Playing with color pictorial rules : Anomalous dimension matrix

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Overview

- Motivation
  - *Perturbation theory and all order analysis*
  - *Color structure for non trivial process*
- Quark-Quark case.
  - *"Intuitive way"*
  - *"Tricky way"*
- Prospects

Motivation

When does the problem arise ?

Computation of soft or/and collinear gluons emissions :

- Soft and collinear radiation brings a double logarithm L^2
- Soft and non-collinear radiation brings a single logarithm L^1.

If those logarithms are big enough $\alpha_s L^2$ (or $\alpha_s L^1$) can be of order $\sim \mathcal{O}(1)$.

Perturbation theory

Inherited from renormalization group procedure for UV divergences, the running of coupling α_s is

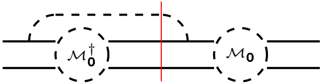
$$\alpha_s(Q) = \frac{2\pi}{b_0 \; log(Q/\Lambda)} \tag{2.1}$$

We need to control emerging logarithm at each order in α_s so we can still use perturbative approach !

Peskin and Schroeder

Motivation - Solve the problem ?

Resuming logarithms, a very long history since pQCD beginning...
The quick way to address all order analysis
Consider (multiple) real emission :



All order analysis

Correction in *"Global observable"* due to large and soft angle gluons.

$$\mathcal{M}_0^{el} \rightarrow \prod_i F_i(Q_i, Q_0) \cdot F_x(\tau_0) \cdot \mathcal{M}_0^{el}, \qquad \tau_0 = \int_{Q_0}^{Q} \frac{dk_\perp}{k_\perp} \frac{\alpha(k_\perp)}{\pi} \tag{2.2}$$

F_x : collinear finite / depend on τ_0 and s, t, u variable of the hard process.

Yu.L. Dokshitzer and G.Marchesini, *JHEP* 0601 (2006) 007 [hep-ph/0509078]

Cross channel form factor F_x

$$\mathcal{M}(\tau_0) = \underbrace{e^{(\lambda-\tau_0)\Gamma_C}}_{Cancel} \underbrace{e^{\tau_0 \Gamma}}_{Talk} \mathcal{M}_0, \qquad -\Gamma \equiv T_t^2 \ln\frac{s}{-t} + T_u^2 \ln\frac{s}{-u} \tag{2.3}$$

Motivation - Anomalous dimension Matrix $\mathcal{Q}$

Anomalous dimension matrix $\mathcal{Q}$

Define as

$$F_x = e^{-\tau_0 (T+U)\cdot \mathcal{Q}} \tag{2.4}$$

Q is a matrix in color space. Valid for any involved partons (quark, gluons, ...)

$$\mathcal{Q} = \frac{\left(T_t^2 + T_u^2\right) + b\left(T_t^2 - T_u^2\right)}{2N_c} \tag{2.5}$$

Complication ?
No color-triviality in $2 \to 2$ process. How to compute the soft anomalous dimension ?
Return to s-channel ?
Gluons case in [DM] $8 \otimes 8 = 27 \oplus 0 \oplus (10 \oplus \overline{10}) \oplus 8s \oplus 8a \oplus 1$
Of the 6 eigenvalues of $\mathcal{Q}$, 3 present an unexpected symmetry :

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3/N_c^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9/N_c^2)}{27} = 0 \tag{2.6}$$

Aim : To get insight of color structure of ($2 \to 2$ process)
How ? We explore this symmetry in various cases

"Intuitive way"

Plan of flight

1 : Find s-channel projection-operator basis

2 : Find t-channel projection-operator basis

3 : Express t-channel color-structures into s-channel color-strucures [!!]

4 : Compute quadratic casimirs for t-channel representations

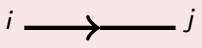
5 : Compute $\mathcal{Q}$ and its eigenvalues

P. Cvitanovic, "Group Theory, Birdtracks, Lie's, and Exceptionnal Groups"
Yu. L. Dokshitzer, "Perturbative QCD (and Beyond)"

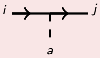
A bit of notation

To construct color multiplet (invariant under $SU(N_c)$), we use pictorial notations :

The primitive invariant of $SU(N_c)$ is

$\delta^i_{\ j} =$ 

The adjoint representation generator

$(t^a)^i_{\ j} =$ 

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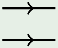
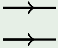
Step 1

From *Young-Tableau* decomposition $\longrightarrow$ $\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-2pt]\square\end{array} = 6 \oplus \bar{3}$

Projection-operators (s-channel) are :

$\mathcal{P}_6 = \frac{1}{2}\Big[$  $+$  $\Big]$

$\mathcal{P}_3 = \frac{1}{2}\Big[$  $-$  $\Big]$

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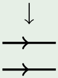
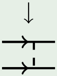
Step 2

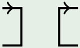
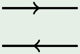
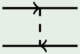
We also get from YT, t-channel decomposition : $\bar{3} \otimes 3 = 8 \oplus 1$

$\mathcal{P}_{\mathbf{1}}^{(t)} = \frac{1}{N_c} \cdot$ 

$\mathcal{P}_{\mathbf{8}}^{(t)} = 2 \cdot$ 

How did we find both coefficients ?

We use *Fierz identity* in its pictorial form (Rotated)

 $= \frac{1}{N_c}$  $+2$ 

"Intuitive way"

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1 : Find s-channel projection-operator basis
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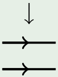
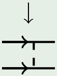
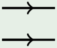
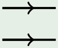
Step 3

Express both

$\mathcal{P}_1^{(t)} = \frac{1}{N_c} \cdot$  $\qquad \mathcal{P}_8^{(t)} = 2 \cdot$ 

as a combination of

$\mathcal{P}_6^{(s)} = \frac{1}{2} \Big[$  $+$  $\Big] \qquad \mathcal{P}_3^{(s)} = \frac{1}{2} \Big[$  $-$  $\Big]$

How ? $\rightarrow$ Use again Fierz identity !

"Intuitive way"

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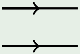
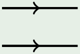
Step 3

In s-channel basis, t-channel operators are

$\mathcal{P}_1 = \frac{1}{N_c}$  $= \frac{1}{N_c}\left[\mathcal{P}_6 + \mathcal{P}_3\right]$

$\mathcal{P}_8 = \frac{-1}{N_c}$  $+$  $= \left[1 - 1/N_c\right]\mathcal{P}_6 + \left[-1 - 1/K_q\right]\mathcal{P}_6$

One can define the following matrix K_{ts} as $\mathcal{P}^{(t)} = \sum K_{ts} \cdot \mathcal{P}^{(s)}$.

$$(K_{ts}) = \begin{pmatrix} 1/N_c & 1/N_c \\ 1 - 1/N_c & -1 - 1/N_c \end{pmatrix}_{\beta\alpha}, \qquad \alpha \equiv \{\mathcal{P}_6, \mathcal{P}_3\}, \beta \equiv \{\mathcal{P}_1, \mathcal{P}_8\}$$

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Step 4 and 5

t-channel casimirs are common knowledge : $c_1 = 0$ and $c_8 = N_c$. And the matrix is in s-channel basis

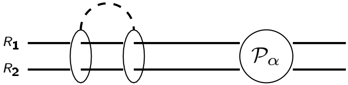
$$T_t^2 = K_{st} c^{(t)} K_{ts} = \frac{1}{2} \begin{pmatrix} N_c - 1 & -1 - N_c \\ 1 - N_c & 1 + N_c \end{pmatrix}$$

The anomalous dimension matrix $\mathcal{Q}$ is given by

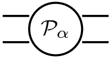
$$\mathcal{Q} = \frac{(T_t^2 + T_u^2) + b(T_t^2 - T_u^2)}{2N_c} = \frac{1}{2N_c} \begin{pmatrix} N_c - 1 & -b(N_c + 1) \\ -b(N_c - 1) & N_c + 1 \end{pmatrix}$$

"Tricky way"

Quark-quark scattering : Another way. Diagrammatic form of casimirs for a given irrep.

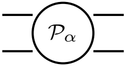


$= c_\alpha$

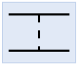


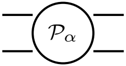
Expand $(T_{R1} + T_{R1})^2$.
It gives us the relation

$\left[-c_\alpha + (c_{R_1} + c_{R_2})\right]$



$= 2$

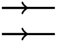




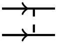
Identity in α irep **Exchange potential $\mathcal{V}$**

Remember t-channel projection operators

$\mathcal{P}_1^{(t)} = \frac{1}{N_c}\cdot$

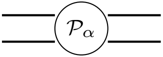


$\mathcal{P}_8^{(t)} = 2\cdot$



K_{ts} matrix is constructed from :

$\times$



$\equiv K_{ts}(\beta\alpha) = \sum_i n_i^{(\beta)} \left(\frac{-c_\alpha + c_{R_1} + c_{R_2}}{2}\right)^i \times \mathcal{P}_\alpha$

We find the following relation for $qq \to qq$

$$K_{ts} = \begin{pmatrix} -2\left(2c_q - c_6\right) & -2\left(2c_q - c_{\bar{3}}\right) \end{pmatrix}$$

We didn't use s-channel projector expression to get K_{ts} !

Compute $\mathcal{Q}$

One can also compute Casimir in t-channel with diagrams or simply use YT-relations.

Remember $c_1 = 0$ and $c_8 = N_c$.

$$\mathcal{Q} = \frac{1}{2N_c} \begin{pmatrix} N_c - 1 & -b(N_c + 1) \\ -b(N_c - 1) & N_c + 1 \end{pmatrix}$$

And

$$\lambda^2 - \lambda Tr(\mathcal{Q}) + Det(\mathcal{Q}) = \lambda^2 - \lambda + \frac{1}{4}\left(1 - 1/N_c^2\right)\left(1 - b^2\right) = 0$$

Also present the same symmetry under $b \leftrightarrow 1/N_c$

In the last 20 minutes :

- Diagrammatic color-rules for partons (valid for generalised partons : think of a compact partonic system).
- Compact expression for K_{ts} and anomalous dimension matrix for $q \otimes q \to q \otimes q$

Have been used for :

- Systematic way to compute $\mathcal{Q}$ for arbitrary large symmetric (or anti-symmetric) irreducible representation. We only use $SU(N_c)$ invariants.
- $\rho \otimes \rho \to \rho \otimes \rho$, 2 weak symmetry for each $\mathcal{Q}$ computed.

Prospects for this project :

- Generalize to arbitrary YT with mixed symmetry...
- Understand $b \leftrightarrow 1/N_c$ symmetry ?
- Applications to phenomenology "induced radiation spectrum", "dijet broadening", ...

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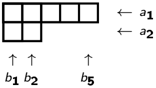
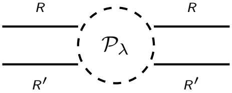
Thank's for your attention !
Any questions ?

Consider a given irep λ in $R \otimes R'$ product :

$\lambda =$ 



Equivalence between YT and Birdtracks. Dimensions are given by

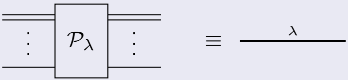
$$d_\lambda = \frac{N_c + d}{y} = Tr\left[\mathcal{P}_\lambda\right] \tag{5.1}$$

Quadratic casimir,

$$\begin{aligned} c_\lambda &= \frac{(2a)}{2}\left(\rho\frac{N_c - \rho}{N_c} + \sum_i a_i^2 - \sum_j b_j^2\right) \\ &= \left[C_R + C_{R'} - 2V(R, R')\right]\mathcal{P}_\lambda \end{aligned} \tag{5.2}$$

Generalised parton

A parton in a non trivial color representation λ (given by its young tableau) different from the fundamental, dual and adjoint representation.
Pictorially :

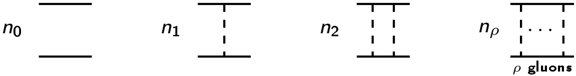


(Wigner-Eckart theorem)

Process

Consider a fully symmetric state of ρ boxes in its YT$\rightarrow$ $\boxed{1}\boxed{2}\cdots\boxed{\rho}$
Process $\rho \otimes \rho \rightarrow \rho \otimes \rho$ in s-channel.

Step B Before looking for a decomposition in t-channel projection-operators, we need a tensor basis.

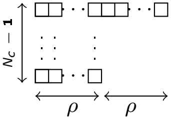


t-channel projection operators are a sum of those terms.
Our job is to find those $n_i^{(\lambda)}$!

Step B, n_i's ?

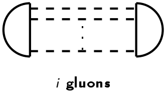
- Dimension, orthogonality, trace.
- And a bit of drawing ...

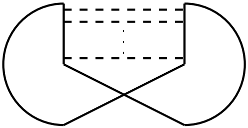
Example



$$\begin{pmatrix} \mathcal{B}_0 & \mathcal{B}_1 & \cdots & \mathcal{B}_\rho \\ \mathcal{C}_0 & \mathcal{C}_1 & \cdots & \mathcal{C}_\rho \\ \mathcal{C}_1 & \mathcal{C}_2 & \cdots & \mathcal{C}_{\rho+1} \\ \vdots & & & \vdots \\ \mathcal{C}_{\rho-1} & \mathcal{C}_\rho & \cdots & \mathcal{C}_{2\rho-1} \end{pmatrix} \cdot \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_\rho \end{pmatrix}^{(\rho)} = \begin{pmatrix} K_\rho \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Where $\mathcal{C}_i$'s and $\mathcal{B}_i$'s are scalars.

$\mathcal{C}_i =$ 

$\mathcal{B}_i =$ 

Step C : Rotate t-channel (previous-YT) to s-channel.

$$K_{ts}(\rho\alpha) = \sum_{i=0}^{\rho} n_i \left(\frac{2c_\rho - c_\alpha}{2}\right)^i \mathcal{P}_\alpha$$

Step D and E Use YT to compute casimirs and done !

$$\mathcal{Q} = \frac{1}{2N_c}\left[\left(K_{st}c^{(t)}K_{ts}\right) + \sigma^{\pm}\left(K_{st}c^{(t)}K_{ts}\right)\sigma^{\pm}\right] + \frac{b}{2N_c}\left[\left(K_{st}c^{(t)}K_{ts}\right) - \sigma^{\pm}\left(K_{st}c^{(t)}K_{ts}\right)\sigma^{\pm}\right]$$

Result for $\rho = 6$

$$\begin{pmatrix}
3-\frac{3}{\mathrm{Nc}} & -\frac{3b(11+\mathrm{Nc})}{11\mathrm{Nc}} & 0 & 0 & 0 & 0 & 0 \\
3b\left(-1+\frac{1}{\mathrm{Nc}}\right) & 3+\frac{3}{\mathrm{Nc}} & -\frac{5b(10+\mathrm{Nc})}{9\mathrm{Nc}} & 0 & 0 & 0 & 0 \\
0 & -\frac{30b}{11} & 3+\frac{8}{\mathrm{Nc}} & -\frac{6b(9+\mathrm{Nc})}{7\mathrm{Nc}} & 0 & 0 & 0 \\
0 & 0 & -\frac{22b(1+\mathrm{Nc})}{9\mathrm{Nc}} & \frac{3(4+\mathrm{Nc})}{\mathrm{Nc}} & -\frac{6b(8+\mathrm{Nc})}{5\mathrm{Nc}} & 0 & 0 \\
0 & 0 & 0 & -\frac{15b(2+\mathrm{Nc})}{7\mathrm{Nc}} & \frac{3(5+\mathrm{Nc})}{\mathrm{Nc}} & -\frac{5b(7+\mathrm{Nc})}{3\mathrm{Nc}} & 0 \\
0 & 0 & 0 & 0 & -\frac{9b(3+\mathrm{Nc})}{5\mathrm{Nc}} & 3+\frac{17}{\mathrm{Nc}} & -\frac{3b(6+\mathrm{Nc})}{\mathrm{Nc}} \\
0 & 0 & 0 & 0 & 0 & -\frac{4b(4+\mathrm{Nc})}{3\mathrm{Nc}} & \frac{3(6+\mathrm{Nc})}{\mathrm{Nc}}
\end{pmatrix}$$

Two eigenvalues have a weaker version of the symmetry $b \leftrightarrow 1/N_c$:

$$\lambda \to 3(1+b) \text{ and } \lambda \to 3(1-b)$$