Lecture 9
Beams and Imperfections

Some slides swiped from Professor Emmanuel Tsesmelis
Others stolen with no remorse from Professor Todd Satogata
Some taken from Professor Waldo MacKay
I even did some by myself!

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What are we talking about?

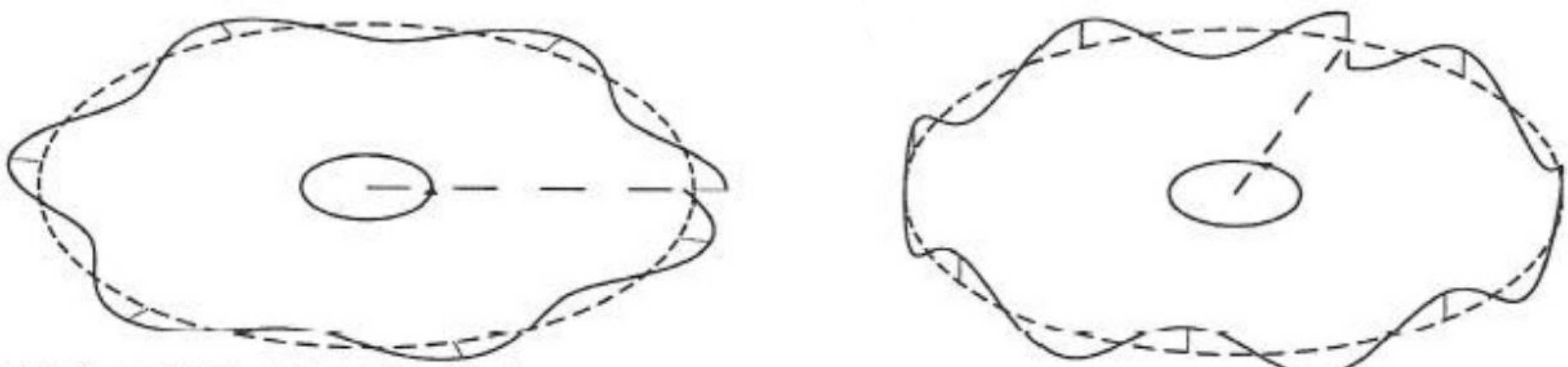
- Introductory stuff…like this.
- Resonance and resonant conditions
  - What is it?
  - Tune
    - Integer Tune
    - Tune Diagrams
- Chromaticity
  - Relationship to Tune
  - Chromaticity Correction
- Dispersion
  - Relationship to chromaticity and tune
  - IF TIME: FODO Dispersion and Dispersion Correction

Obligatory Introductory Stuff

- Been working on accelerators in some way, shape, or form since 2004.
  - SRF technology, then beam dynamics, then beamline design, now feedback system simulation.
  - Always worked on linear machines
- There’s a lot to talk about, and only an hour to talk about it
  - I’ll do what I can, but I HIGHLY recommend you check out:
    - Helmut Wiedemann – Particle Accelerator Physics (AKA the Bible of Accelerator Physics Textbooks)
    - S.Y. Lee – Accelerator Physics (some treatments are excellent, others are confusing)
    - www.toddsatogata.net – USPAS instructor, current professor and senior research staff scientist at Jefferson Lab
    - http://uspas.fnal.gov/ - Website for USPAS, which contains many class notes and useful information.
- OK, now let’s see how well I do with this…

Resonant Conditions: What are they?

- A resonance can be excited through various imperfections in the beamline.
- The phase advance of the betatron oscillation around the machine will repeat itself after a certain number of turns around the machine.
  - Ex/ If phase advance/turn = 120°, repeats after 3 turns

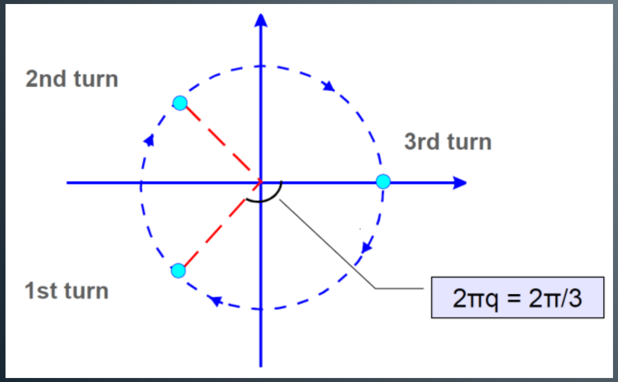


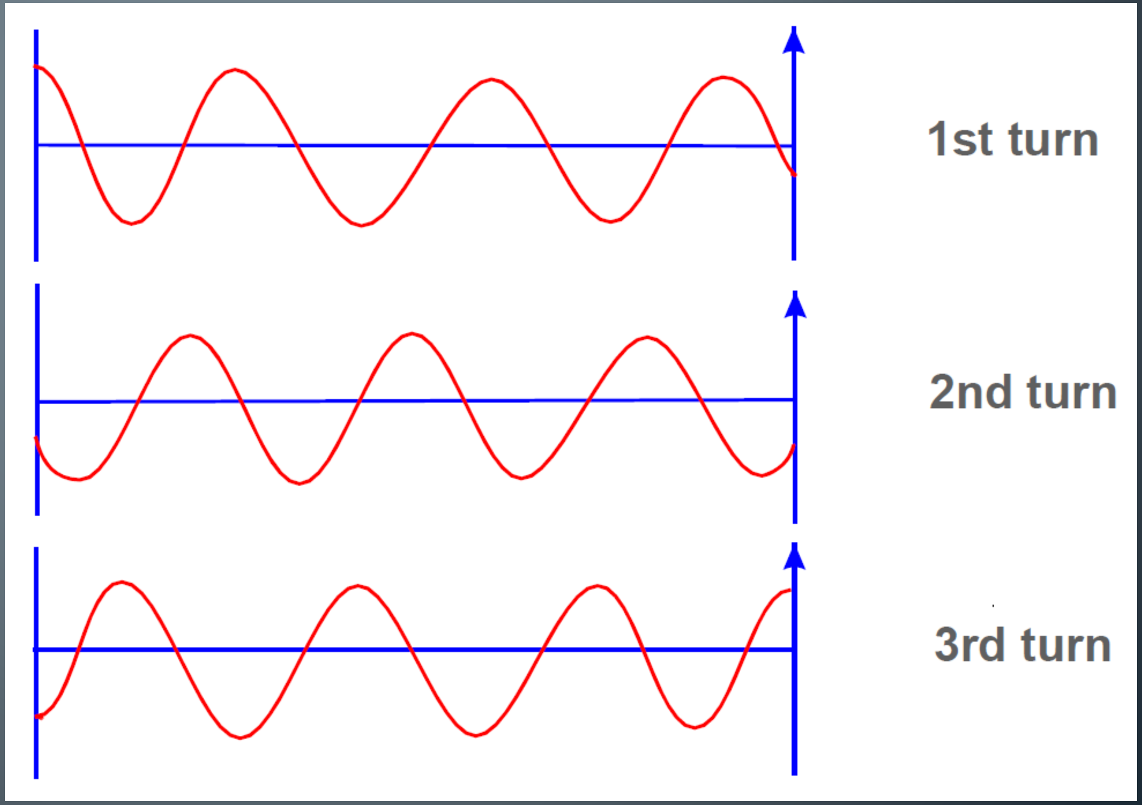
Horizontal Betatron Oscillation with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.

Vertical Betatron Oscillation with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.

Resonant Conditions: What are they?

- Simply put, let's say we have a Q = 3.333
  - This can also be stated as 3Q = 10
  - We can define the order of a resonance as "n" where n x Q = integer
- For Q = 3.333:
  - 3 x Q = 10
  - q = 0.333
- On the normalized phase ellipse:





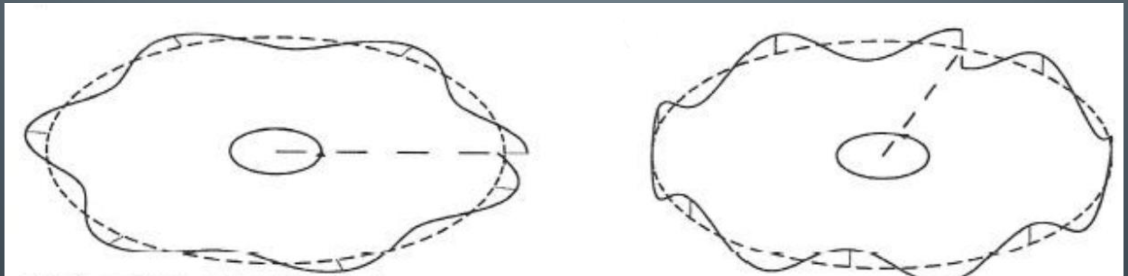
Resonant Conditions: A bit more detail

- Synchrotron is a periodic focusing system, often made up of smaller periodic regions.
  - Can write down a periodic one-turn matrix as

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

- Tune is defined as the total betatron phase advance in one revolution around the ring, divided by 2π

$$Q_{x,y} = \frac{\Delta\phi_{x,y}}{\Delta\theta} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$



Horizontal Betatron Oscillation with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.

Vertical Betatron Oscillation with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.

Resonant Conditions: A bit more detail

- Tunes are both horizontal and vertical
- They are a direct indication of the amount of focusing in an accelerator
  - Higher tune means tighter focusing, lower $< \beta_{x,y}(s) >$
- Tunes are critical for accelerator performance
  - Linear stability depends upon phase advance
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted using groups of quadrupoles

$$M_{one-turn} = I\cos(2\pi Q) + J\sin(2\pi Q)$$

- There's another way to describe all this…
  - http://www.toddsatogata.net/2013-USPAS/2013-01-23-Resonances1.pdf

Resonant Conditions: Integer resonance

Integer resonances

Let's start with a simplified formalism for the horizontal motion equation:

$$\frac{d^2x}{d\theta^2} + {Q_{\mathrm{H}}}^2 x = f(\theta),$$

where

- $\theta = s/R$ is the azimuthal angle around the ring with
- R being the average radius of the ring,
  i. e. we approximate by a circular ring.
- $f(\theta)$ is some source of perturbations from errors.

Fourier transform the function f and let's look at the m^{th} harmonic term:

$$\frac{d^2x}{d\theta^2} + {Q_{\mathrm{H}}}^2 x = \varepsilon \cos(m\theta). \qquad (1)$$

Resonant Conditions: Integer resonance

Solution to Eq. (1) is of the form

$$x = \tilde{x} + \bar{x},$$

with homogeneous part

$$\tilde{x} = A\cos(Q_\mathrm{H}\theta) + B\sin(Q_\mathrm{H}\theta),$$

an inhomogeneous part

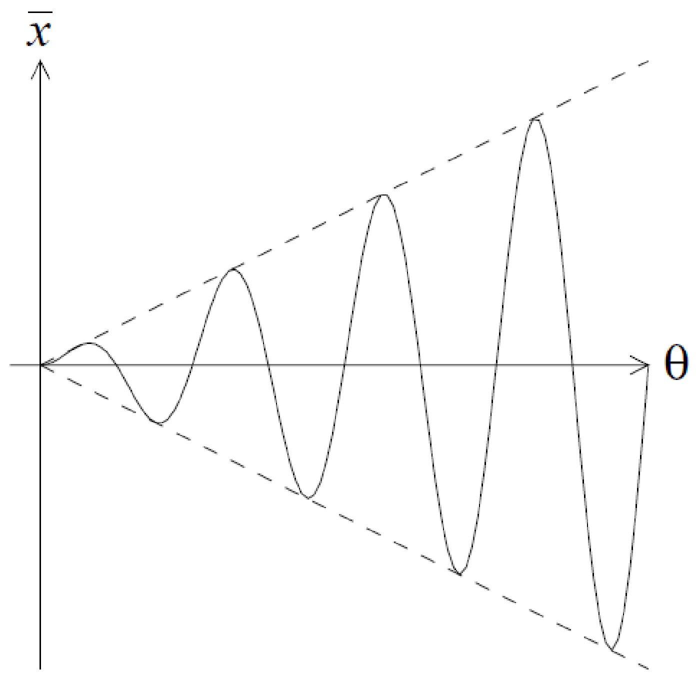
$$\bar{x} = \frac{\varepsilon}{{Q_\mathrm{H}}^2 - m^2}[\cos(m\theta) - \cos(Q_\mathrm{H}\theta)]$$

$$\bar{x} = \frac{\varepsilon\theta}{Q_\mathrm{H} + m}\,\sin\left(\frac{Q_\mathrm{H} + m}{2}\,\theta\right)\,\frac{2}{(Q_\mathrm{H} - m)\theta}\,\sin\left(\frac{Q_\mathrm{H} - m}{2}\,\theta\right).$$

which reduces to $\qquad \bar{x} \simeq \dfrac{\varepsilon\theta}{2Q_\mathrm{H}}\sin(Q_\mathrm{H}\theta), \qquad$ for $Q_\mathrm{H} = m$.

Resonant Conditions: Blown Up!

Linear growth from integer resonance



$$\bar{x}(\theta) = \frac{\varepsilon}{2Q_{\mathrm{H}}}\theta \sin(Q_{\mathrm{H}}\theta).$$

Resonant Conditions: Integer Resonance = BAD

- Putting it more simply:

- On an integer resonance, μ is a multiple of 2π, we expect $\mathbf{M} = \mathbf{I}$.

If there is a small path-length error δl in one drift section, then the 1-turn matrix becomes

$$\mathbf{M} = \begin{pmatrix} 1 & \delta l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \delta l \\ 0 & 1 \end{pmatrix}.$$

Any particle with $x'_0 \neq 0$ will propagate as

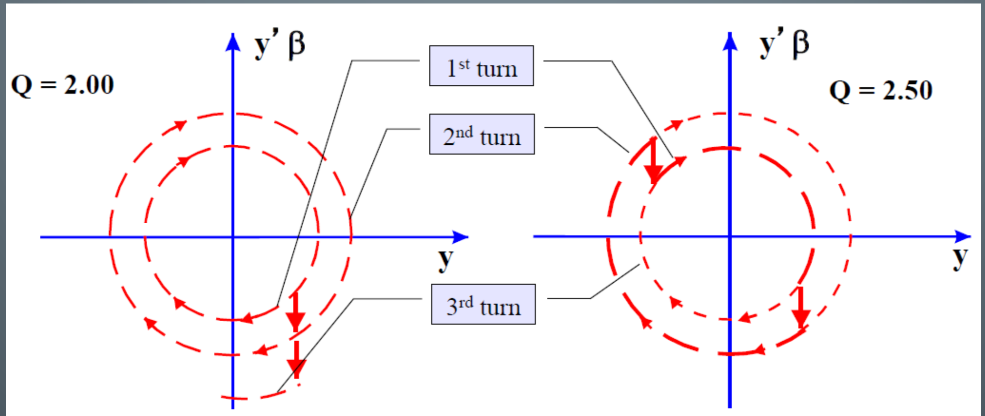
$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} 1 & \delta l \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} x_0 + n\, x'_0\, \delta l \\ x'_0 \end{pmatrix}.$$

This grows linearly with turn number n.

Before the ugly math, here's what I'm describing

- Various imperfections in the beamline will alter the tune in a periodic machine.
- One way to visualize the influence of these imperfections is by looking at what happens on the normalized phase space plot.
  - However, without knowing what is happening, it is hard to understand WHY these are helpful.
- For example, a dipole may have the following:

Note: deflection is independent of position.

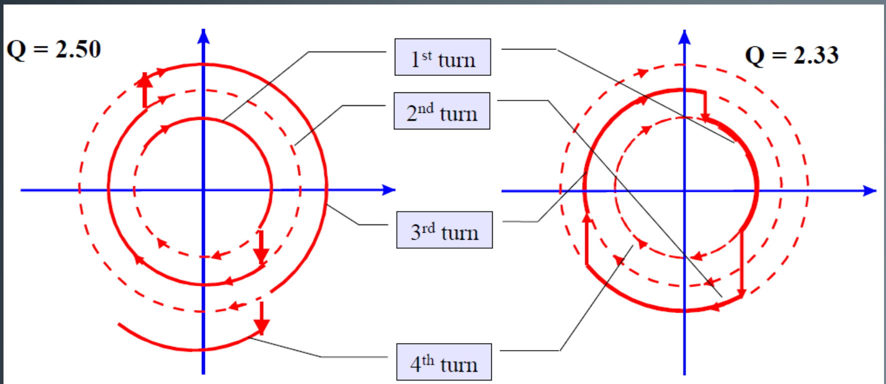


For **Q = 2.00**: Oscillation induced by the **dipole kick** grows on each turn and the particle is lost (**1st order resonance Q = 2**).

For **Q = 2.50**: Oscillation is cancelled out **every second turn**, and therefore the particle **motion is stable**.

A bit more visualization

- For a quadrupole



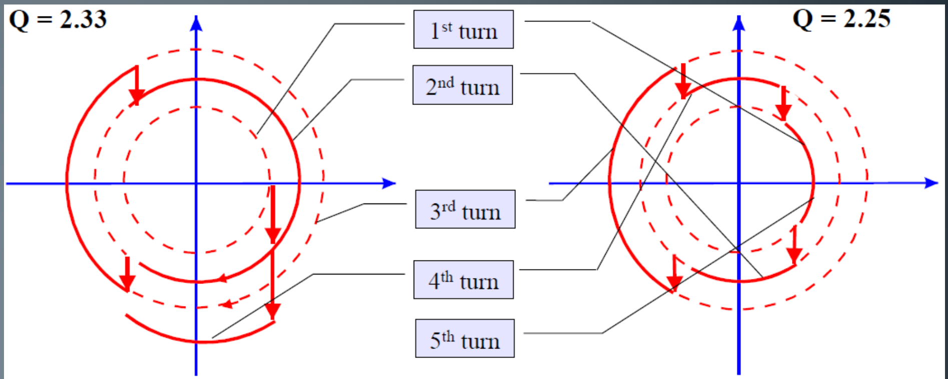
For Q = 2.50: Oscillation induced by the **quadrupole kick** grows on each turn and the particle is lost

(2nd order resonance 2Q = 5)

For Q = 2.33: Oscillation is cancelled out **every third turn**, and therefore the particle **motion is stable**.

Note: deflection is proportional to position.

- For a sextupole



For Q = 2.33: Oscillation induced by the **sextupole kick** grows on each turn and the particle is lost

(3rd order resonance 3Q = 7)

For Q = 2.25: Oscillation is cancelled out **every fourth turn**, and therefore the particle **motion is stable**.

Note: deflection is proportional to the square of the position.

Resonant Conditions: Some math

- If you include linear coupling between the planes (small amount, due to the slight roll of a quadrupole by angle θ (and trying to avoid most of the math).

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon\cos(m\theta)\, y$$
$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 y = \varepsilon\cos(m\theta)\, x$$

- Assuming ε is small, and making some substitutions:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \frac{1}{2}\varepsilon_y\left[\cos(Q_\mathrm{V}+m)\theta + \cos(m-Q_\mathrm{V})\theta\right]$$
$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 y = \frac{1}{2}\varepsilon_x\left[\cos(Q_\mathrm{H}+m)\theta + \cos(m-Q_\mathrm{H})\theta\right]$$

Resonant Conditions: A bit more math

- Running through somewhat obscene amounts of mathematics, you can eventually find your way to the resonance conditions

Linear Sum Resonance $\Rightarrow$ $Q_\mathrm{H} + Q_\mathrm{V} = m$

Linear Difference Resonance $\Rightarrow$ $|Q_\mathrm{H} - Q_\mathrm{V}| = m$

- Let's take an uncoupled one-turn matrix:

$$\mathbf{T} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_2 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 & 0 & 0 \\ -\gamma_1 \sin\mu_1 & \cos\mu_1 - \alpha_1 \sin\mu_1 & 0 & 0 \\ 0 & 0 & \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \\ 0 & 0 & -\gamma_2 \sin\mu_2 & \cos\mu_2 - \alpha_2 \sin\mu_2 \end{pmatrix}$$

- Diff. res. condition: $\sin\mu_1 = \sin\mu_2$,
- Sum res. condition: $\sin\mu_1 = -\sin\mu_2$.

Resonant Conditions: I know…too much math

- If the last element in T is a thin lens quadrupole with a small roll:

Estimate effect of rolled thin quad:

$$\mathbf{T}' = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} = \mathbf{R}\mathbf{Q}\mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{T}$$

$$= \begin{pmatrix} \mathbf{I}\cos\theta & \mathbf{I}\sin\theta \\ -\mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{I}\cos\theta & -\mathbf{I}\sin\theta \\ \mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_2 \end{pmatrix}.$$

Fast forward skipping a bit of algebra:

$$\mathbf{T}' = \begin{pmatrix} (\mathbf{I}\cos^2\theta + \mathbf{D}^2\sin^2\theta)\mathbf{u}_1 & (\mathbf{I} - \mathbf{F}^2)\mathbf{u}_2\cos\theta\sin\theta \\ (\mathbf{D}^2 - \mathbf{I})\mathbf{u}_1\cos\theta\sin\theta & (\mathbf{I}\cos^2\theta + \mathbf{F}^2\sin^2\theta)\mathbf{u}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{f}\sin^2\theta & 1 \end{pmatrix}\mathbf{u}_1 & \begin{pmatrix} 0 & 0 \\ \frac{2}{f}\cos\theta\sin\theta & 0 \end{pmatrix}\mathbf{u}_2 \\ \begin{pmatrix} 0 & 0 \\ \frac{2}{f}\cos\theta\sin\theta & 0 \end{pmatrix}\mathbf{u}_1 & \begin{pmatrix} 1 & 0 \\ -\frac{2}{f}\sin^2\theta & 1 \end{pmatrix}\mathbf{u}_2 \end{pmatrix}.$$

Resonant Conditions: This is leading somewhere

- Some unavoidable math:

$$\mathbf{M} = \mathbf{u}_1 \cos^2\theta + \frac{2\sin^2\theta}{f} \begin{pmatrix} 0 & 0 \\ \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \end{pmatrix},$$

$$\mathbf{N} = \mathbf{u}_2 \cos^2\theta - \frac{2\sin^2\theta}{f} \begin{pmatrix} 0 & 0 \\ \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \end{pmatrix},$$

$$\mathbf{m} = \begin{pmatrix} 0 & 0 \\ \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \end{pmatrix} \frac{\sin 2\theta}{f},$$

$$\mathbf{n} = \begin{pmatrix} 0 & 0 \\ \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \end{pmatrix} \frac{\sin 2\theta}{f}.$$

- And then accepting (let's just call it a definition):

$$\kappa = \lambda + \lambda^{-1} = \frac{\mathrm{tr}(\mathbf{M}+\mathbf{N})}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2}\right)^2 + |\mathbf{m}+\tilde{\mathbf{n}}|}.$$

- We can find that for both resonant conditions, $\cos\mu_1 = \cos\mu_2$, so the $Tr(\boldsymbol{u}_1)$=Tr($\boldsymbol{u}_2$)

Resonant Conditions: Almost there

- A bit more ugliness:

$$\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2} = \frac{\beta_1 \sin\mu_1 - \beta_2 \sin\mu_2}{f} \sin^2\theta,$$

$$|\mathbf{m}+\tilde{\mathbf{n}}| = \frac{\beta_1\beta_2}{f^2}\sin^2(2\theta)\sin\mu_1\sin\mu_2,$$

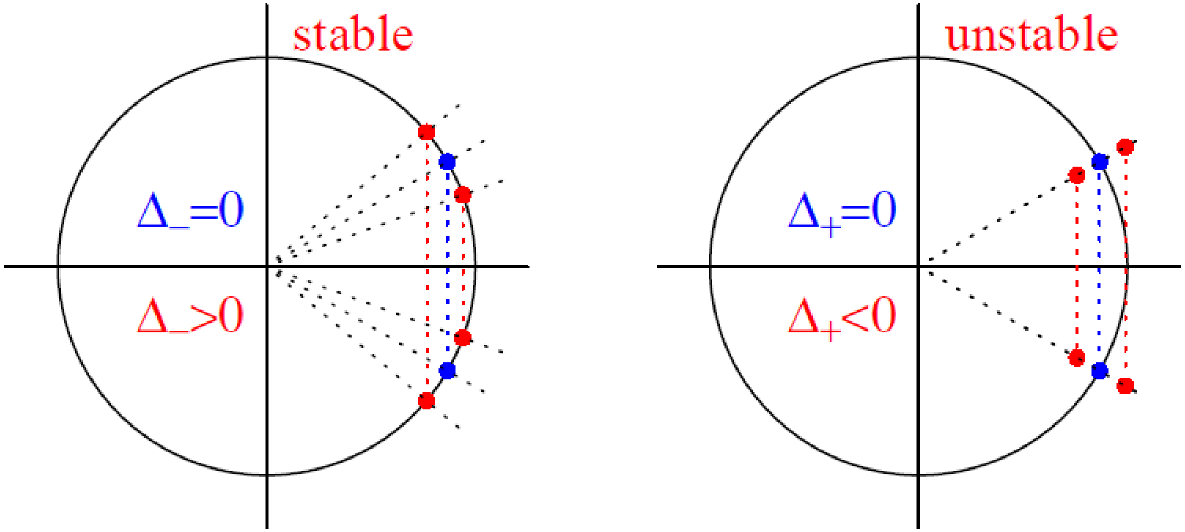
- Notice that $|\mathbf{m}+\tilde{\mathbf{n}}| \neq 0$ if there is a slight roll of the quadrupole.
- The sign of $|\mathbf{m}+\tilde{\mathbf{n}}|$ is determined solely by the product $\sin\mu_1\sin\mu_2$. For the slightly coupled $\mathbf{T}'$, the argument of the radical is

$$\begin{aligned}
\Delta_\pm &= \left(\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2}\right)^2 + |\mathbf{m}+\tilde{\mathbf{n}}| \\
&= \frac{\sin^4\theta}{f^2}(\beta_1\sin\mu_1 - \beta_2\sin\mu_2)^2 + \frac{\beta_1\beta_2}{f^2}\sin^2(2\theta)\sin\mu_1\sin\mu_2 \\
&= \frac{\sin^4\theta}{f^2}(\beta_1 \pm \beta_2)^2\sin^2\mu_1 \mp \frac{\beta_1\beta_2}{f^2}\sin^2(2\theta)\sin^2\mu_1 \\
&\simeq \mp\frac{4\beta_1\beta_2\sin^2\mu_1}{f^2}\,\theta^2, \qquad \text{for small } \theta.
\end{aligned}$$

Resonant Conditions: See? Makes a little sense.

- So, as θ increases, the degenerate eigenvalues separate:

1. In the case of a **difference resonance**, $\Delta_- > 0$, and the degenerate λ_j eigenvalue pairs split apart by moving along the unit circle in the complex plane. Since the eigenvalues stay on the circle, the motion remains **stable** with $\lambda_j^* = \lambda_j^{-1}$.
2. For a **sum resonance**, $\Delta_+ < 0$, and the λ_j eigenvalues move away from the unit circle out into the complex plane resulting in **unstable** motion with $\lambda_j^* \neq \lambda_j^{-1}$.



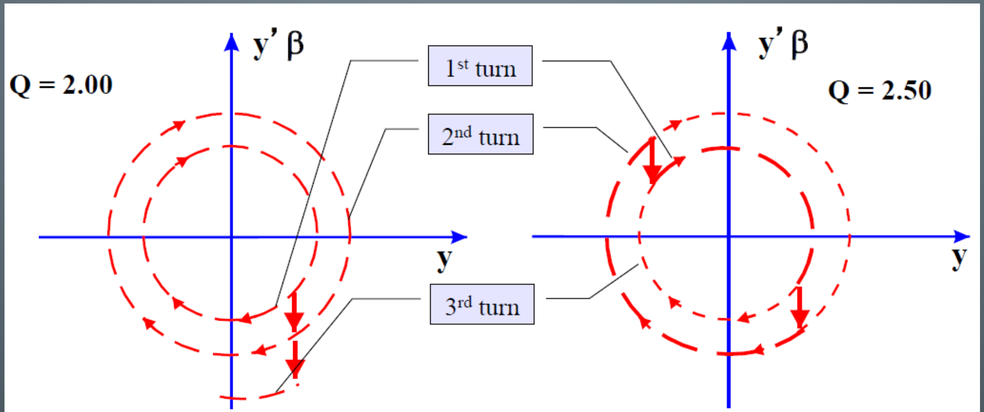
Difference Resonance

Sum Resonance

So, that's why I showed you that math

- Various imperfections in the beamline will alter the tune in a periodic machine.
- One way to visualize the influence of these imperfections is by looking at what happens on the normalized phase space plot.
  - However, without knowing what is happening, it is hard to understand WHY these are helpful.
- For example, a dipole may have the following:

Note: deflection is independent of position.



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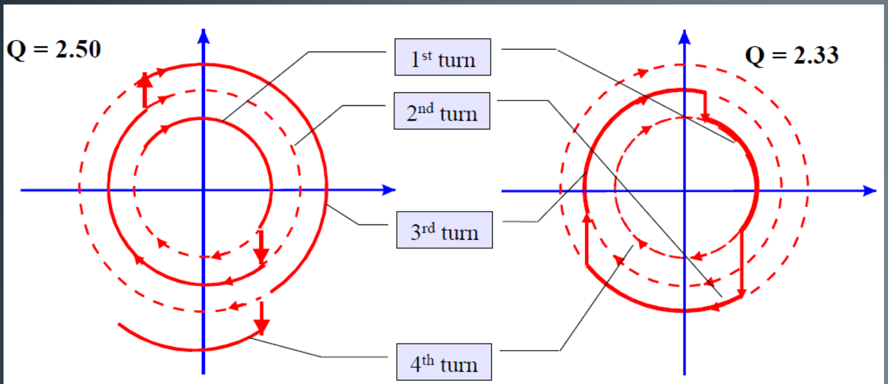
For **Q = 2.50**: Oscillation is cancelled out **every second turn**, and therefore the particle **motion is stable**.

Worthwhile, right?

- For a quadrupole



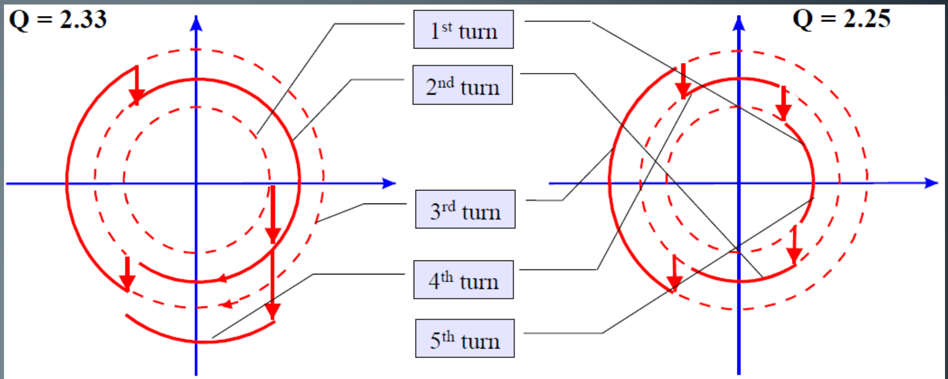
For Q = 2.50: Oscillation induced by the quadrupole kick grows on each turn and the particle is lost

(2nd order resonance 2Q = 5)

For Q = 2.33: Oscillation is cancelled out every third turn, and therefore the particle motion is stable.

Note: deflection is proportional to position.

- For a sextupole



For Q = 2.33: Oscillation induced by the sextupole kick grows on each turn and the particle is lost

(3rd order resonance 3Q = 7)

For Q = 2.25: Oscillation is cancelled out every fourth turn, and therefore the particle motion is stable.

Note: deflection is proportional to the square of the position.

I know you're confused

- Because of the various imperfections due to the many different elements in the beamline, making sure you do not excite a resonant condition gets very difficult to control.

- Instead of keeping track of these circles for each case, and remembering which resonances you excite for each element, a tune plot is often used (sometimes called the necktie diagram).

- Avoiding the mathematical derivation for each case (see the MacKay lecture of which I am drawing much of this), one can construct these necktie diagrams as demonstrated on the following slides.

The results!

- A normal quadrupole excites half-integer resonances:

$$2Q_{\mathrm{H}} = \pm m, \quad \text{and} \quad 2Q_{\mathrm{V}} = \pm m.$$

- A normal octopole:

$$\begin{aligned} \pm 4Q_{\mathrm{H}} &= m, \\ \pm 4Q_{\mathrm{V}} &= m, \\ \pm 2Q_{\mathrm{H}} &= m, \\ \pm 2Q_{\mathrm{V}} &= m, \\ \pm 2Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m. \end{aligned}$$

- A normal sextupole:

$$\begin{aligned} \pm 3Q_{\mathrm{H}} &= m, \\ \pm Q_{\mathrm{H}} &= m, \\ \pm Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m. \end{aligned}$$

- A normal decapole:

$$\begin{aligned} \pm 5Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m, \\ \pm 5Q_{\mathrm{H}} &= m, \\ \pm 3Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m, \\ \pm 3Q_{\mathrm{H}} &= m, \\ \pm Q_{\mathrm{H}} \pm 4Q_{\mathrm{V}} &= m, \\ \pm Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m, \\ \pm Q_{\mathrm{H}} &= m. \end{aligned}$$

The results!

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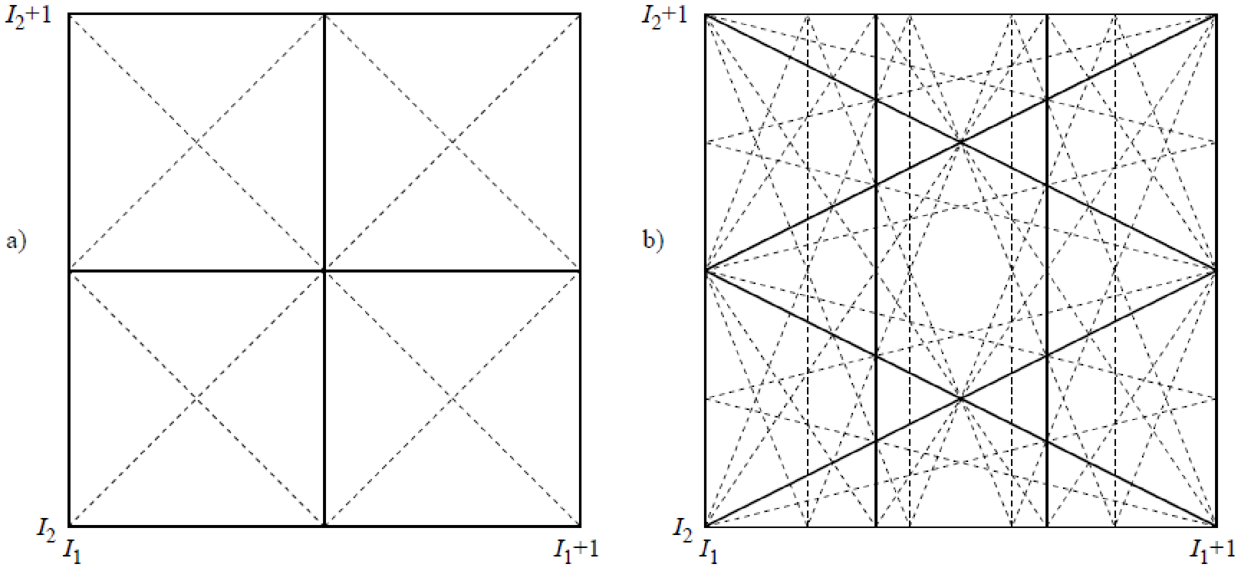
Translating to a graphical means…

Lines from normal multipoles



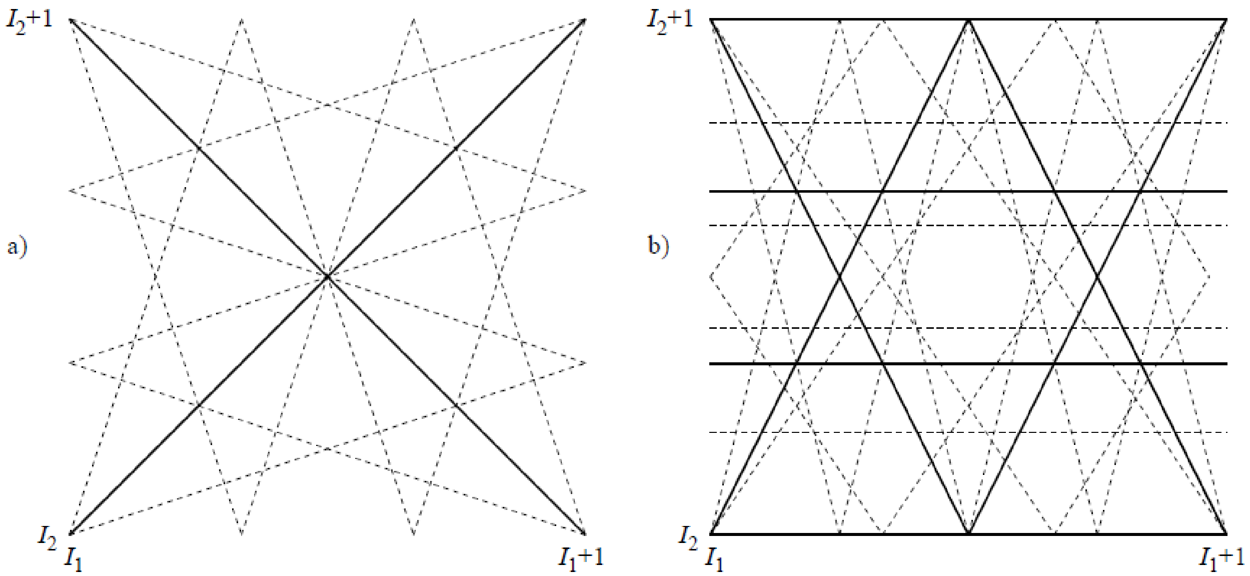
a) A tune plot showing the resonance lines driven by a normal quadrupole perturbation (heavy lines), and a normal octopole perturbation (all lines). I_1 and I_2 are arbitrary integers.

b) A tune plot showing the resonance lines driven by a normal sextupole (heavy lines), and a normal decapole (heavy and dashed lines).

- Typically: Positive slopes (diff res) OK; Negative slopes (sum res) bad.

And including skews…

Lines from skew multipoles



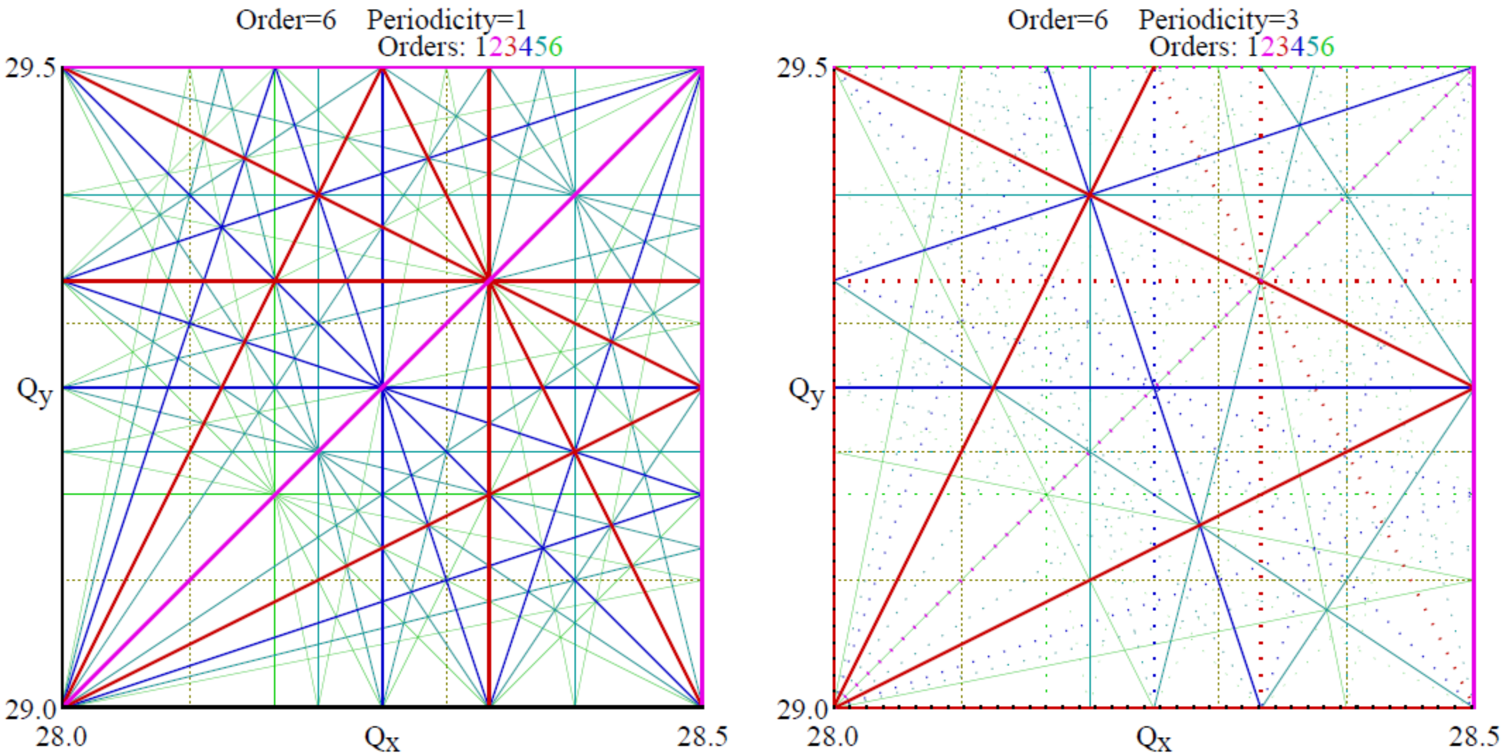
a) Skew quad lines (solid) and skew octopole lines (bold and dashed).

b) Skew sextupole (bold) and skew decapole (bold and dashed) lines.

- Again: Positive slopes (diff res) OK; Negative slopes (sum res) bad.

And then including periodicity…

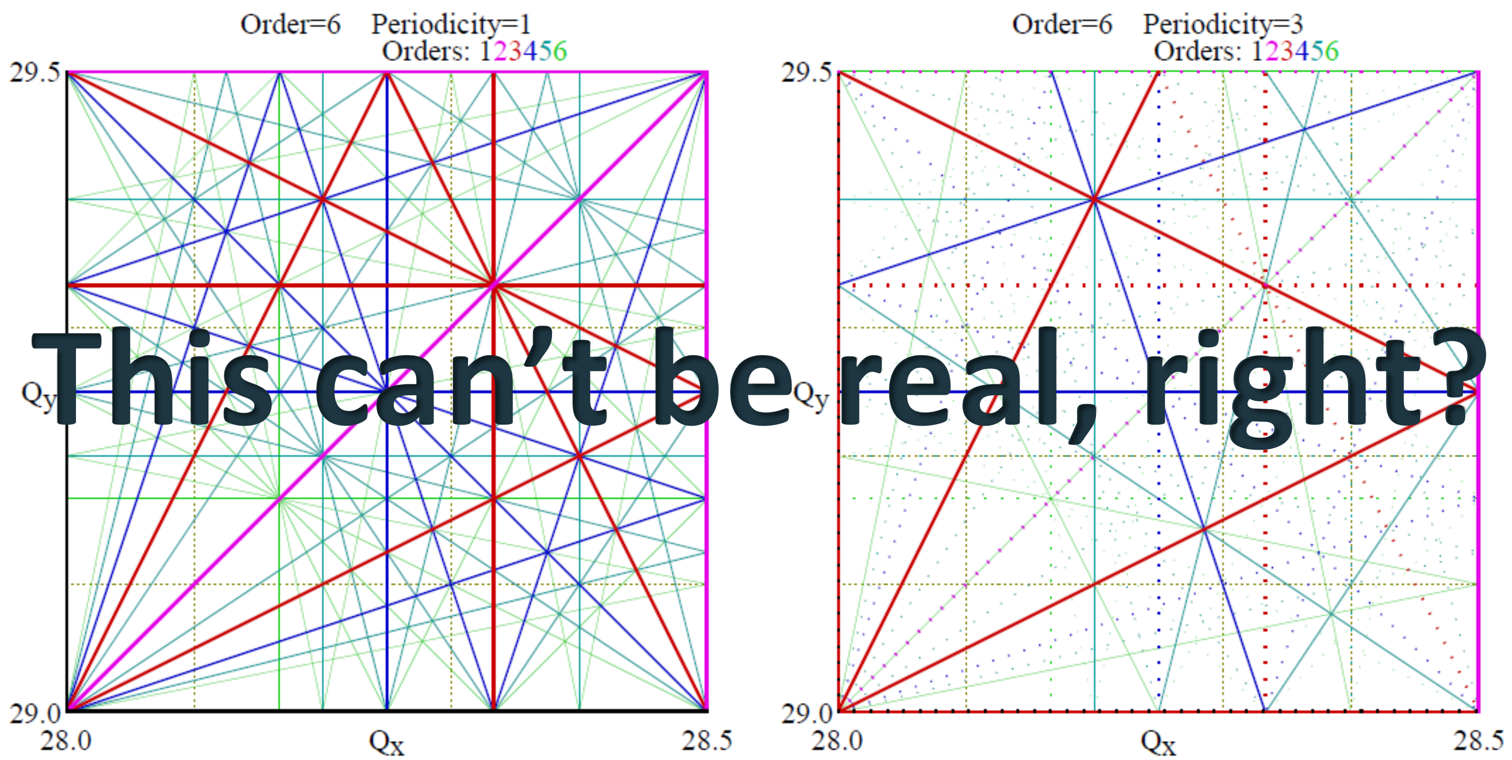
Periodicity



USPAS: Lectures on Resonances
Waldo MacKay January, 2013

And then including periodicity…

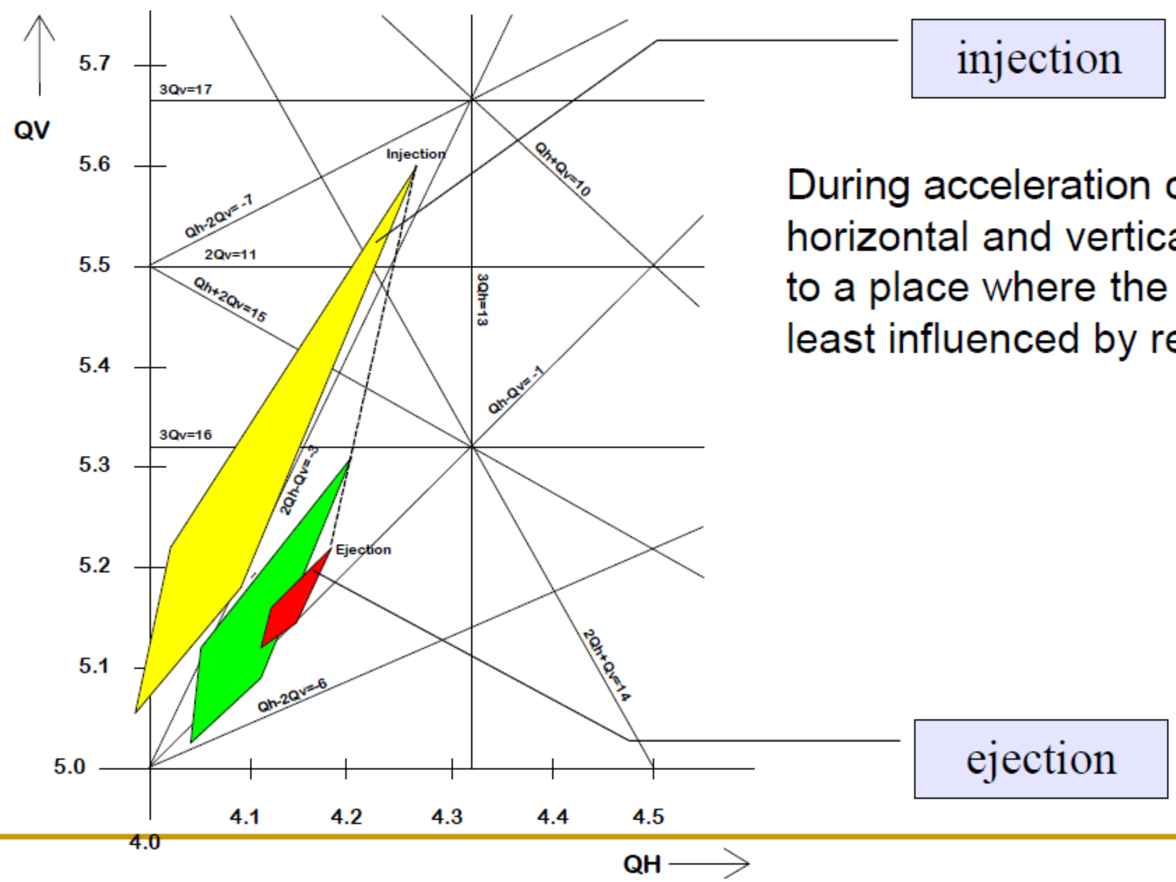
Periodicity



This can’t be real, right?

It’s real, and here’s an example

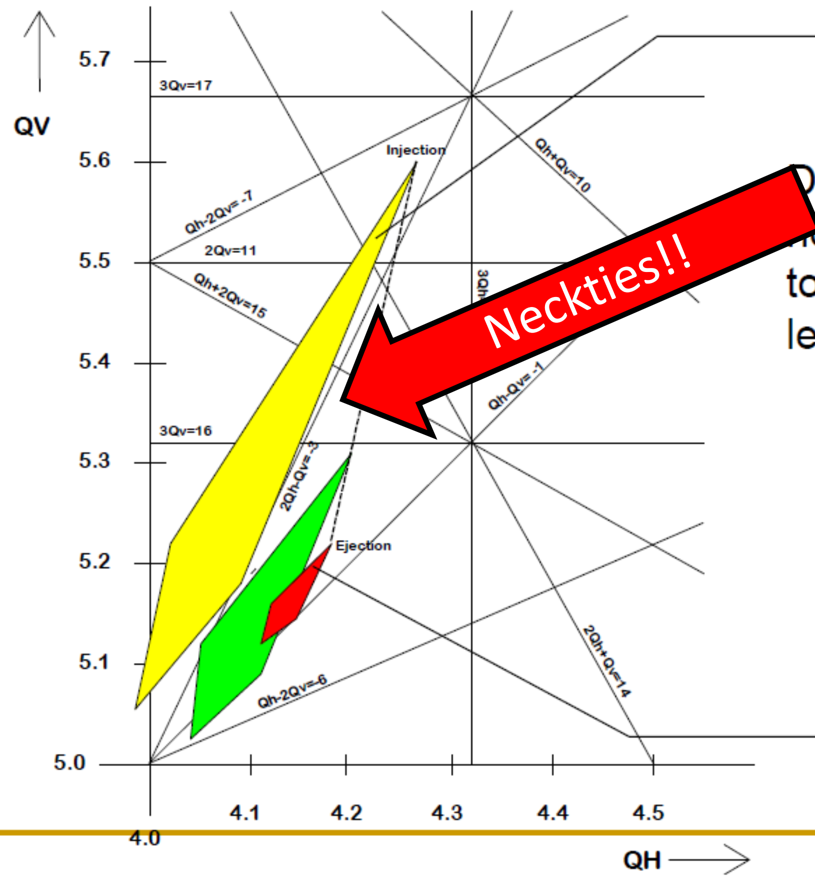
P.S. Booster Tune Diagramme



During acceleration change the horizontal and vertical tune to a place where the beam is least influenced by resonances.

It’s real, and here’s an example

P.S. Booster Tune Diagramme



injection

During acceleration change the horizontal and vertical tune to a place where the beam is least influenced by resonances.

ejection

One more thing before moving on…

- The tune does not stay constant in the machine. This leads to a variation of Q for each turn.
- This variation can go up and down, giving a range of possible values for Q, which we can call ΔQ.
- This range of values has a width, which is called the stopband of the resonance.
- Not only do you want to avoid the resonances, but you want to avoid being in the stopband of a resonance as well, as it may pull you into the resonance itself.

Another Imperfection: Chromaticity

- The focusing in a machine (and thus tune) depends on the momentum.
- The variation of the tune with momentum offset ($\delta \stackrel{\text{def}}{=} \Delta p / p_0$) is called chromaticity.
  - Inserting a momentum perturbation is akin to adding a bit of extra focusing to the one-turn matrix which depends on the unperturbed focusing, K_0.

$$M_{\text{one turn}}(\delta) = \begin{pmatrix} 1 & 0 \\ K_0 \delta ds & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi Q) + \alpha \sin(2\pi Q) & \beta \sin(2\pi Q) \\ -\gamma \sin(2\pi Q) & \cos(2\pi Q) - \alpha \sin(2\pi Q) \end{pmatrix}$$

$$M_{\text{one turn}}(\delta) = \begin{pmatrix} \cos(2\pi Q) + \alpha \sin(2\pi Q) & \beta \sin(2\pi Q) \\ -\gamma \sin(2\pi Q) + K_0 \delta [\cos(2\pi Q) + \alpha \sin(2\pi Q)] ds & \cos(2\pi Q) - \alpha \sin(2\pi Q) + K_0 \delta \beta \sin(2\pi Q) ds \end{pmatrix}$$

- The trace is related to the new tune:

$$\cos(2\pi Q_{\text{new}}) = \frac{1}{2} \text{Tr } M = \cos(2\pi Q) + \frac{K_0 \delta}{2} \beta \sin(2\pi Q) ds$$

Chromaticity and Tune

- Going through a bit of math:

$$\cos(2\pi Q_{\mathrm{new}}) = \frac{1}{2}\mathrm{Tr}\ M = \cos(2\pi Q) + \frac{K_0\delta}{2}\beta\sin(2\pi Q)ds$$

$$\cos(2\pi Q_{\mathrm{new}}) = \cos(2\pi(Q+dQ)) \approx \cos(2\pi Q) - 2\pi\sin(2\pi Q)dQ$$

- Last two terms must be equal, therefore

$$dQ = -\frac{K(s)\delta}{4\pi}\beta(s)ds \xrightarrow{\text{Integrate around ring}} \Delta Q = -\frac{\delta}{4\pi}\oint K(s)\beta(s)\,ds$$

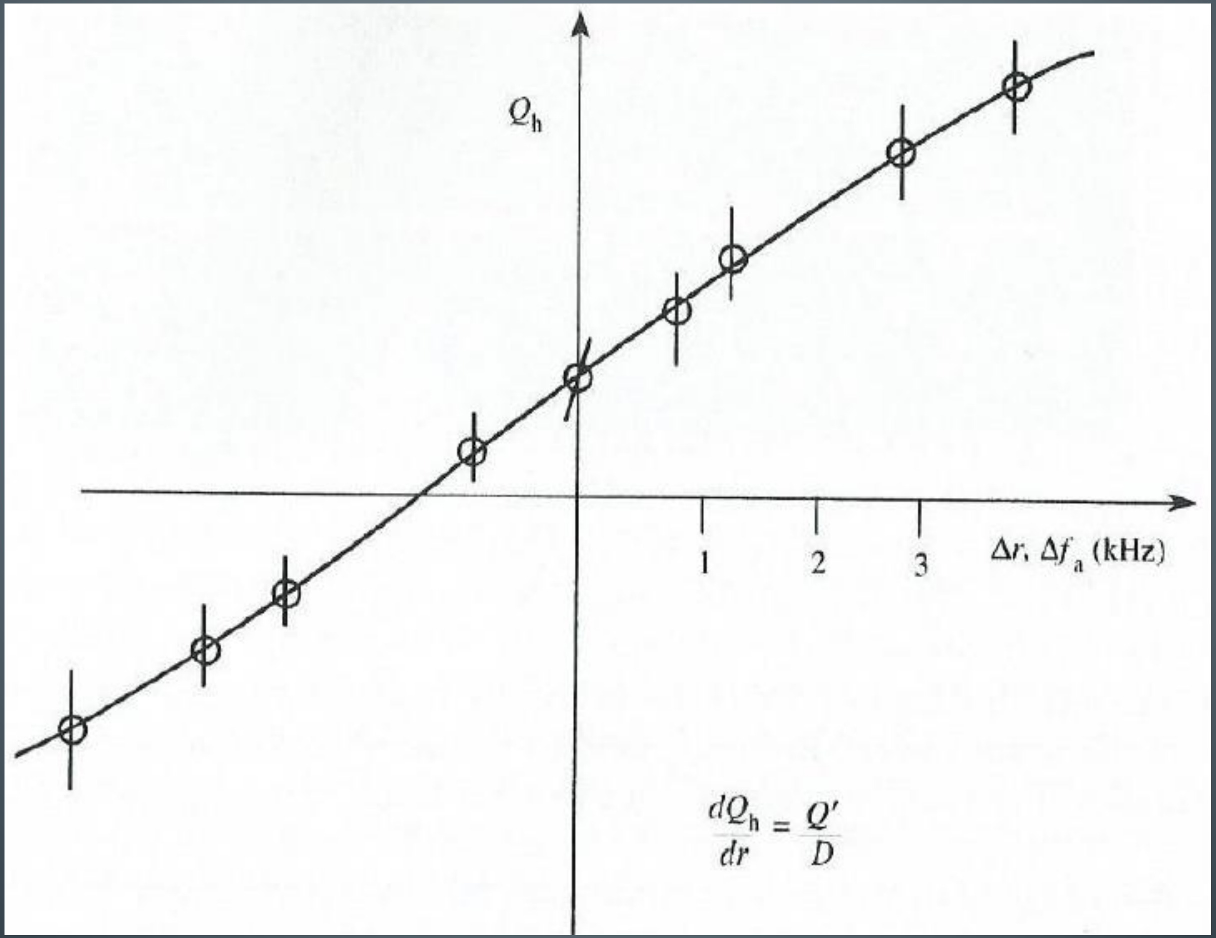
Total change in tune

- The tune will always have a bit of a spread due to the momentum spread. You can define the natural chromaticity as:

$$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\,ds$$

Chromaticity is measureable

- Steering the beam to a new mean radius, and adjusting the RF frequency to vary the momentum, you can measure the Q



Chromaticity is correctable

- Need a way to connect the momentum offset, δ, to focusing.
- We can do this using sextupoles, which give us nonlinear focusing dependent on position) and dispersion (momentum-dependent position).
- This is going to require an aside, so we can discuss dispersion, which actually deserves more than an aside.

More than a small aside: Dispersion

- Dispersion, η(s), is defined as the change in particle position with fractional momentum offset, δ.
  - This originates from the momentum dependence of dipole bends.

- Add explicit momentum dependence to EOM: $$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

Particular sol'n inhomog. DE w/ periodic ρ(s).

- The trajectory has two parts: $x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

More than a small aside: Dispersion

- Noting that dispersion is periodic $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta_0 \end{pmatrix}$$

- In an achromat, D = D' = 0. If we let $\delta_0 = 0$ we can simplify the process and solve to find

$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

$$(I - M) \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}}$$

  - Solving gives

$$\boxed{\begin{aligned} \eta(s) &= \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\Delta\phi)} \\ \eta'(s) &= \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\Delta\phi)} \end{aligned}}$$

- More on this later.

So how do we correct the chromaticity?

- Recall that we define the natural chromaticity as

$$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\,ds$$

- And that the trajectory goes as

$$x(s) = x_{\mathrm{betatron}}(s) + \eta_x(s)\delta$$

- If we describe the sextupole B field as $B_y = b_2 x^2$, we can then break it down as

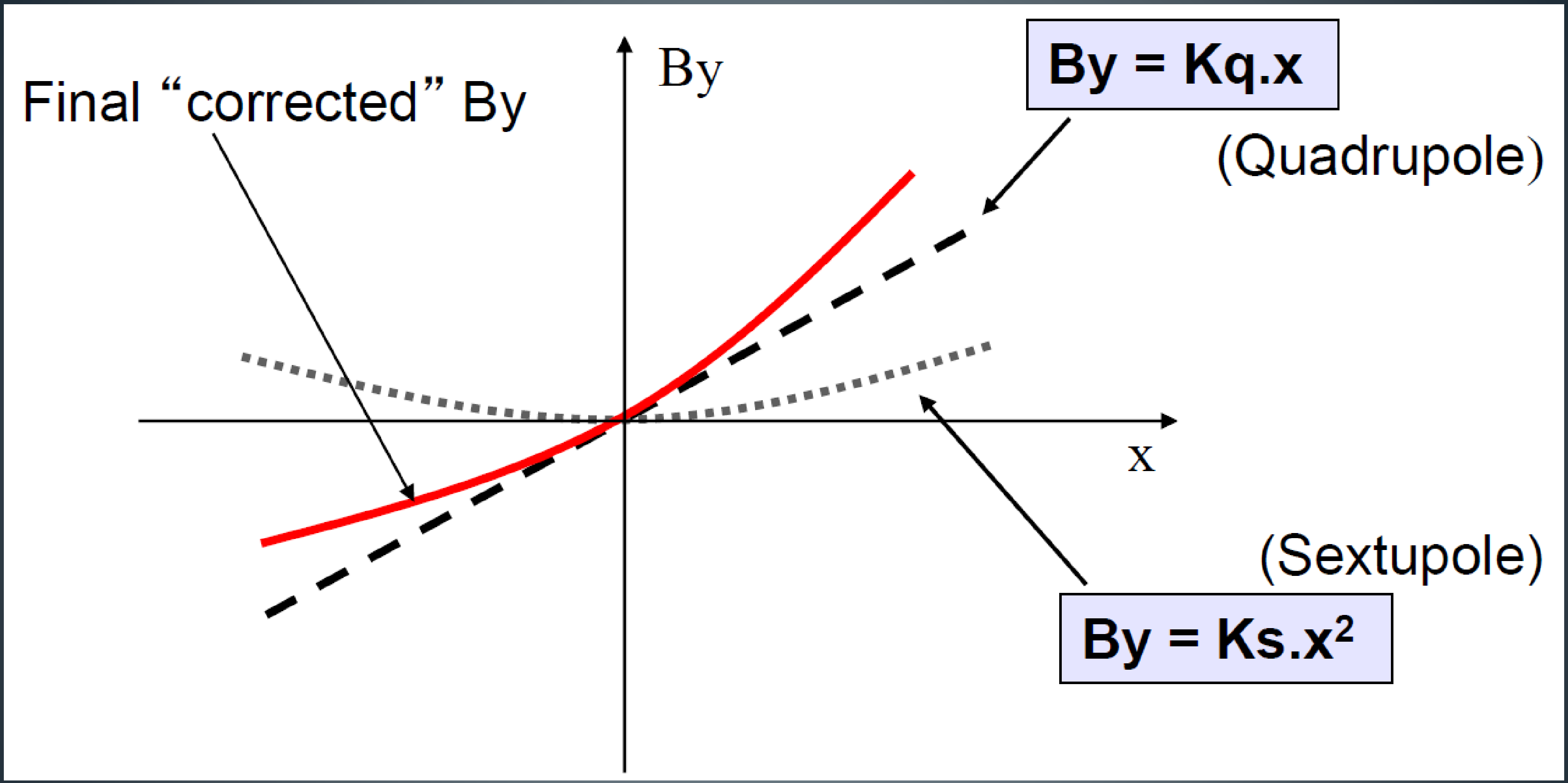
$$B_y(\mathrm{sext}) = b_2[x_{\mathrm{betatron}}(s) + \eta_x(s)\delta]^2 \approx \underbrace{b_2 x^2_{\mathrm{betatron}}}_{\text{Nonlinear}} + \underbrace{2b_2 x_{\mathrm{betatron}}(s)\eta_x(s)\delta}_{\text{Like quad: K(s)}}$$

- You end up getting a total chromaticity from all sources as

$$\xi = -\frac{1}{4\pi Q}\oint [K(s) - b_2(s)\eta_x(s)]ds$$

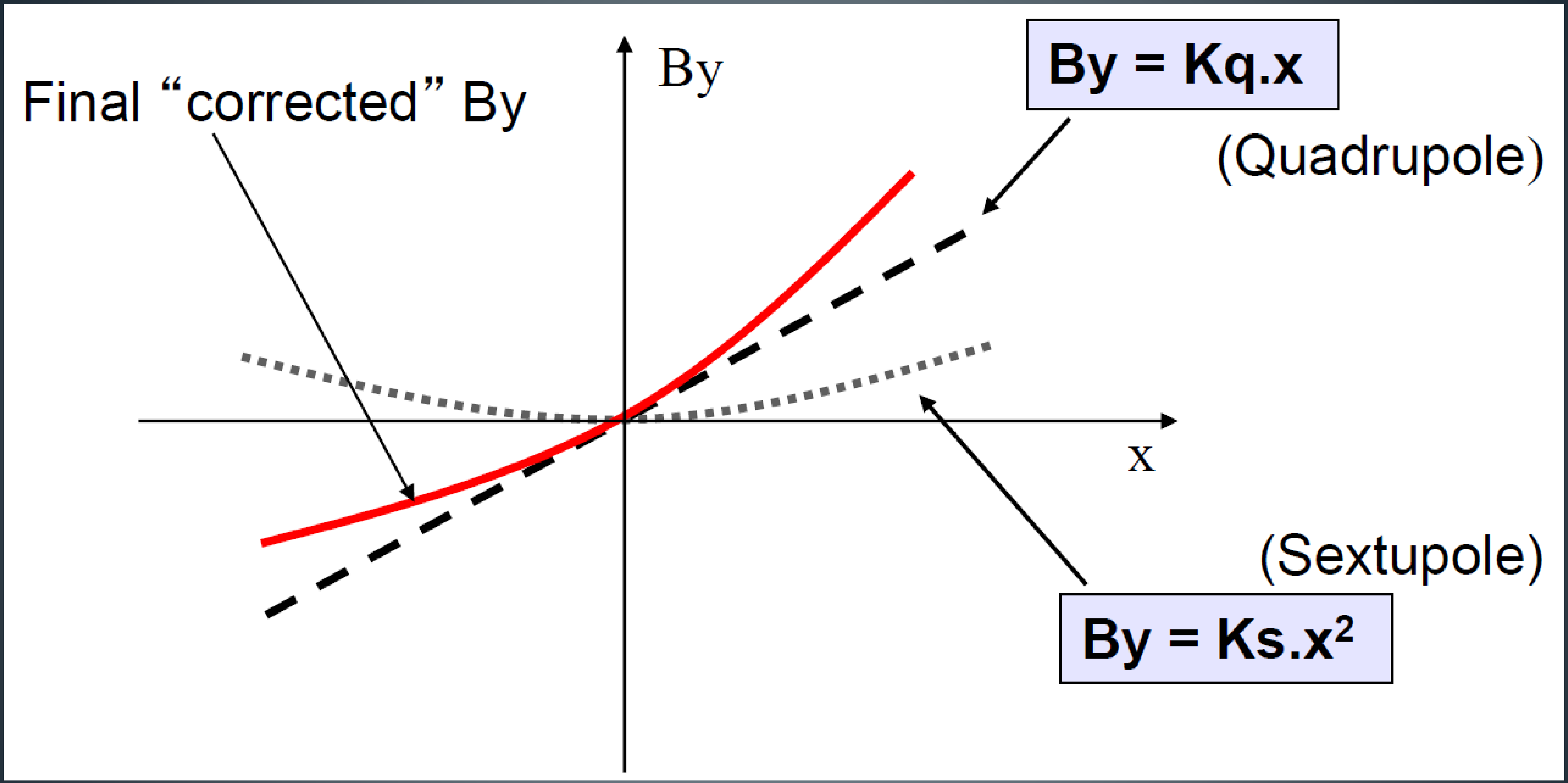
Notice that this means strong focusing (large K) requires large sextupoles!

Maybe with a bit less math?



- Here, the sextupole field acts to increase the quadrupole magnetic field for particles that have a positive displacement, and decrease the field for particles with negative displacements.

Maybe with a bit less math?



- Since the dispersion describes how the momentum changes the radial position of the particles, the sextupoles will alter the focusing field seen by the particles as a function of momentum.

How are we for time?

Got some more in you?

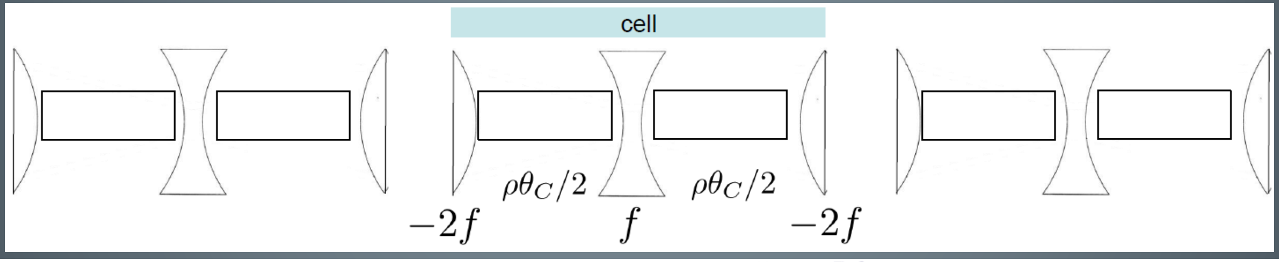
A bit more on Dispersion – For a FODO Cell

- Recall from before

$$\eta(s) = \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\Delta\phi)}$$

$$\eta'(s) = \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\Delta\phi)}$$

- A periodic lattice without dipoles has no **intrinsic** dispersion. If we consider a FODO lattice with long dipoles and thin quads, which is one of many in a large accelerator so that $\theta_C \ll 1$:



$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\text{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\text{FODO}} = M_{-2f} M_{\text{dipole}} M_f M_{\text{dipole}} M_{-2f}$$

$$M_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

A bit more on Dispersion – For a FODO Cell

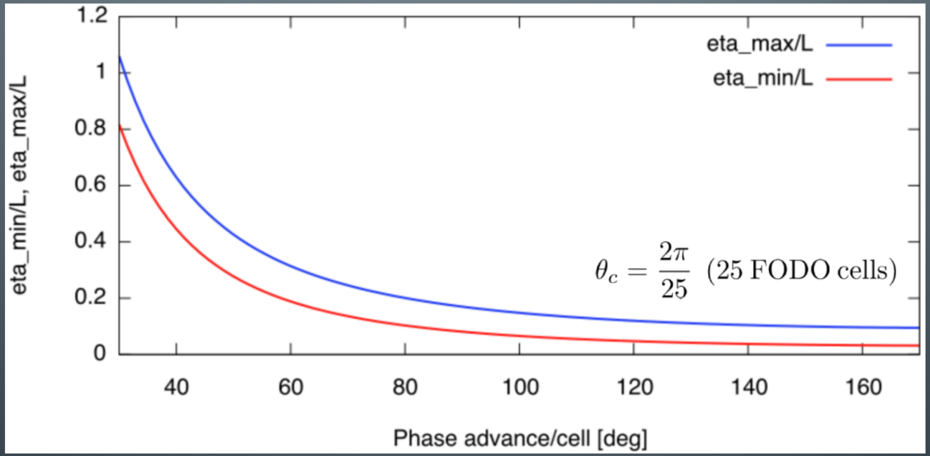
- Since we defined the periodicity based from a focusing quad center, we get

$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right] \qquad \eta'_x = 0 \text{ at max}$$

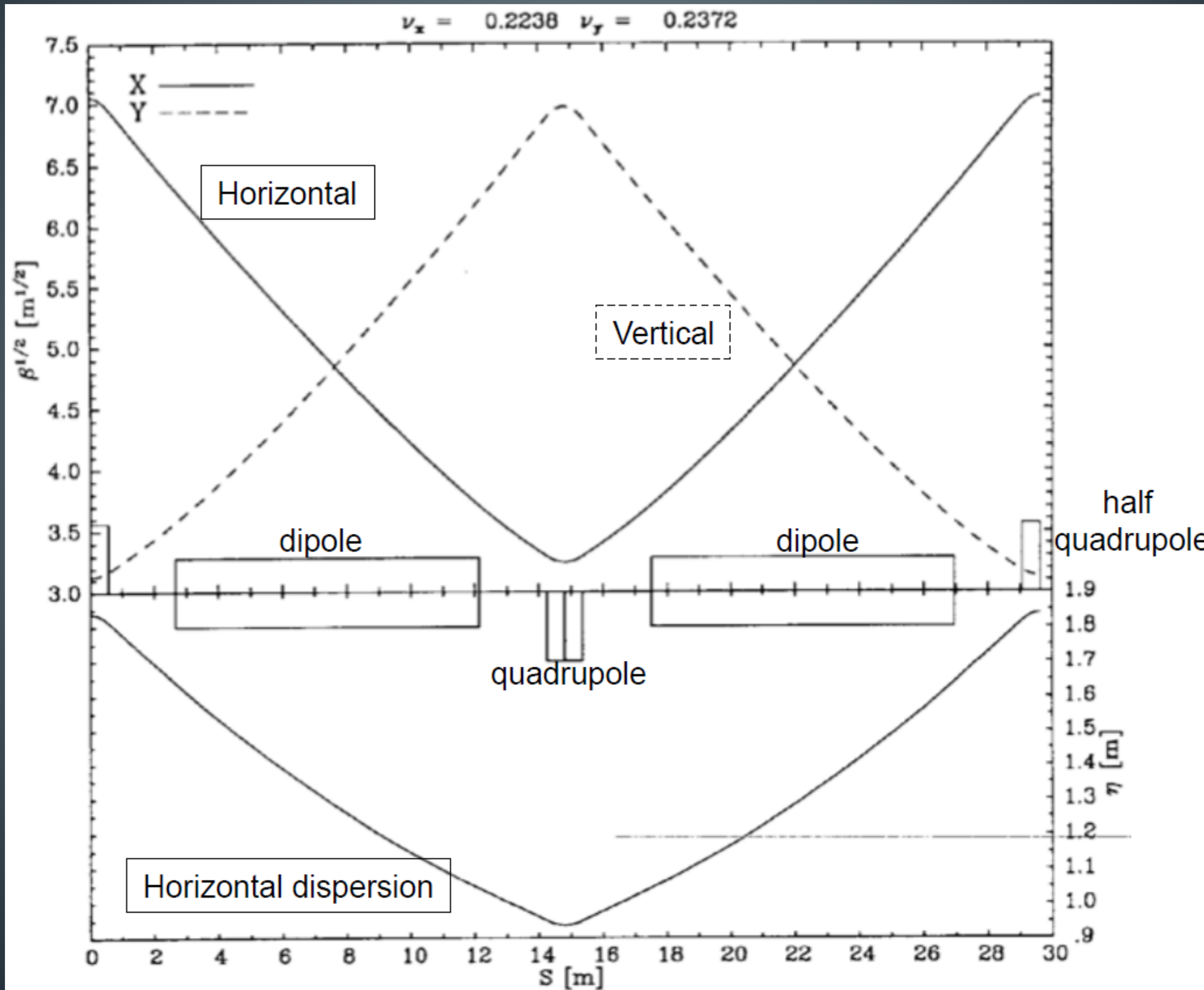
- If we change the periodicity to be based from defocusing quad centers, this becomes

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right]$$

- If we plot $\eta_{max}/_L$ and $\eta_{min}/_L$ against the phase advance/cell:



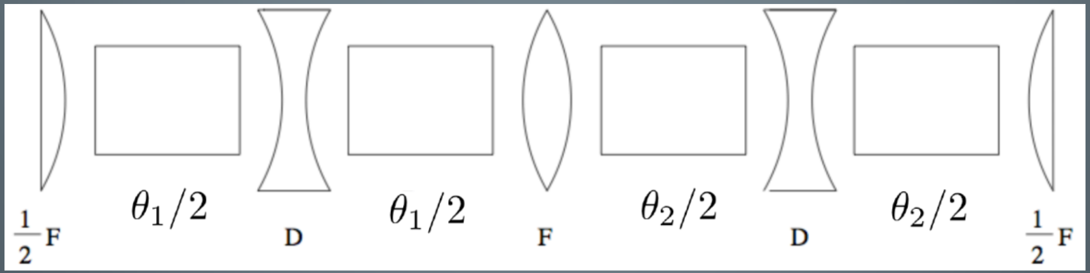
An example: RHIC FODO Cell



How to suppress dispersion

- Often, in straight sections, you want to make $\eta_x = \eta'_x = 0$.
  - Perhaps to keep the beam small in a wiggler in a light source

- However, the FODO dispersion solution is non-zero everywhere. In order to address this, you can match between these two sections with a non-periodic set of magnets called a dispersion suppressor.



- Here we have two FODO cells with different bend angles, but the same quad focusing so that β and Δϕ remain (mostly) the same.
- The goal is to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0{,}0)$.
- For simplicity, let's make $\alpha_x = 0$ at each end.

A little math for the dispersion suppressor

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\Delta\phi_x & \beta_x \sin 2\Delta\phi_x & D(s) \\ -\frac{\sin 2\Delta\phi_x}{\beta_x} & \cos 2\Delta\phi_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

$$D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \qquad D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

$$\hat{\eta}_x = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)(\theta_1 + \theta_2)$$

$$\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta$$

$$\theta = \theta_1 + \theta_2$$

- So, we have two cells, but one FODO bend angle with reduced bending.

What did we talk about?

- Introductory stuff.
- Resonance and resonant conditions
  - What is it?
  - Tune
    - Integer Tune
    - Tune Diagrams
- Chromaticity
  - Relationship to Tune
  - Chromaticity Correction
- Dispersion
  - Relationship to chromaticity and tune
  - IF TIME: FODO Dispersion and Dispersion Correction

Wrapping up

- This is only an introduction to what is a very complex topic.
  - You should DEFINITELY read more about this in the references listed before, as well as your book.
  - Look at these notes, plus Emmanuel's notes and the ones on Todd's website.

- Hopefully, these overlap well with your assignments.

References

- Lecture notes from MePAS 2011, Lectures 4, 5, 6, and 7 - http://www.toddsatogata.net/2011-MePAS/
- Lecture notes from USPAS 2013 (Waldo MacKay), Lecture "Resonances I" - http://www.toddsatogata.net/2013-USPAS/2013-01-23-Resonances1.pdf
- JAI Graduate Physics 2015 – Lecture 8 Notes from Emmanuel Tsesmelis
- Particle Accelerator Physics – Helmut Wiedemann
- Accelerator Physics – SY Lee
- Handbook of Accelerator Physics and Engineering – Alex Chao, Maury Tigner