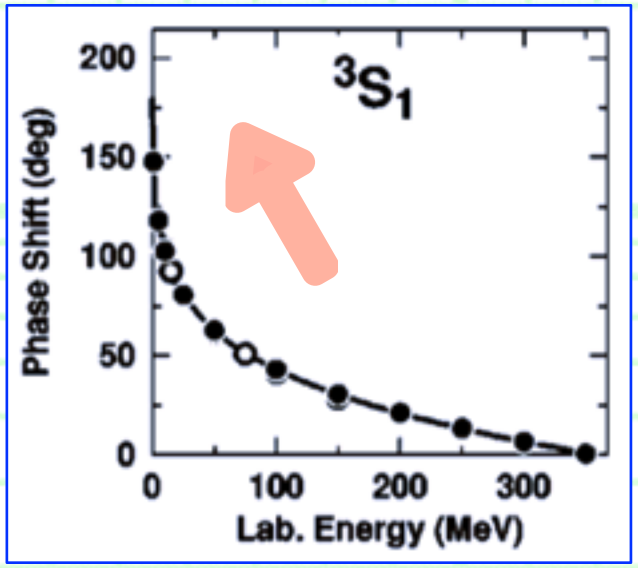
- Hadronic molecules should be unique, because they are composed of hadrons themselves, which are color singlet states.

  --> Large spatial size, compositeness, ...

1. Introduction

++ Hadronic molecules and quantum mechanics ++

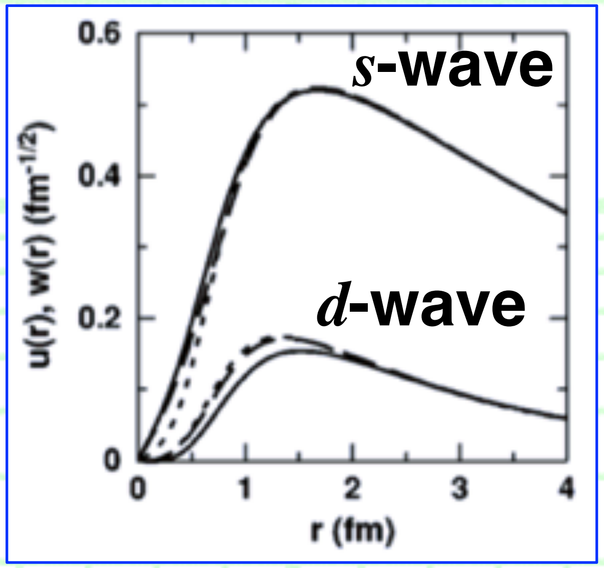
- An example of hadronic molecule: deuteron.

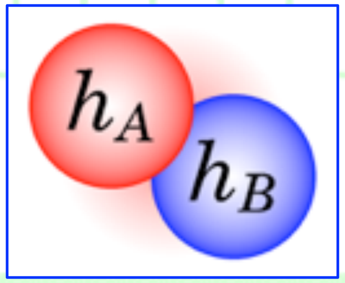


NN phase shift

Machleidt, *Phys. Rev.* C63 (2001) 024001.

Wave function of deuteron





□ Deuteron is a proton-neutron bound state. <-- Who proved this ?

--- Weinberg proved this by using general wave equations in quantum mechanics in the weak binding limit ($B_{\mathrm{E}} \ll E_{\mathrm{typical}}$).

<-- Without using QCD !

Weinberg (1965).

□ Introduce field renormalization constant Z: $\boxed{Z \equiv \langle B|B_0\rangle\langle B_0|B\rangle}$

--> "Bare" component | B_0 > in the total wave function | B >.

$$a = \frac{2(1-Z)}{2-Z}R + \mathcal{O}(m_\pi^{-1}), \quad r_e = -\frac{Z}{1-Z}R + \mathcal{O}(m_\pi^{-1}), \quad R \equiv \frac{1}{\sqrt{2\mu B}} = 4.318\ \mathrm{fm}$$

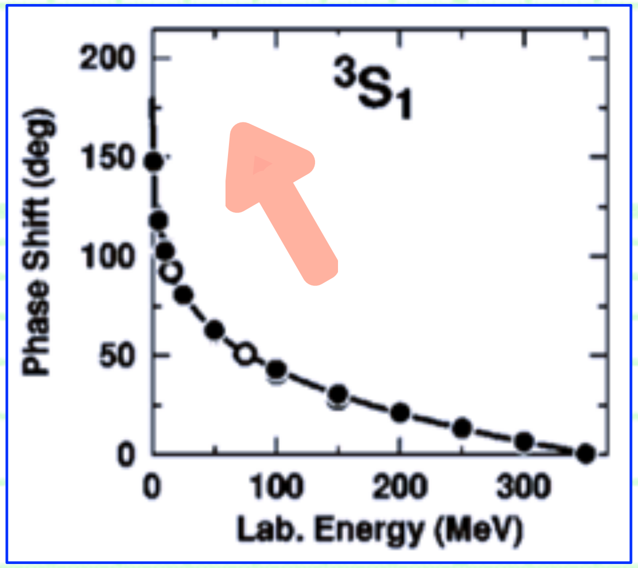
$$a = 5.419 \pm 0.007\ \mathrm{fm}, \quad r_e = 1.7513 \pm 0.008\ \mathrm{fm}$$

--> Consistent with $Z \approx 0$!

1. Introduction

++ Hadronic molecules and quantum mechanics ++

- An example of hadronic molecule: deuteron.



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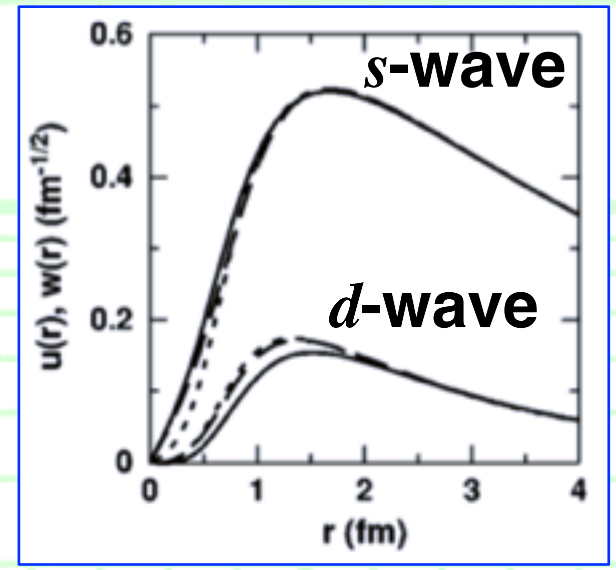
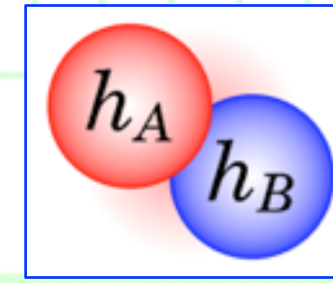




□ Lesson: In a similar manner, we may study the structure of general hadronic molecules.

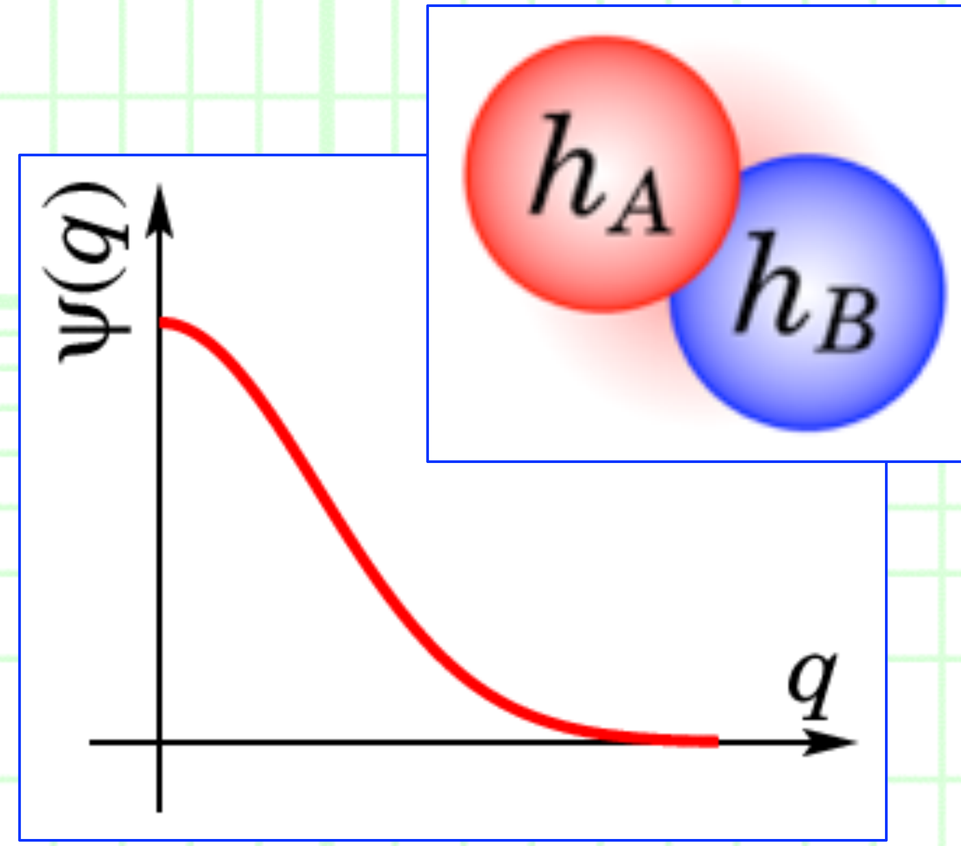
--- We can use quantum mechanics to investigate them:
Two-body wave function, its norm = compositeness,
scattering amplitude, ...

<--> In contrast, for hadrons of other configurations, we have to treat color multiplet states explicitly and appropriately.

1. Introduction

++ How to clarify their structure ? ++

- How can we use quantum mechanics to clarify the structure of hadronic molecule candidates ?



- We evaluate the wave function of hadron-hadron composite contribution.

--- *cf.* Wave function for relative motion of two nucleons inside deuteron.

- How to evaluate the wave function ?

<-- We employ a fact that the two-body wave function appears in the residue of the scattering amplitude of the two particles at the resonance pole.

  - The WF and compositeness (= norm) are automatically scaled.

$$T(E;\, \mathbf{q}', \mathbf{q}) = \langle \mathbf{q}' | \hat{T}(E) | \mathbf{q} \rangle \approx \frac{\gamma(q')\gamma(q)}{E - E_{\text{pole}}}$$

$$\langle \mathbf{q} | \Psi \rangle = \tilde{\psi}(q) = \frac{\gamma(q)}{E_{\text{pole}} - \mathcal{E}(q)}$$

$$X \equiv \int \frac{d^3 q}{(2\pi)^3} \left[\tilde{\psi}(\mathbf{q}) \right]^2$$

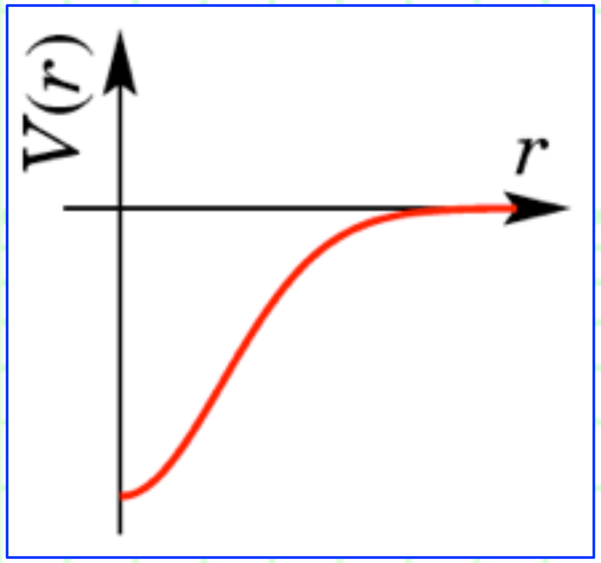
2. Wave functions from amplitudes

++ How to calculate the wave function ++

- There are **several approaches to calculate the wave function**.

Ex.) A bound state in a NR single-channel problem.

□ Usual approach: Solve the Schrödinger equation.



$$\boxed{\hat{H}|\Psi\rangle = (\hat{H}_0 + \hat{V})|\Psi\rangle = E_{\rm pole}|\Psi\rangle}$$

--- Wave function in coordinate / momentum space:

$$\boxed{\langle \mathbf{r}|\Psi\rangle = \psi(r)} \qquad \boxed{\langle \mathbf{q}|\Psi\rangle = \tilde{\psi}(q)}$$

$$\boxed{\left[M_{\rm th} - \frac{\nabla^2}{2\mu} + V(r) \right] \psi(r) = E_{\rm pole}\psi(r)}$$

--- | q > is an eigenstate of free Hamiltonian H_0:

$$\boxed{\hat{H}_0|\mathbf{q}\rangle = \mathcal{E}(q)|\mathbf{q}\rangle} \qquad \boxed{\mathcal{E}(q) = M_{\rm th} + \frac{q^2}{2\mu}}$$

--> After solving the Schrödinger equation, we have to **normalize the wave function by hand**.

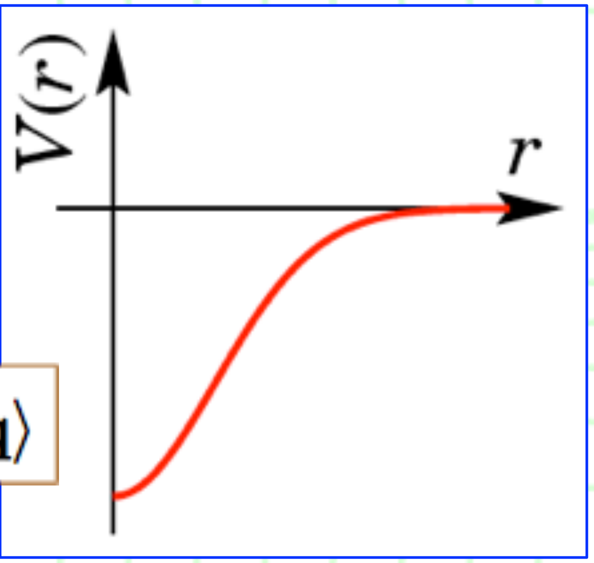
$$\boxed{\int d^3 r\, [\psi(r)]^2 = 1} \quad \text{or} \quad \boxed{\int \frac{d^3 q}{(2\pi)^3} \left[\tilde{\psi}(q) \right]^2 = 1}$$ <-- We require !

2. Wave functions from amplitudes

++ How to calculate the wave function ++

- There are **several approaches to calculate the wave function**.

Ex.) A bound state in a NR single-channel problem.



- ▫ Our approach: Solve the Lippmann-Schwinger equation at **the pole position** of the bound state.

$$\hat{T}(E) = \hat{V} + \hat{V}\frac{1}{E - \hat{H}_0}\hat{T} = \hat{V} + \hat{V}\frac{1}{E - \hat{H}}\hat{V}$$

$$T(E; \mathbf{q}', \mathbf{q}) = \langle \mathbf{q}' | \hat{T}(E) | \mathbf{q} \rangle$$

--- Near **the resonance pole position E_{pole}**, amplitude is dominated by the pole term in the expansion by the eigenstates of H as

$$\langle \mathbf{q}' | \hat{T}(E) | \mathbf{q} \rangle \approx \langle \mathbf{q}' | \hat{V} | \Psi \rangle \frac{1}{E - E_{\text{pole}}} \langle \tilde{\Psi} | \hat{V} | \mathbf{q} \rangle$$

$|\Psi\rangle, \; |\mathbf{q}_{\text{full}}\rangle, \ldots$ $\langle \tilde{\Psi}|, \; \langle \mathbf{q}_{\text{full}}|, \ldots$

$$\Rightarrow \quad \mathbb{1} = |\Psi\rangle\langle\tilde{\Psi}| + \cdots$$

--- **The residue of the amplitude at the pole position has information on the wave function !**

$$\langle \mathbf{q} | \hat{V} | \Psi \rangle = \langle \mathbf{q} | (\hat{H} - \hat{H}_0) | \Psi \rangle = [E_{\text{pole}} - \mathcal{E}(q)] \tilde{\psi}(q)$$

$$\langle \tilde{\Psi} | \hat{V} | \mathbf{q} \rangle = [E_{\text{pole}} - \mathcal{E}(q)] \tilde{\psi}(q)$$

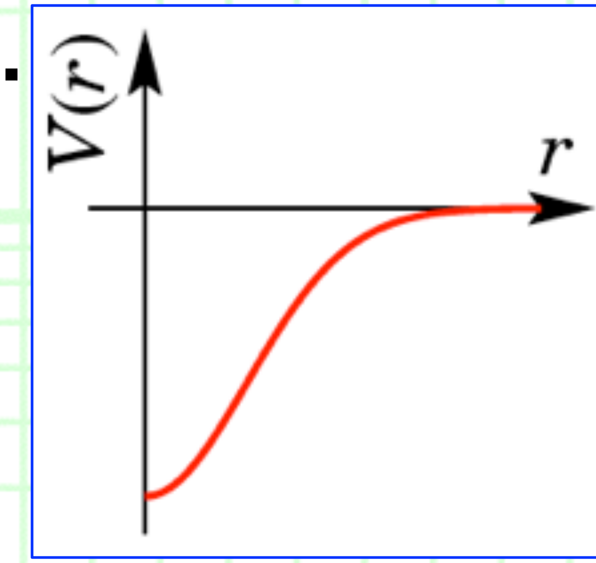
$$\mathcal{E}(q) = M_{\text{th}} + \frac{q^2}{2\mu}$$

2. Wave functions from amplitudes

++ How to calculate the wave function ++

- There are several approaches to calculate the wave function.

Ex.) A bound state in a NR single-channel problem.



□ Our approach: Solve the Lippmann-Schwinger equation at the pole position of the bound state.

--- The wave function can be extracted from the residue of the amplitude at the pole position:

$$T(E;\, \mathbf{q}', \mathbf{q}) = \langle \mathbf{q}' | \hat{T}(E) | \mathbf{q} \rangle \approx \frac{\gamma(q')\gamma(q)}{E - E_{\mathrm{pole}}}$$

<-- Off-shell Amp.

$$\gamma(q) \equiv \langle \mathbf{q} | \hat{V} | \Psi \rangle = [E_{\mathrm{pole}} - \mathcal{E}(q)] \tilde{\psi}(q)$$

--> Because the scattering amplitude cannot be freely scaled due to the optical theorem, the wave function from the residue of the amplitude is automatically scaled as well !

If purely molecule -->

$$\int \frac{d^3 q}{(2\pi)^3} \left[\frac{\gamma(q)}{E_{\mathrm{pole}} - \mathcal{E}(q)} \right]^2 = 1$$

<-- We obtain !

E. Hernandez and A. Mondragon, *Phys. Rev.* C29 (1984) 722.

--> Therefore, from hadron-hadron scattering amplitudes with resonance poles, we can calculate their two-body wave function.

2. Wave functions from amplitudes

++ Example: Stable bound state ++

- A Λ hyperon in $A \sim 40$ nucleus.

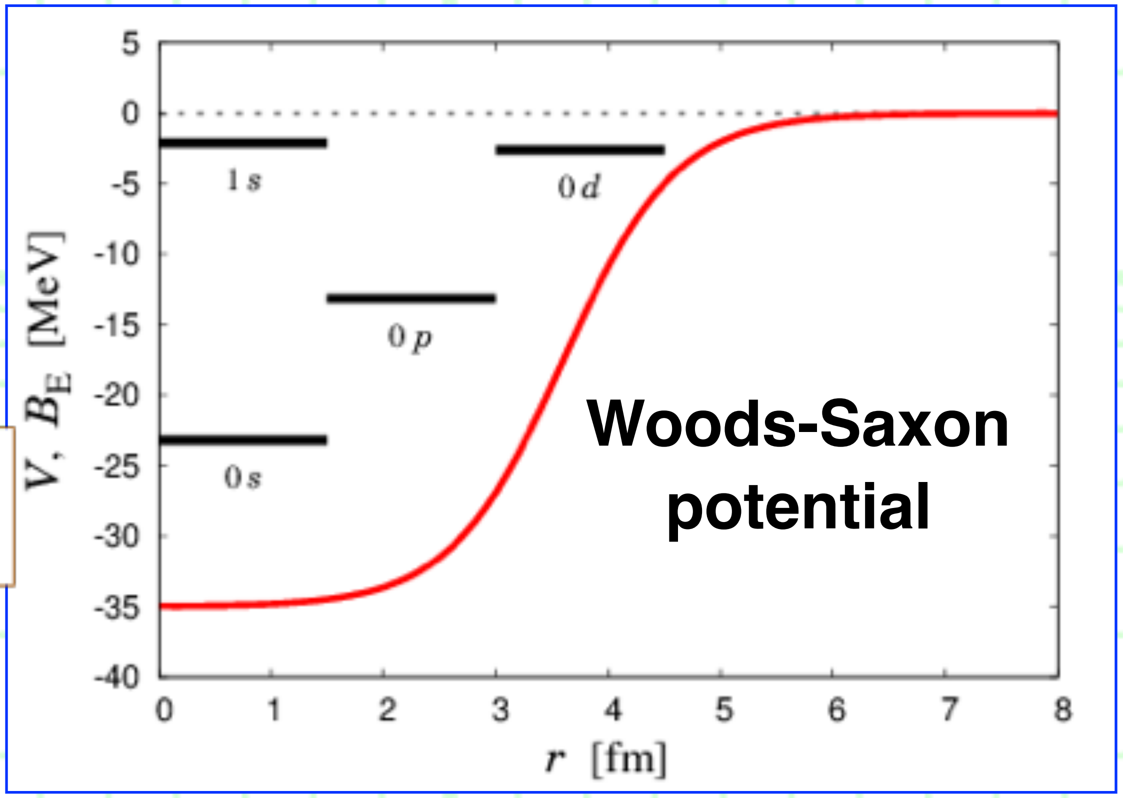
--> Calculate wave functions in 2 ways.

1. Solve Schrödinger equation:

$$\mathcal{E}(q)\tilde{\psi}(q) + \int \frac{d^3 q}{(2\pi)^3} \tilde{V}(\mathbf{q}, \mathbf{q}')\tilde{\psi}(q') = E_{\text{pole}}\tilde{\psi}(q)$$

--> Normalize ψ by hand !

$$\int \frac{d^3 q}{(2\pi)^3} \left[\tilde{\psi}(q)\right]^2 = 1$$



2. Solve Lippmann-Schwinger equation:

$$T(E; \mathbf{q}', \mathbf{q}) = V(\mathbf{q}', \mathbf{q}) + \int \frac{d^3 k}{(2\pi)^3} \frac{\tilde{V}(\mathbf{q}', \mathbf{k})T(E; \mathbf{k}, \mathbf{q})}{E - \mathcal{E}(k)}$$

--> Extract WF from the residue:

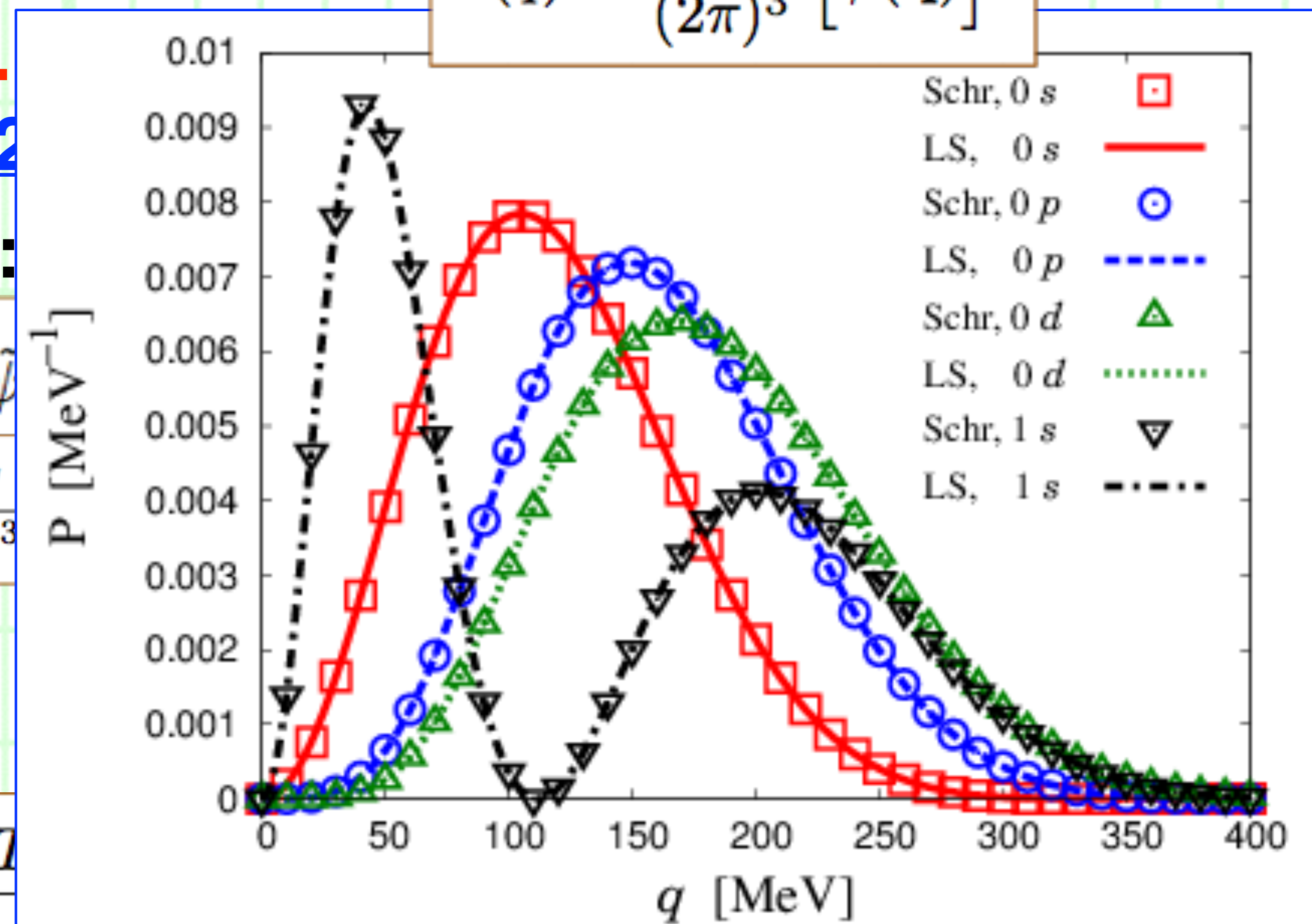
$$T(E; \mathbf{q}', \mathbf{q}) \approx \frac{\gamma(q')\gamma(q)}{E - E_{\text{pole}}} \quad \text{-->} \quad \tilde{\psi}(q) = \frac{\gamma(q)}{E_{\text{pole}} - \mathcal{E}(q)}$$

--- Without normalizing by hand !

2. Wave functions from amplitudes

++ Example: Stable boun

$$\mathrm{P}(q) = \frac{4\pi q^2}{(2\pi)^3}\left[\tilde{\psi}(\mathbf{q})\right]^2$$



- A Λ hyperon in $A \sim 40$ nucleus.

--> Calculate wave functions in 2

1. Solve Schrödinger equation:

$$\mathcal{E}(q)\tilde{\psi}(q) + \int \frac{d^3q}{(2\pi)^3}\tilde{V}(\mathbf{q},\mathbf{q}')\tilde{\psi}(q') = E_{\mathrm{pole}}\tilde{\psi}$$

--> Normalize ψ by hand !

$$\int \frac{d^3q}{(2\pi)^3}$$

2. Solve Lippmann-Schwinger equation:

$$T(E;\mathbf{q}',\mathbf{q}) = V(\mathbf{q}',\mathbf{q}) + \int \frac{d^3k}{(2\pi)^3}\frac{\tilde{V}(\mathbf{q}',\mathbf{k})T}{E - \mathcal{E}(k)}$$

--> Extract WF from the residue:

$$T(E;\mathbf{q}',\mathbf{q}) \approx \frac{\gamma(q')\gamma(q)}{E - E_{\mathrm{pole}}}$$ --> $$\tilde{\psi}(q) = \frac{\gamma(q)}{E_{\mathrm{pole}} - \mathcal{E}(q)}$$

--- Without normalizing by hand !

- In 1st way: Points.
  2nd way: Lines.

- Exact coincidence !

--- We obtain auto-matically normalized WF from the Amp. !

2. Wave functions from amplitudes

++ Example: Stable bound state ++

- We define **the compositeness X as the norm of the wave function**:

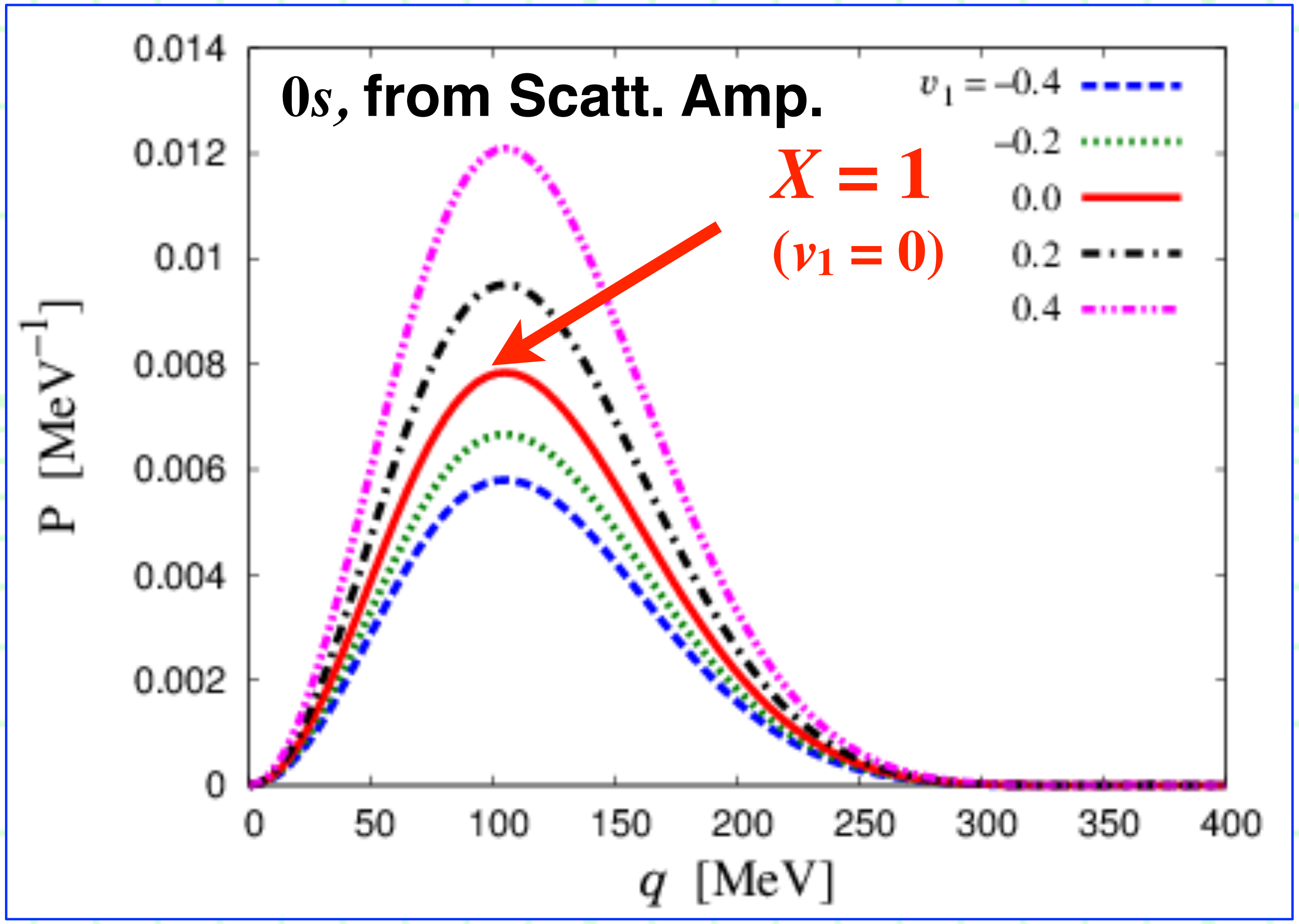
$$X \equiv \int \frac{d^3q}{(2\pi)^3} \langle \tilde{\Psi} | \mathbf{q} \rangle \langle \mathbf{q} | \Psi \rangle = \int_0^\infty dq \, \mathrm{P}(q)$$

$$\mathrm{P}(q) = \frac{4\pi q^2}{(2\pi)^3} \left[\frac{\gamma(q)}{E_{\mathrm{pole}} - \mathcal{E}(q)} \right]^2$$

--- In the following, we calculate X from the scattering amplitude.

□ The compositeness is **unity for energy independent interaction**.

Hernandez and Mondragon (1984).



□ However, if the interaction depends on the energy, the compositeness from the scattering amplitude **deviates from unity**.

$$V(r;\, E) \propto [v_0 + v_1 (E - E_{\mathrm{pole}})]$$

2. Wave functions from amplitudes

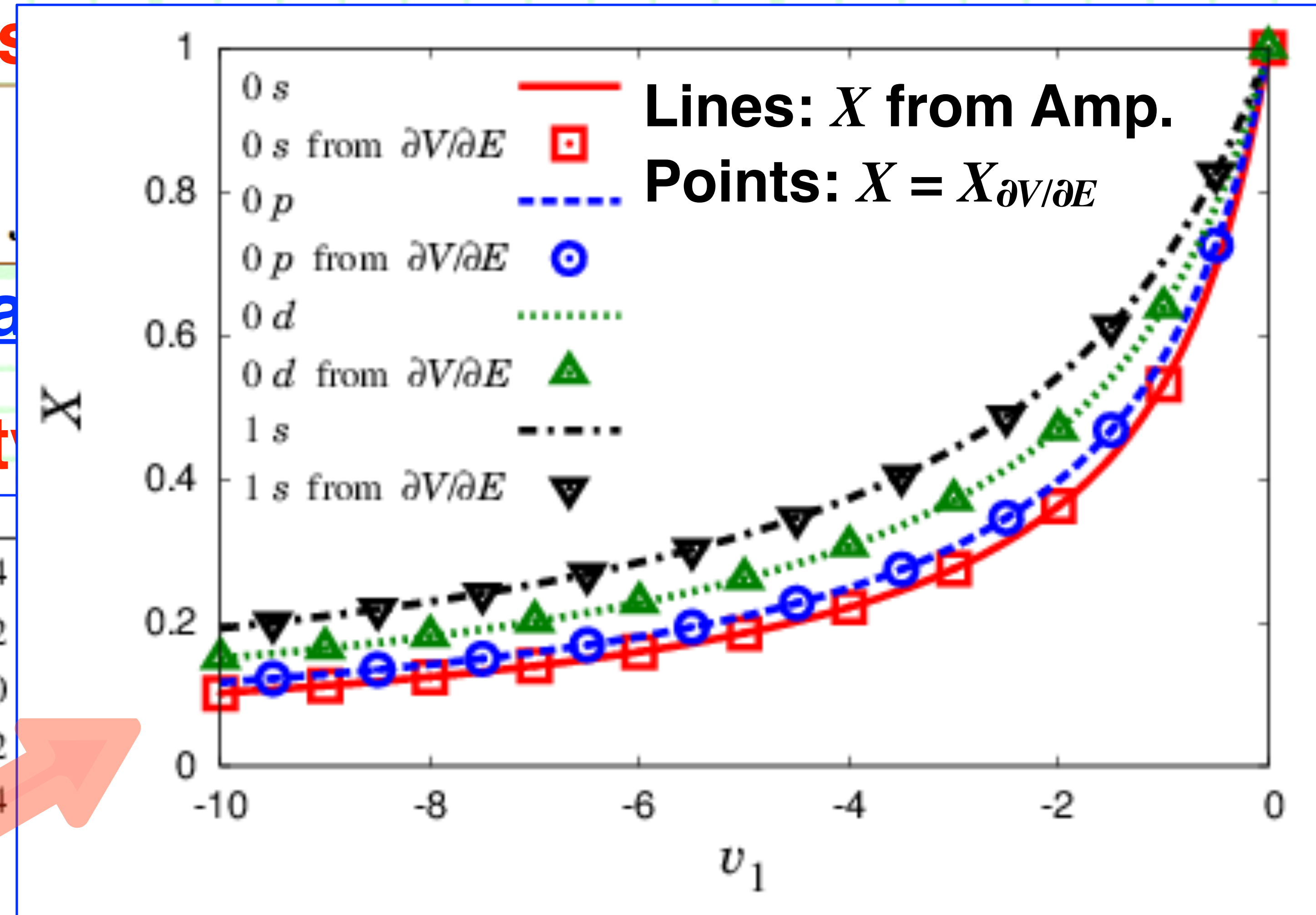
++ Example: Stable bound state ++

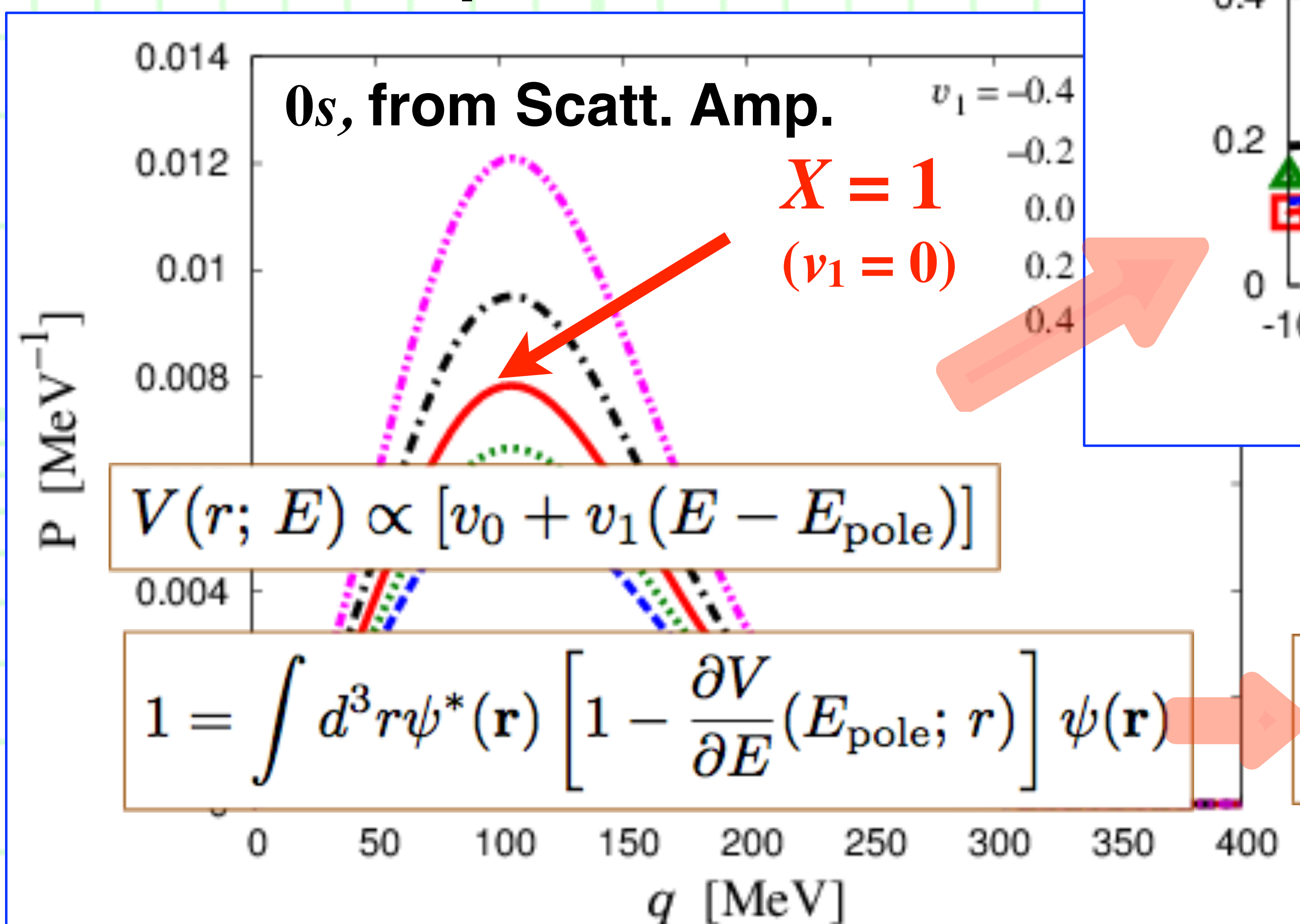
- We define **the compositenes**

$$X \equiv \int \frac{d^3q}{(2\pi)^3} \langle \tilde{\Psi} | \mathbf{q} \rangle \langle \mathbf{q} | \Psi \rangle =$$

--- In the following, we calcula

- The compositeness is unit





$$V(r;\,E) \propto [v_0 + v_1(E - E_{\text{pole}})]$$

$$1 = \int d^3 r \psi^*(\mathbf{r}) \left[1 - \frac{\partial V}{\partial E}(E_{\text{pole}};\, r) \right] \psi(\mathbf{r})$$

- **Consistent with the norm with energy-dep. interaction.**

$$X_{\partial V/\partial E} = 1 + \int d^3 r \psi^*(\mathbf{r}) \frac{\partial V}{\partial E}(E_{\text{pole}};\, r) \psi(\mathbf{r})$$

Formanek, Lombard and Mares (2004); Miyahara and Hyodo (2016).

2. Wave functions from amplitudes

++ Example: Stable bound state ++

- We define **the compositenes**

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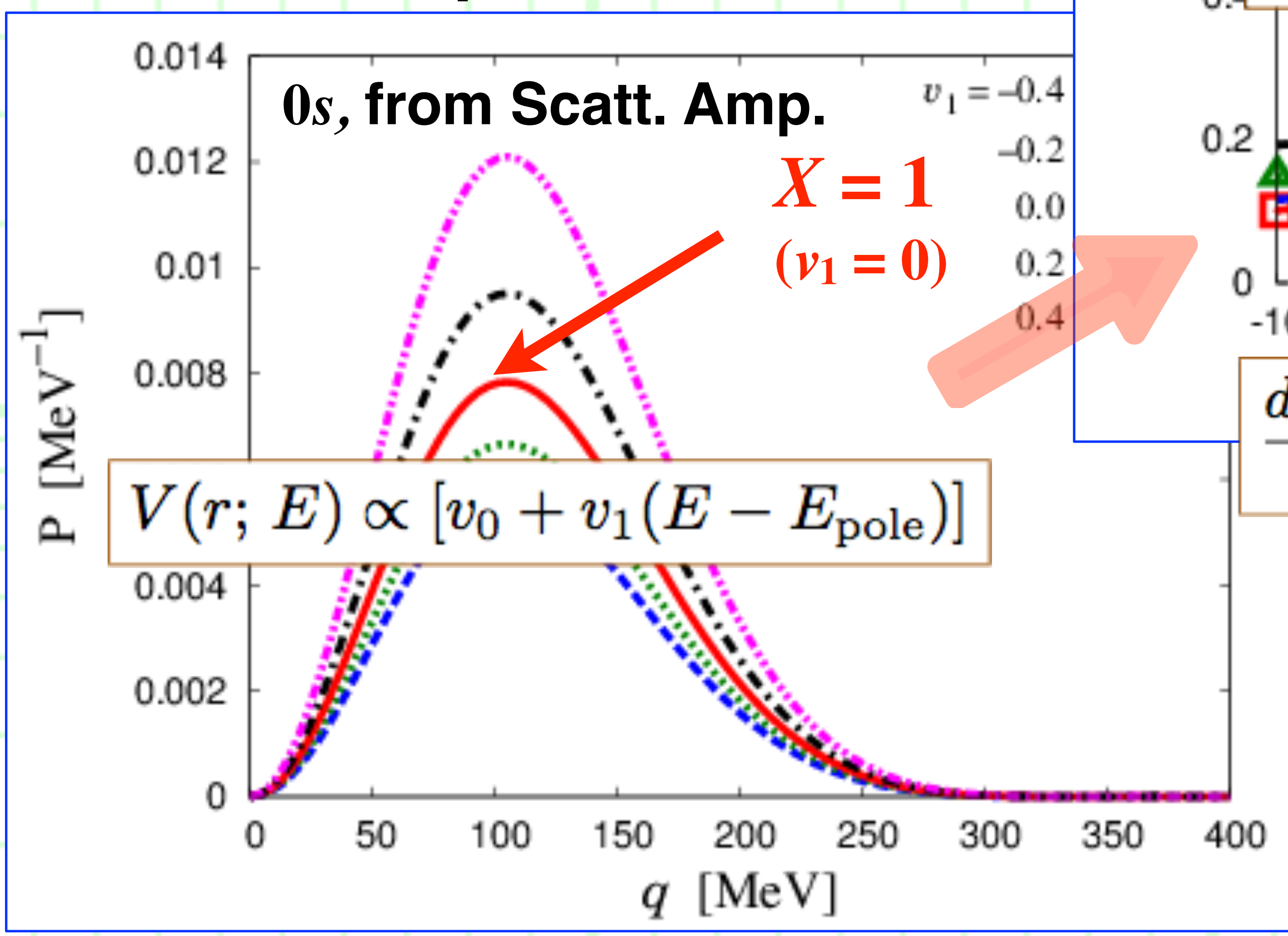
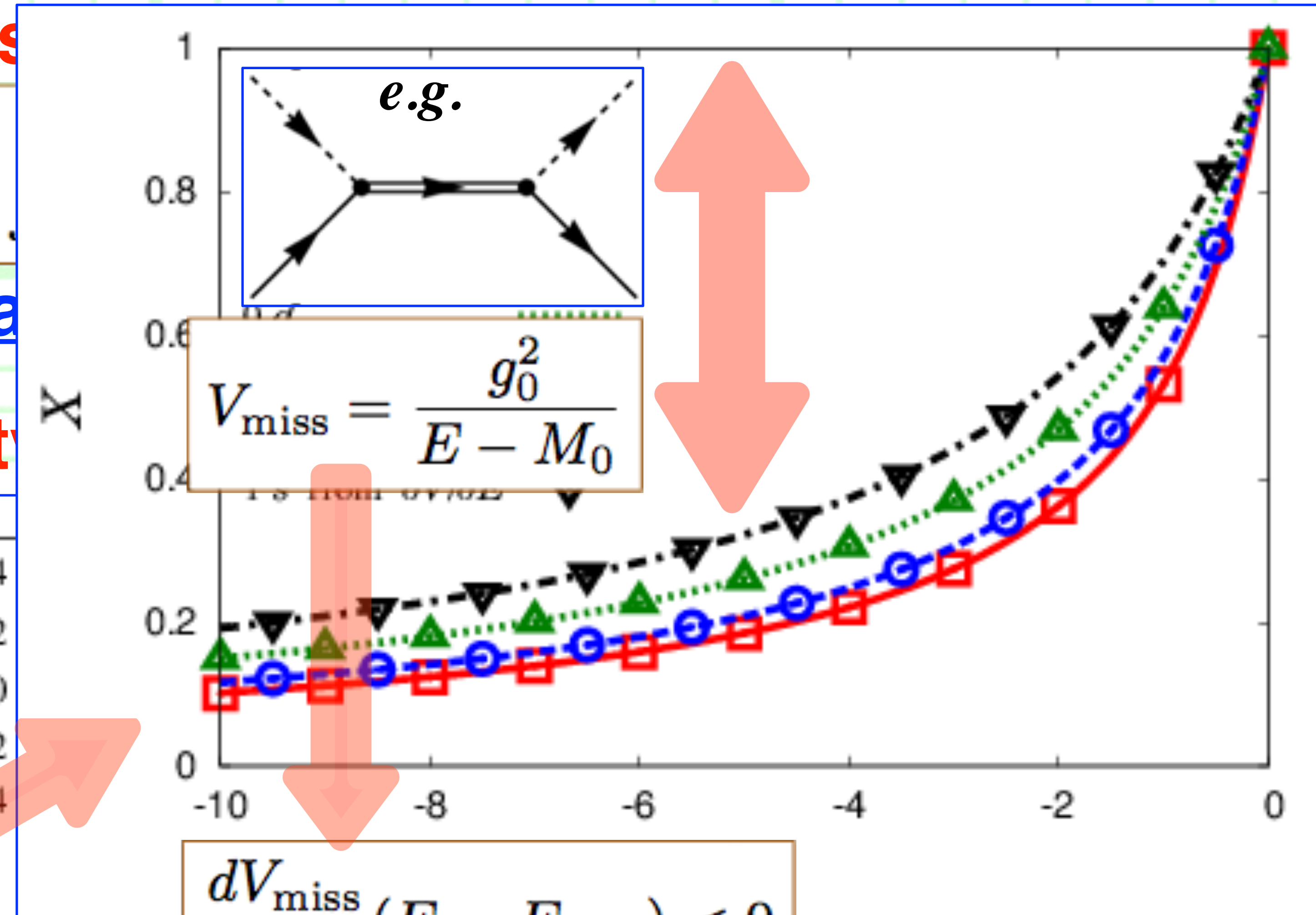


$$V(r;\, E) \propto [v_0 + v_1 (E - E_{\text{pole}})]$$



$$V_{\text{miss}} = \frac{g_0^2}{E - M_0}$$

$$\frac{dV_{\text{miss}}}{dE}(E = E_{\text{pole}}) < 0$$

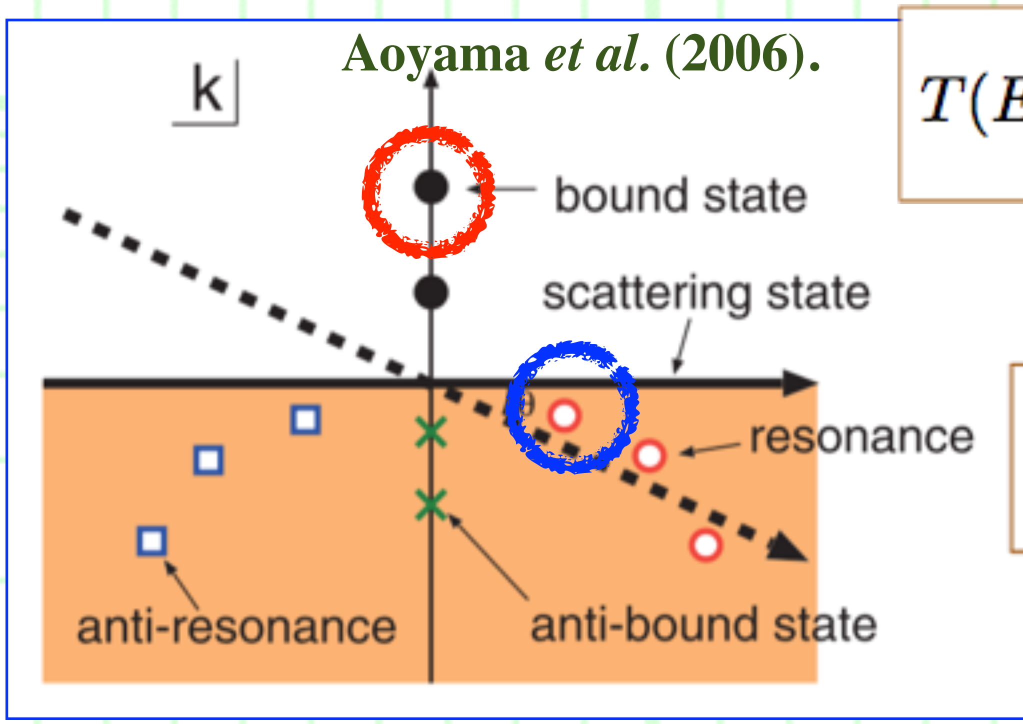
- **Deviation of compositeness from unity** can be interpreted **as a missing-channel part.**

T. S. , Hyodo and Jido, *PTEP* 2015 063D04.

2. Wave functions from amplitudes

++ Lessons from schematic models ++

- We can **extract the two-body wave function** from **the residue of the scattering amplitude at the pole position**, both stable and unstable.



Aoyama *et al.* (2006).

$$T(E; \mathbf{q}', \mathbf{q}) \approx \frac{\gamma(q')\gamma(q)}{E - E_{\mathrm{pole}}}$$

$$\tilde{\psi}(\mathbf{q}) = \frac{\gamma(\mathbf{q})}{E_{\mathrm{pole}} - E(q)}$$

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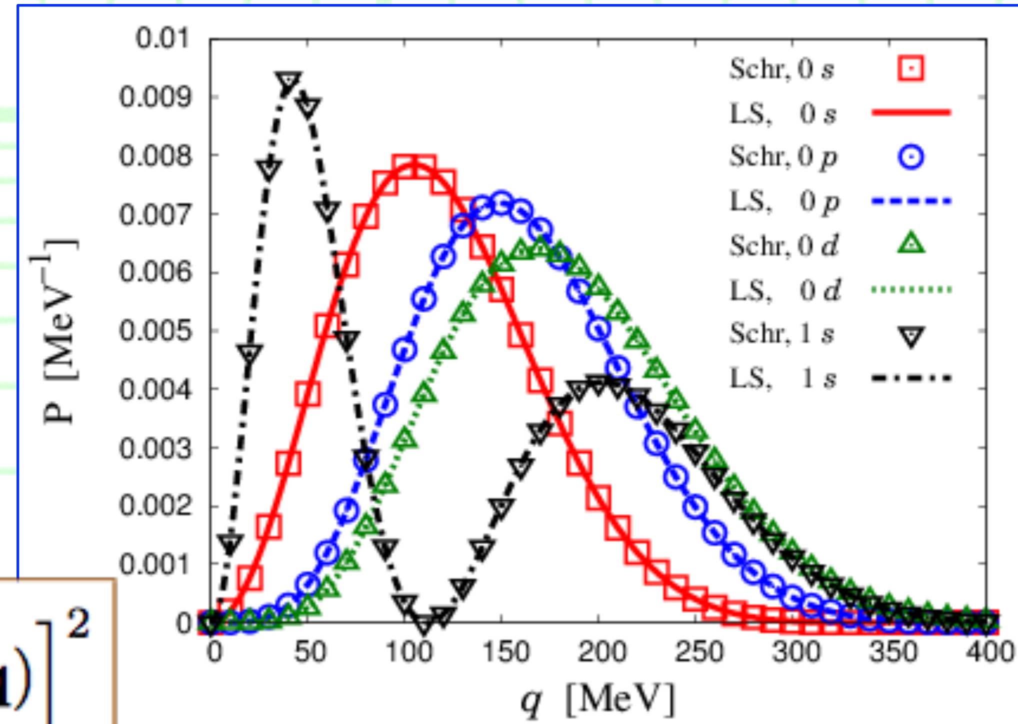


T. S., arXiv:1609.09496 [quant-ph].

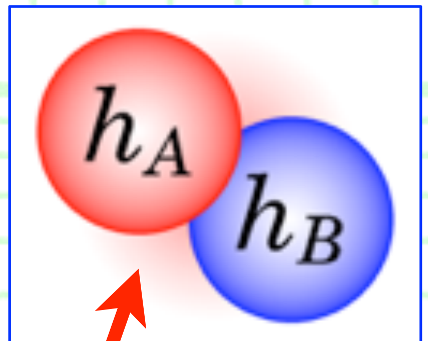
- **The WF from the scattering amplitude is automatically scaled.**
  - **The compositeness** (= norm of the two-body WF) **is unity** for a bound state in **an energy independent interaction**.
  - For an energy dependent interaction, **the compositeness deviates from unity**, reflecting **a missing channel contribution**.

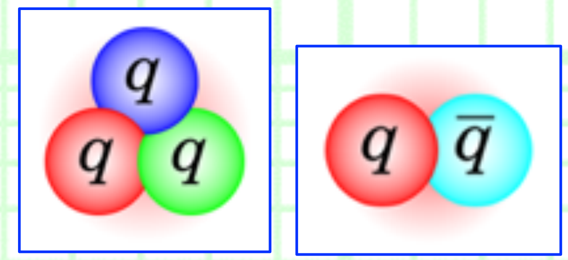
3. Compositeness of DGRs

++ Wave functions for hadrons ++

- By using **the two-body wave function and compositeness** (norm), we can **distinguish a certain configuration** of hadrons in a model.

Hadronic molecules as a bound state of hadrons (*cf.* deuteron)



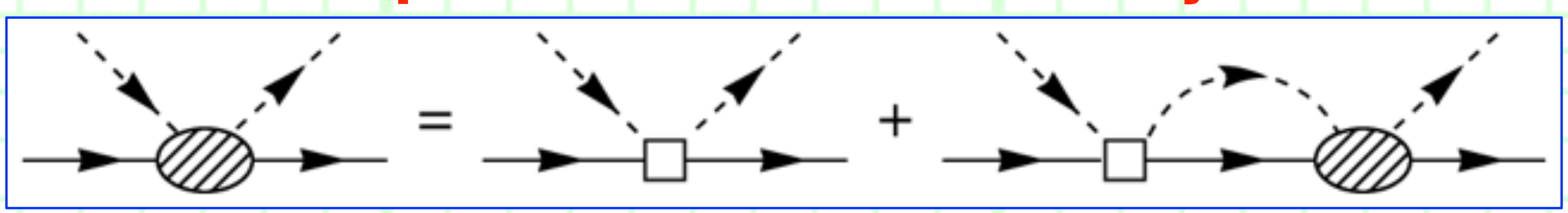


Ordinary hadrons

$$\langle \tilde{\Psi} | \Psi \rangle = X + Z = 1$$

$$X = \int \frac{d^3 q}{(2\pi)^3} \langle \tilde{\Psi} | \mathbf{q} \rangle \langle \mathbf{q} | \Psi \rangle = \int \frac{d^3 q}{(2\pi)^3} \left[\tilde{\psi}(\mathbf{q}) \right]^2$$

- Actually, several excited hadrons are **dynamically generated as resonance poles in the meson-baryon scattering amplitudes**.



Kaiser-Siegel-Weise ('95);
Oset-Ramos ('98);
...

--- In particular, **the chiral unitary approach** is successful !

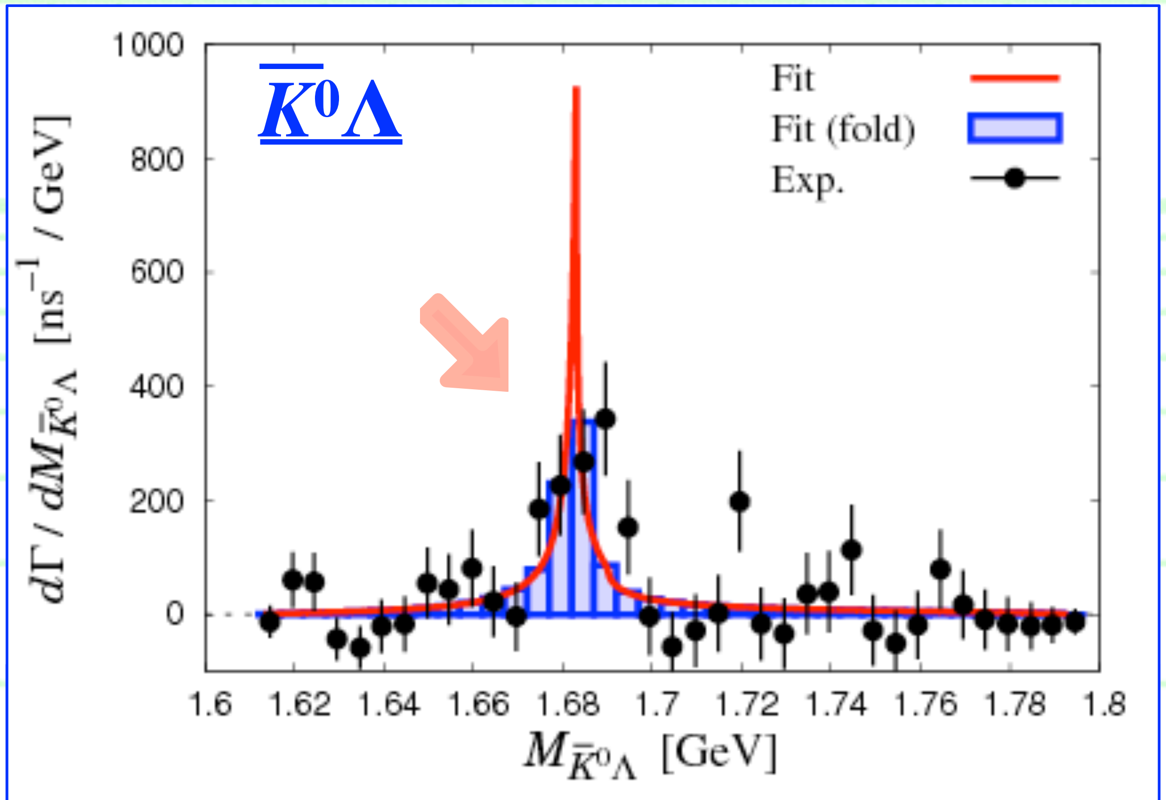
□ $\Lambda(1405)$. □ $\Xi(1690)$. □ $N(1535)$ & $N(1650)$. □ ...

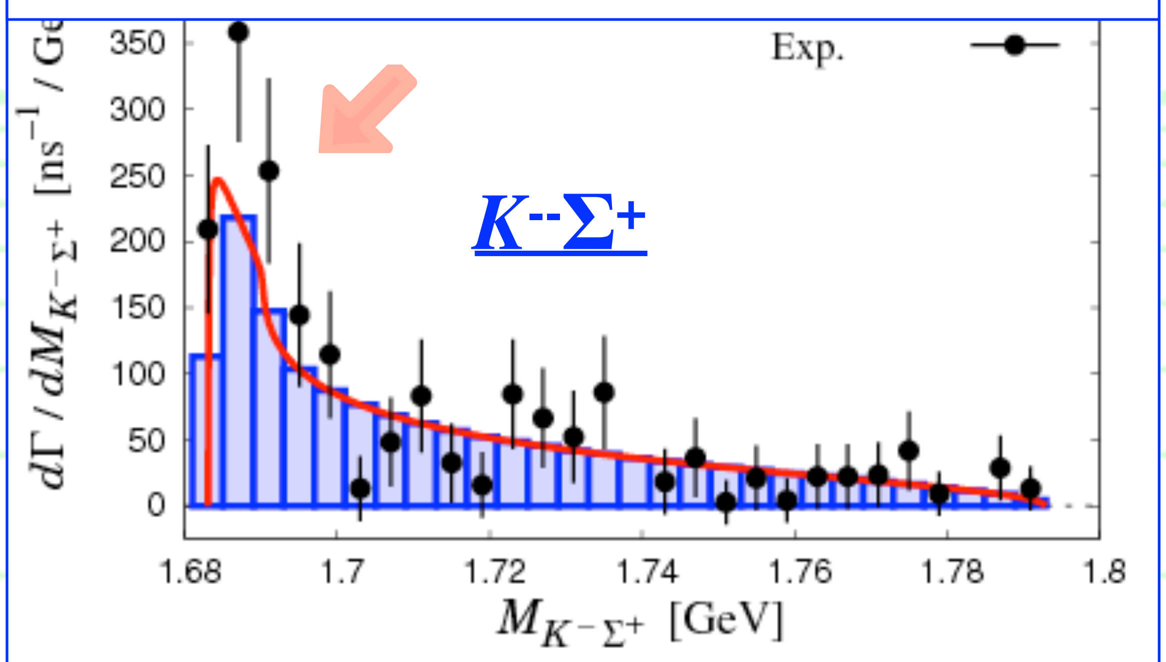
--> We can **evaluate X for these dynamically generated resonances**.

3. Compositeness of DGRs

++ Compositeness for Ξ(1690) ++

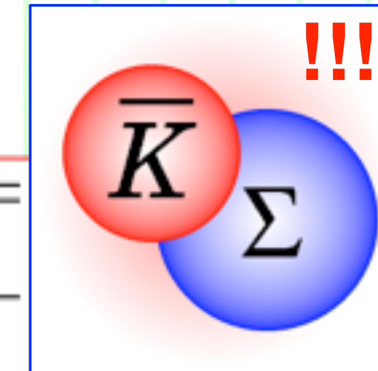
- Compositeness X for Ξ(1690) in the chiral unitary approach.





T. S. , *PTEP* 2015 091D01.

	$\Xi(1690)^0$
$\sqrt{s_{\rm pole}}$	$1684.3 - 0.5i$ MeV
$X_{K^-\Sigma^+}$	$0.83 - 0.31i$
$X_{\bar{K}^0\Sigma^0}$	$0.12 + 0.17i$
$X_{\bar{K}^0\Lambda}$	$-0.02 + 0.00i$
$X_{\pi^+\Xi^-}$	$0.00 + 0.00i$
$X_{\pi^0\Xi^0}$	$0.00 + 0.00i$
$X_{\eta\Xi^0}$	$0.01 + 0.02i$
Z	$0.06 + 0.11i$



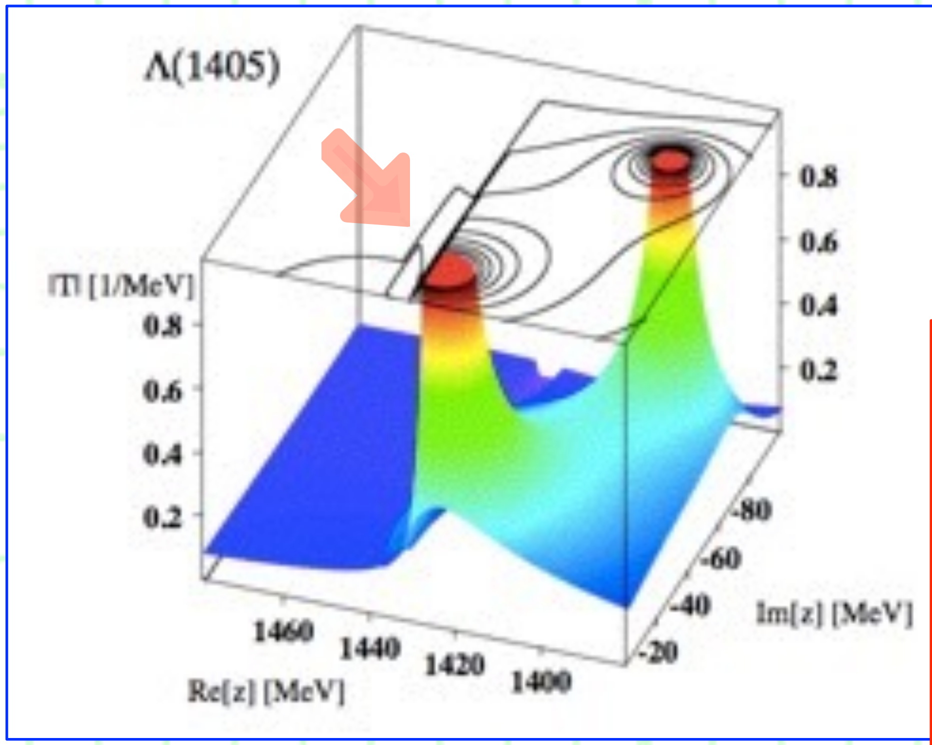
--- Large $\bar{K}\Sigma$ component for Ξ(1690), since $X_{K\Sigma}$ is almost unity with small imaginary parts.

3. Compositeness of DGRs

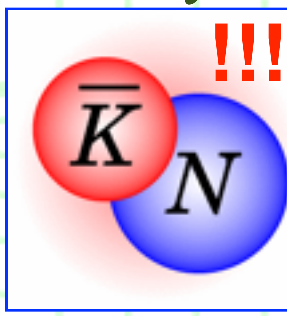
++ Compositeness for Λ(1405) ++

- **Compositeness X for Λ(1405) in the chiral unitary approach.**

Ikeda, Hyodo and Weise, *Phys. Lett.* B706, (2011) 63;
Nucl. Phys. A881 (2012) 98.



Hyodo and Jido ('12).



	Λ(1405), higher pole	Λ(1405), lower pole
$\sqrt{s_{\rm pole}}$	$1424 - 26i$ MeV	$1381 - 81i$ MeV
$X_{\bar{K}N}$	$1.14 + 0.01i$	$-0.39 - 0.07i$
$X_{\pi\Sigma}$	$-0.19 - 0.22i$	$0.66 + 0.52i$
$X_{\eta\Lambda}$	$0.13 + 0.02i$	$-0.04 + 0.01i$
$X_{K\Xi}$	$0.00 + 0.00i$	$-0.00 + 0.00i$
Z	$-0.08 + 0.19i$	$0.77 - 0.46i$

T. S. , Hyodo and Jido, *PTEP* 2015, 063D04.

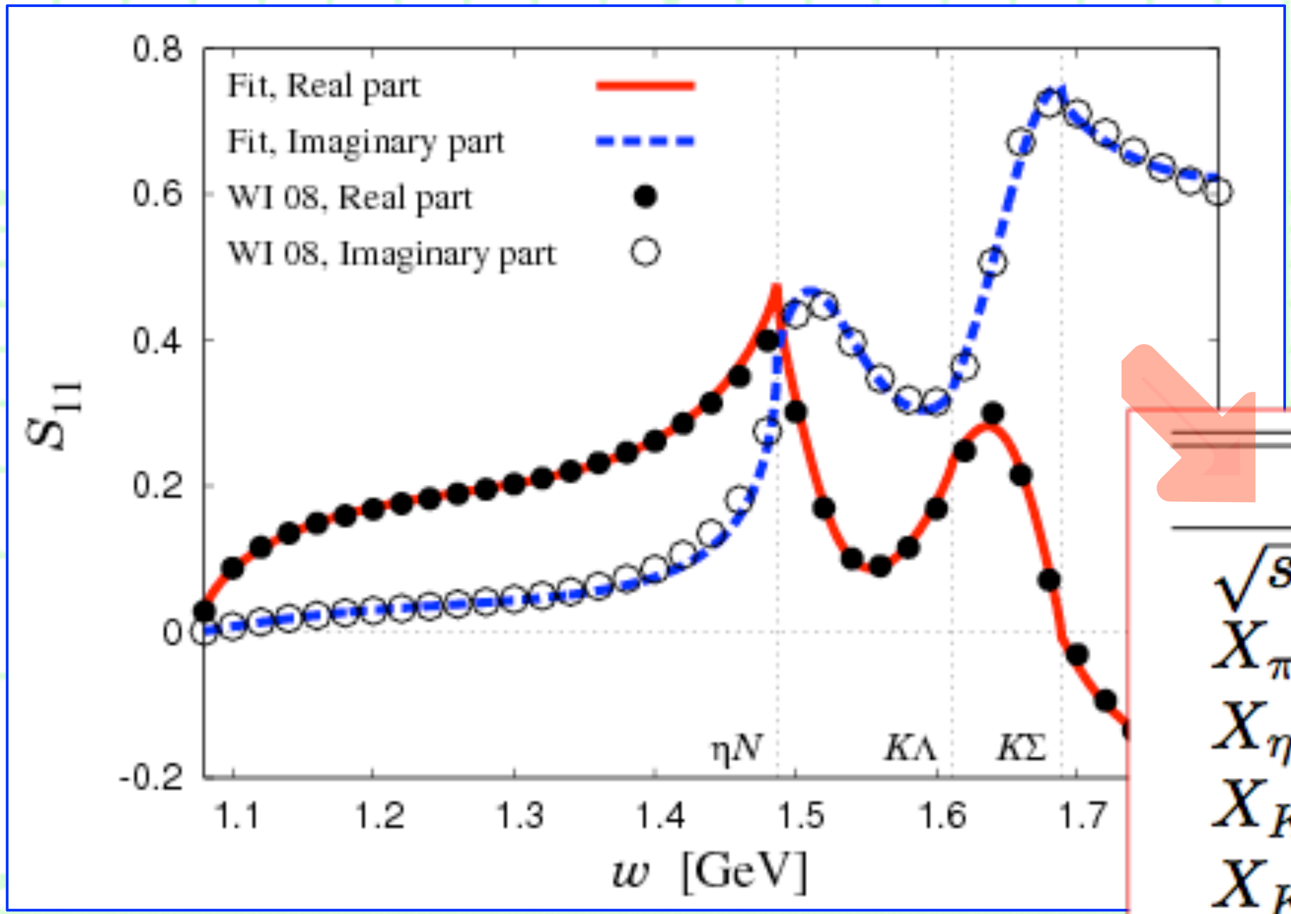
--- **Large $\bar{K}N$ component for (higher pole) Λ(1405), since $X_{\bar{K}N}$ is almost unity with small imaginary parts.**

3. Compositeness of DGRs

++ Compositeness for $N(1535)$ and $N(1650)$ ++

- **Compositeness X for $N(1535)$ & $N(1650)$ in chiral unitary approach.**

T. S., T. Arai, J. Yamagata-Sekihara and S. Yasui,
Phys. Rev. C93 (2016) 035204.



	$N(1535)$	$N(1650)$
$\sqrt{s_{\rm pole}}$ [MeV]	$1496.4 - 58.7i$	$1660.7 - 70.0i$
$X_{\pi N}$	$-0.02 + 0.03i$	$0.00 + 0.04i$
$X_{\eta N}$	$0.04 + 0.37i$	$0.00 + 0.01i$
$X_{K\Lambda}$	$0.14 + 0.00i$	$0.08 + 0.05i$
$X_{K\Sigma}$	$0.01 - 0.02i$	$0.09 - 0.12i$
Z	$0.84 - 0.38i$	$0.84 + 0.01i$

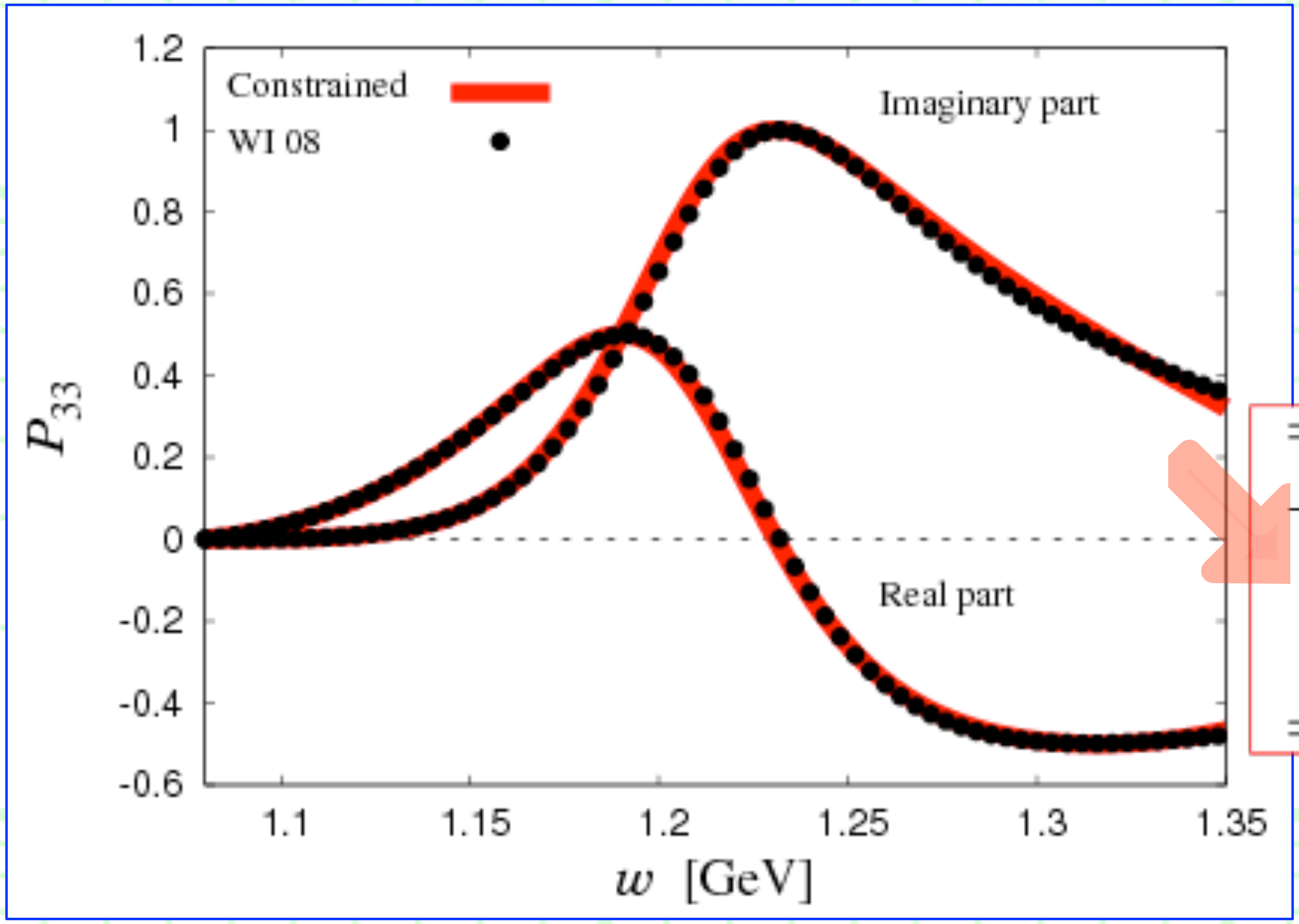
- **For both N^* resonances, the missing-channel part Z is dominant.**

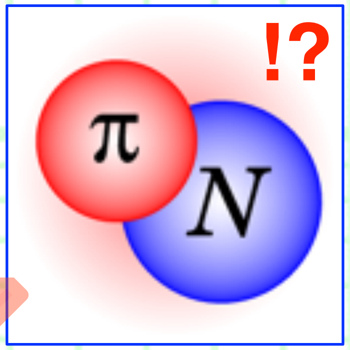
--> $N(1535)$ and $N(1650)$ have large components originating from contributions other than πN, ηN, $K\Lambda$, and $K\Sigma$.

3. Compositeness of DGRs

++ Compositeness for Δ(1232) ++

- **Compositeness X for Δ(1232) in chiral unitary approach.**





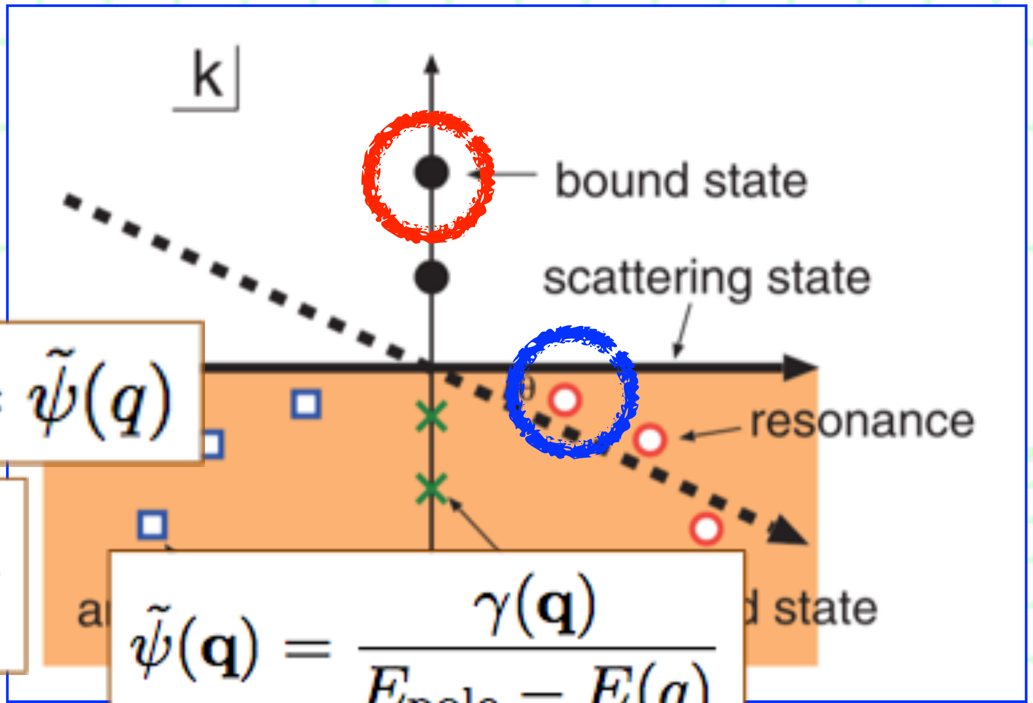
Constrained	Δ(1232)	$N(940)$
$\sqrt{s_{\rm pole}}$ [MeV]	$1206.9 - 49.6i$	938.9
$X_{\pi N}$	$0.87 + 0.35i$	0.00
Z	$0.13 - 0.35i$	1.00

□ **The πN compositeness $X_{\pi N}$ takes large real part !** **But non-negligible imaginary part as well.**

--> Large πN component in the Δ(1232) resonance !?

4. Summary

- We can extract the two-body WF from the residue of the scattering amplitude at the pole position, both stable and unstable states.



$$\langle \mathbf{q} | \Psi \rangle = \tilde{\psi}(q)$$

Scattering amplitude:
$$T(E; \mathbf{q}', \mathbf{q}) = \langle \mathbf{q}' | \hat{T}(E) | \mathbf{q} \rangle \approx \frac{\gamma(q')\gamma(q)}{E - E_{\text{pole}}}$$

$$\tilde{\psi}(\mathbf{q}) = \frac{\gamma(\mathbf{q})}{E_{\text{pole}} - E(q)}$$

- The WF from the scattering amplitude is automatically scaled.
  - The compositeness (= norm of the two-body WF) is unity for a bound state in an energy independent interaction.
  - For an energy dependent interaction, the compositeness deviates from unity, reflecting a missing channel contribution.

- In the chiral unitary approach, as an effective model, we evaluate the compositeness of dynamically generated resonances.
  - $\Lambda(1405)$ as $\bar{K}N$!
  - $\Xi(1690)$ as $\bar{K}\Sigma$!
  - $\Delta(1232)$ as πN !?

Thank you very much for your kind attention !

Appendix

Appendix

++ Example 2: Unstable resonance state ++

- **Unstable resonance in $\bar{K}N$-$\pi\Sigma$ system**

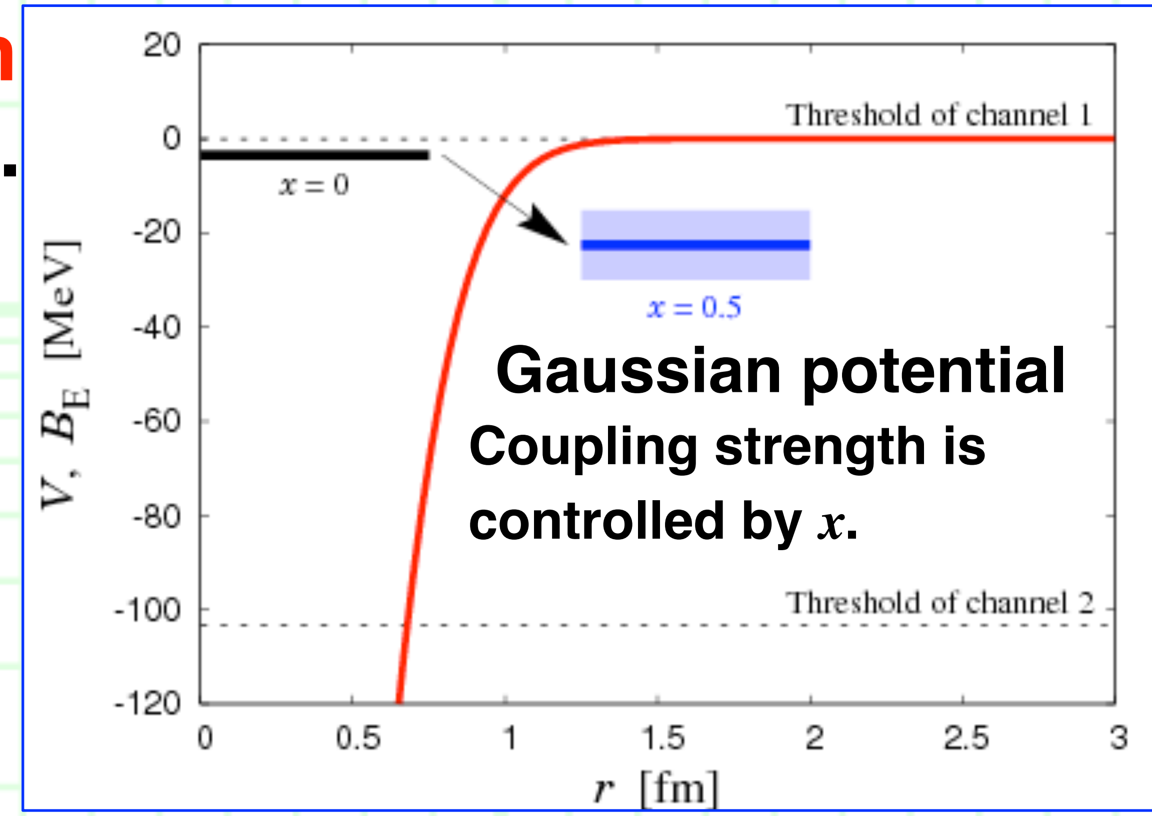
--> Calculate wave functions in 2 ways.

1. Solve Schrödinger equation:

$$E_j(q)\tilde{\psi}_j(\mathbf{q}) + \sum_k \int \frac{d^3q'}{(2\pi)^3} \tilde{V}_{jk}(\mathbf{q}, \mathbf{q}')\tilde{\psi}_k(\mathbf{q}') = E_{\rm pole}\tilde{\psi}_j(\mathbf{q})$$

--> Normalize ψ_j by hand !

$$X_1 + X_2 = 1, \quad X_j \equiv \int \frac{d^3q}{(2\pi)^3} \left[\tilde{\psi}_j(\mathbf{q})\right]^2$$



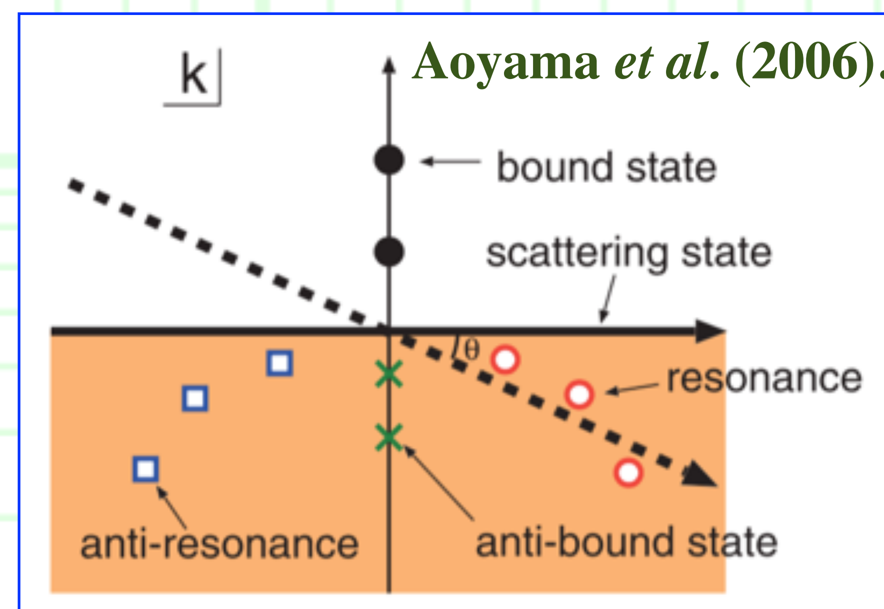
2. Solve Lippmann-Schwinger equation:

$$T_{jk}(\mathbf{q}', \mathbf{q};\, E) = \tilde{V}_{jk}(\mathbf{q}', \mathbf{q}) + \sum_l \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{V}_{jl}(\mathbf{q}', \mathbf{k})T_{lk}(\mathbf{k}, \mathbf{q};\, E)}{E - E_l(k)}$$

--> Extract WF from the residue:

$$T_{jk}(\mathbf{q}', \mathbf{q};\, E) \approx \frac{\gamma_j(\mathbf{q}')\gamma_k(\mathbf{q})}{E - E_{\rm pole}} \quad \text{-->} \quad \tilde{\psi}_j(\mathbf{q}) = \frac{\gamma_j(\mathbf{q})}{E_{\rm pole} - E_j(q)}$$

--- Without normalizing by hand !



Appendix

++ Example 2: Unstable resonan

▪ Unstable resonance in $\bar{K}N$-π

--> Calculate wave functions

1. Solve Schrödinger equati

$$E_j(q)\tilde{\psi}_j(\mathbf{q}) + \sum_k \int \frac{d^3q'}{(2\pi)^3}\tilde{V}_{jk}(\mathbf{q},\, \mathbf{q}')\tilde{\psi}_k(\mathbf{q}') =$$

--> Normalize ψ_j by hand !

$$X_1 + X_2 = 1, \quad X_j \equiv \int \frac{d^3q}{(2\pi)^3}\left[\tilde{\psi}_j(\mathbf{q})\right]^2$$

2. Solve

equati

$$T_{jk}(\mathbf{q}',\, \mathbf{q};\, E \quad \cdots \quad E - E_l(k)$$

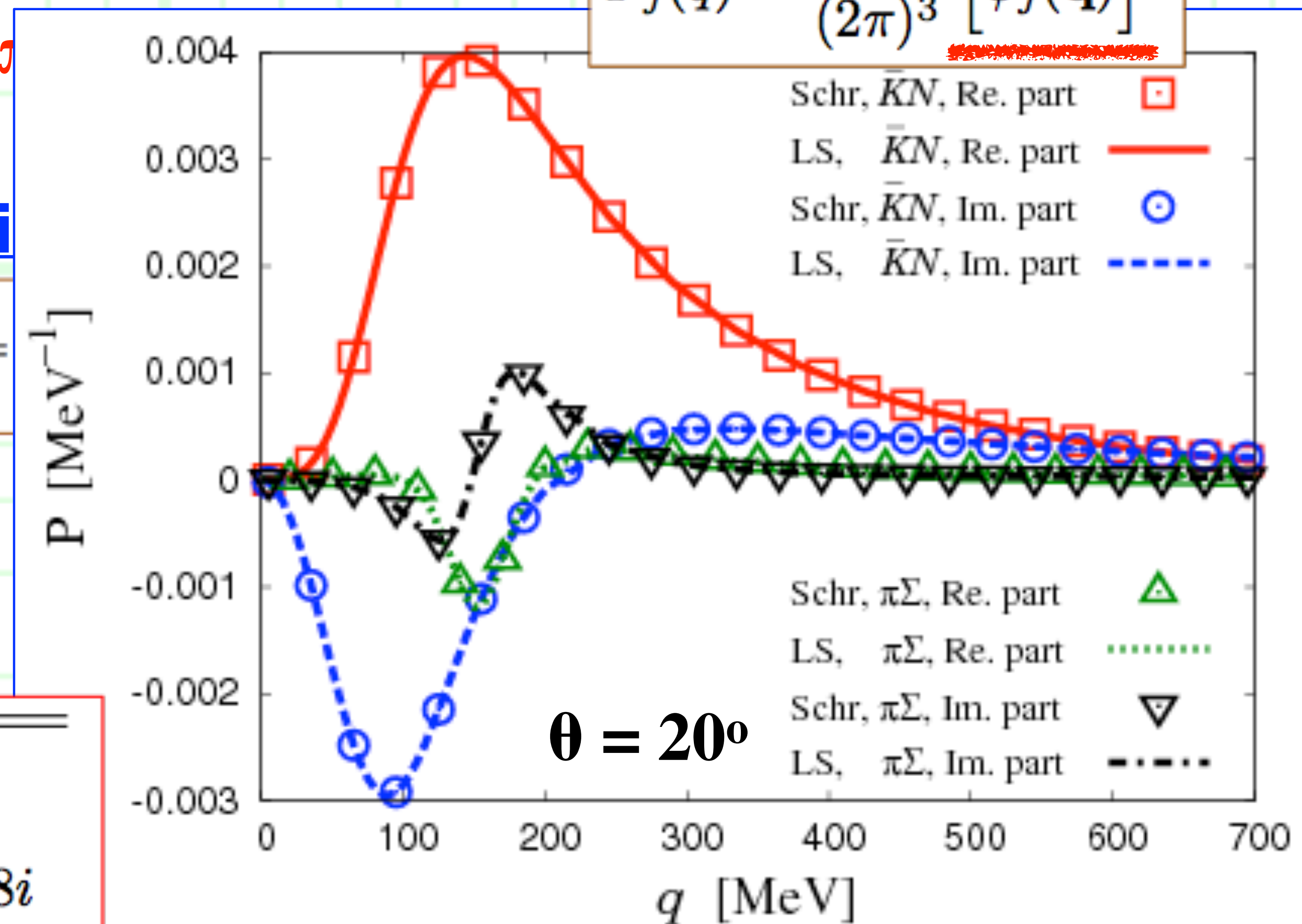
$B_{\rm E}$ [MeV]	22.6
Γ [MeV]	14.7
X_1	$0.99 - 0.08i$
X_2	$0.01 + 0.08i$
$X_1 + X_2$	$1.00 + 0.00i$

--> Extra e:

$$T_{jk}(\mathbf{q}',\, \mathbf{q};\, E) \approx \frac{\gamma_j(\mathbf{q}')\gamma_k(\mathbf{q})}{E - E_{\rm pole}} \quad \text{-->} \quad \tilde{\psi}_j(\mathbf{q}) = \frac{\gamma_j(\mathbf{q})}{E_{\rm pole} - E_j(q)}$$

--- Without normalizing by hand !

$$\mathrm{P}_j(q) = \frac{4\pi q^2}{(2\pi)^3}\left[\tilde{\psi}_j(\mathbf{q})\right]^2$$



□ In 1st way: Points.
2nd way: Lines.

□ Coincidence again !
--- Our method is valid
even for resonances !

Appendix

++ Example 2: Unstable resonance state ++

- We define **the compositeness X as the norm of the wave function**:

$$X \equiv \int \frac{d^3q}{(2\pi)^3} \langle \Psi^* | \mathbf{q} \rangle \langle \mathbf{q} | \Psi \rangle = \int_0^\infty dq \, \mathrm{P}(q)$$

$$\mathrm{P}(q) = \frac{4\pi q^2}{(2\pi)^3} \left[\tilde{\psi}(\mathbf{q}) \right]^2$$

--- **θ Indep. !**

--- In the following, we **calculate X from the scattering amplitude**.

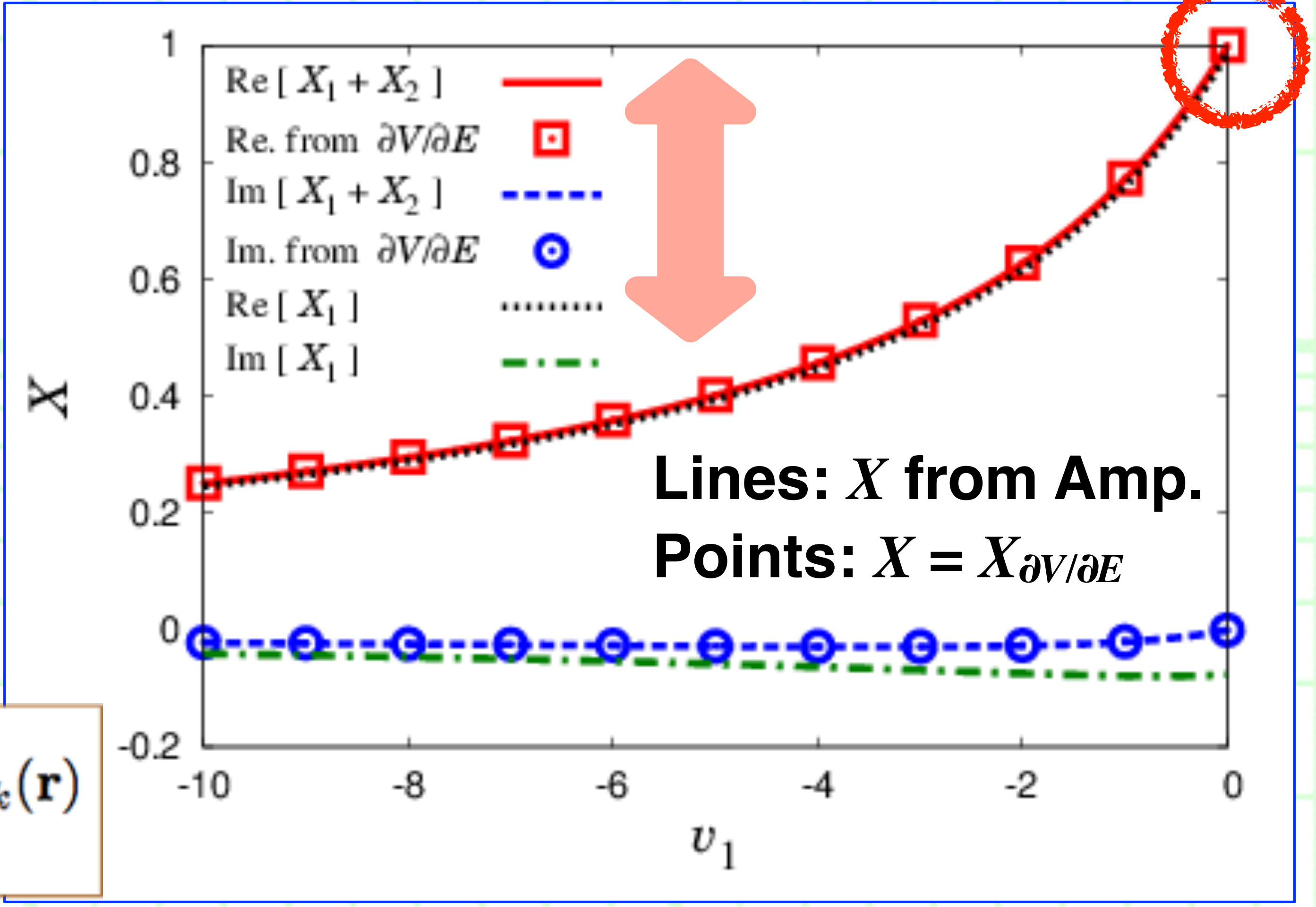
<-- The compositeness is **unity for energy independent interaction**.

Hernandez and Mondragon (1984).

- When we consider **the energy dependence of the interaction**, **the compositeness from the scattering amplitude deviates from unity** because of **missing channel contribution**.

--- *e.g.*: $V_{\mathrm{miss}} = \dfrac{g_0^2}{E - M_0}$

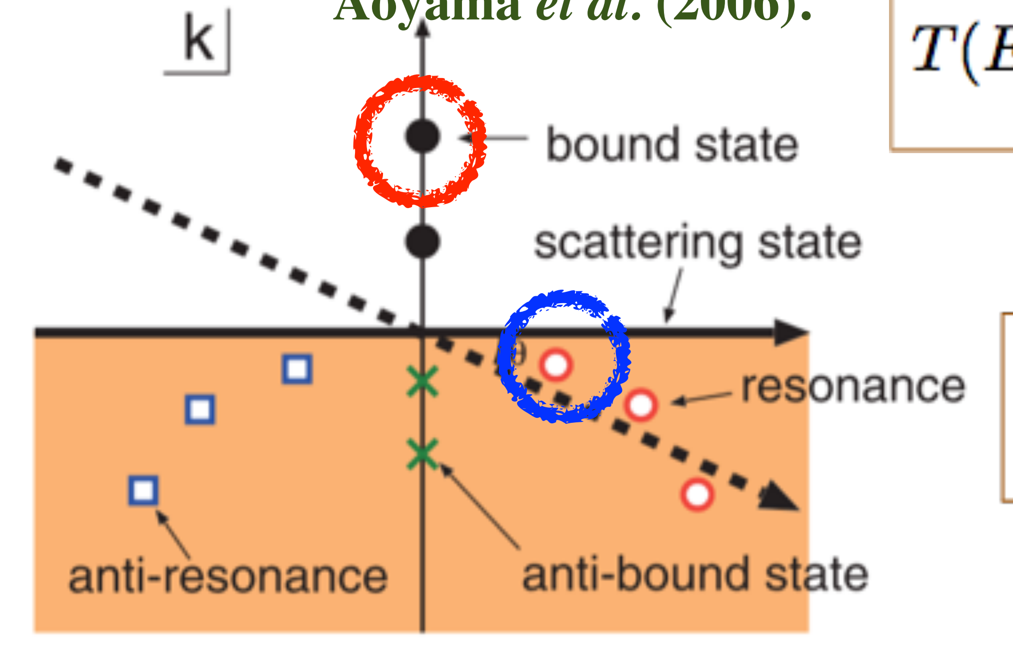
$$X_{\partial V/\partial E} = 1 + \sum_{j,k} \int d^3 r \psi_j(\mathbf{r}) \frac{\partial V_{jk}}{\partial E}(E_{\mathrm{pole}}; r) \psi_k(\mathbf{r})$$



Appendix

++ Lessons from schematic models ++

- We can **extract the two-body wave function** from **the residue of the scattering amplitude at the pole position**, both stable and unstable.

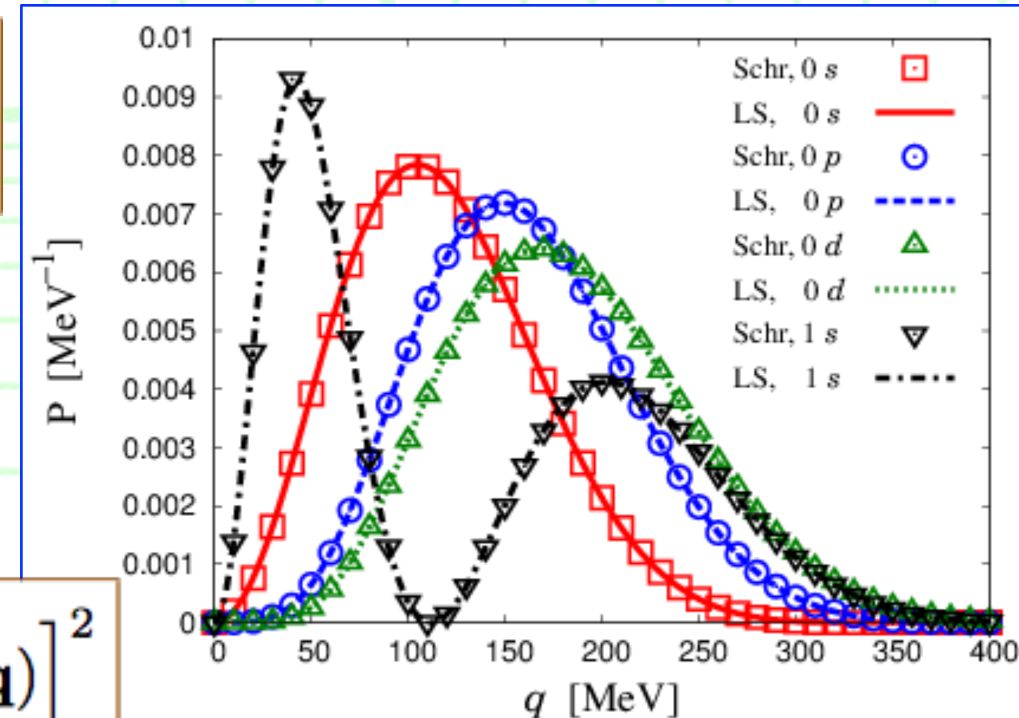


Aoyama *et al.* (2006).

$$T(E; \mathbf{q}', \mathbf{q}) \approx \frac{\gamma(q')\gamma(q)}{E - E_{\rm pole}}$$

$$\tilde{\psi}(\mathbf{q}) = \frac{\gamma(\mathbf{q})}{E_{\rm pole} - E(q)}$$

$$\mathrm{P}(q) = \frac{4\pi q^2}{(2\pi)^3}\left[\tilde{\psi}(\mathbf{q})\right]^2$$



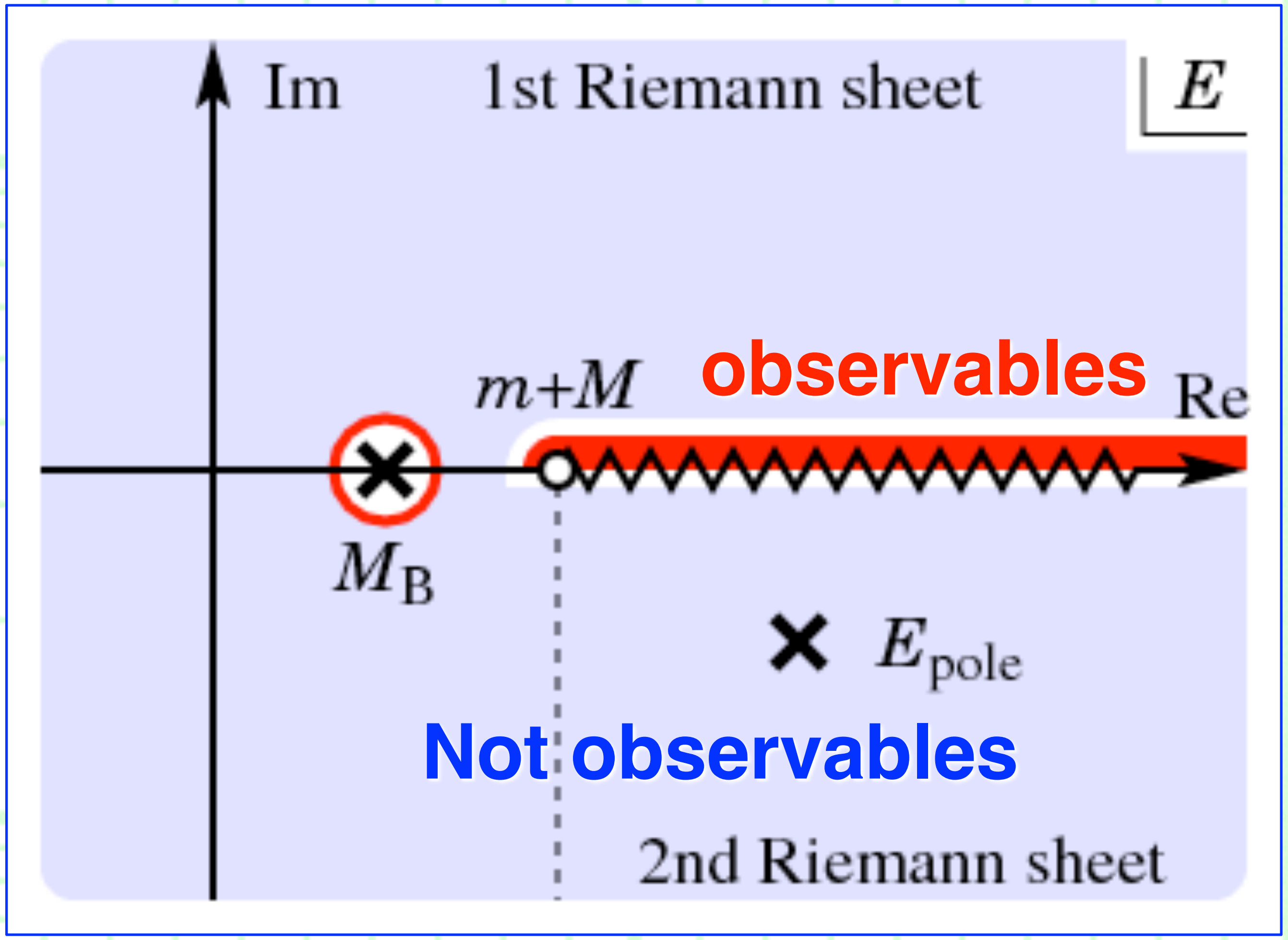
T. S., arXiv:1609.xxxx [hep-ph].

- **The WF from the scattering amplitude is automatically scaled.**
  - □ **The compositeness** (= norm of the two-body WF) **is unity** for a bound state in **an energy independent interaction**.
  - □ For an energy dependent interaction, **the compositeness deviates from unity**, reflecting **a missing channel contribution**.

Appendix

++ Observable and model (in)dependence ++

- Here we comment on the observables and non-observables.
  - Observables:
    Cross section.
    Its partial-wave decomposition.
    --> On-shell Scatt. amplitude
    via the optical theorem.
    Mass of bound states.
  - NOT observables:
    Wave function and potential.
    Resonance pole position.
    Residue at pole.
    Off-shell amplitude.



--> Since the residue of the amplitude at the resonance pole is NOT observable, the wave function and its norm = compositeness are also not observable and model dependent.

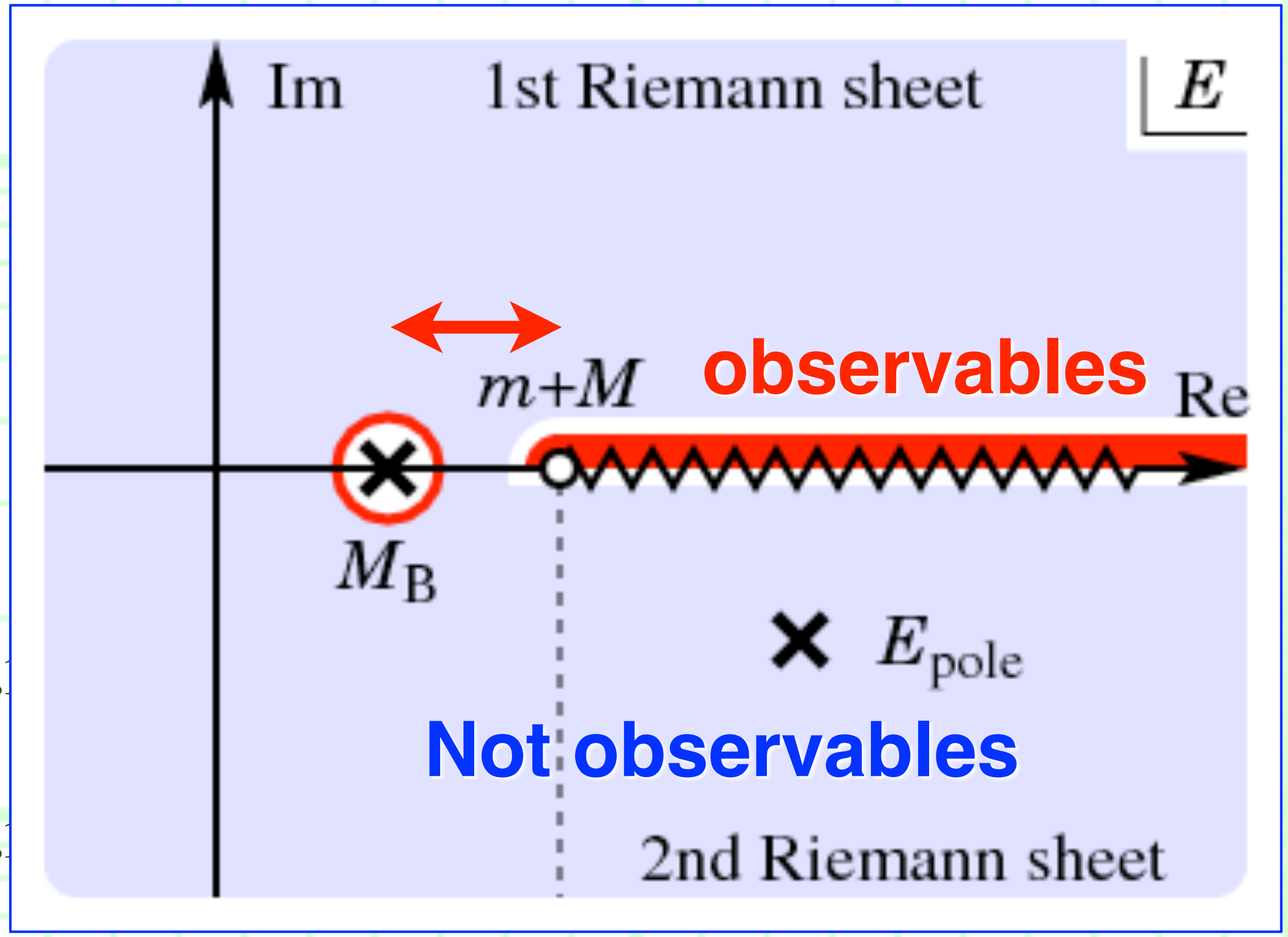
--- Exception: Compositeness for near-threshold poles.

Appendix

++ Observable and model (in)dependence ++

- Special case: Compositeness for near-threshold poles.

--- Compositeness can be expressed with threshold parameters such as scattering length and effective range.



- □ Deuteron.
  Weinberg ('65).
- □ $f_0(980)$ and $a_0(980)$.
  Baru et al. ('04),
  Kamiya-Hyodo, *Phys. Rev.* C93 (2016) 035203.
- □ Λ(1405).
  Kamiya-Hyodo, *Phys. Rev.* C93 (2016) 035203.
- □ ...

- General case: Compositeness are model dependent quantity.

--> Therefore, we have to employ appropriate effective models (V) to construct precise hadron-hadron scattering amplitude, in order to discuss the structure of hadronic molecule candidates !