# Transverse Dynamics I

JAI Accelerator Physics Course
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# Acknowledgements

These lectures have been produced with the advice and some content from Ted Wilson, whose book is the main text for this course.

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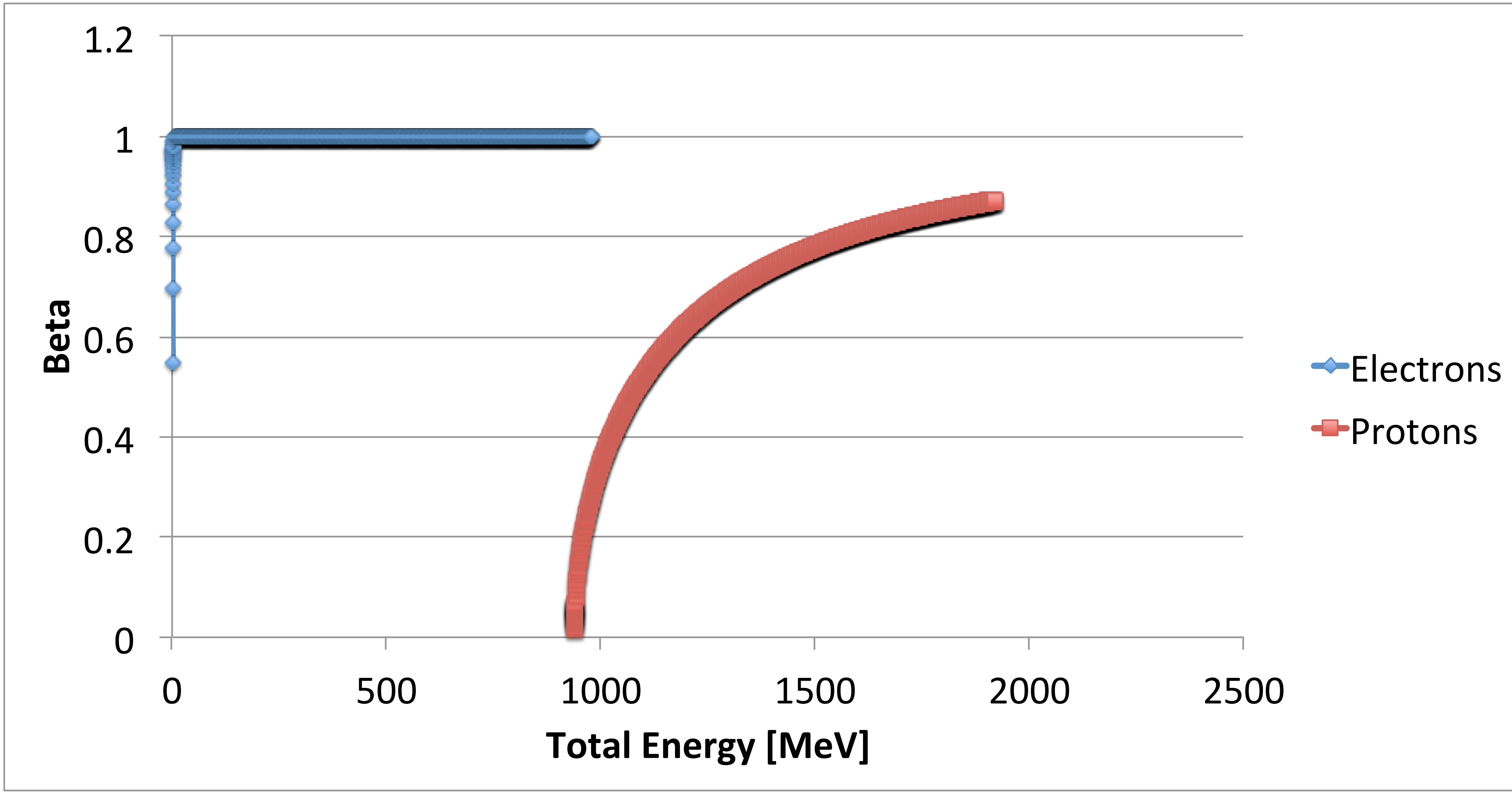
- Reminder: relativity
- Magnetic rigidity
- Transverse dynamics in a cyclotron
- AVF cyclotrons
- Synchrotrons - weak focusing
- Magnet types and multipoles
- Synchrotrons - strong focusing

# Reminder: relativity




Can keep gaining in energy, but the velocity no longer increases…

Question: To calculate the bending in an accelerator, do we care about energy, velocity or momentum??

# Reminder: relativity

| | $\beta$ | $cp$ | $T$ | $E$ | $\gamma$ |
|---|---|---|---|---|---|
| $\beta =$ | $\beta$ | $\frac{cp/E}{\sqrt{(E_0/cp)^2+1}}$ | $\sqrt{1-(1+T/E_0)^{-2}}$ | $\sqrt{1-(E_0/E)^2}=\frac{cp}{E}$ | $\sqrt{1-\gamma^{-2}}$ |
| $cp =$ | $E_0/\sqrt{\beta^{-2}-1}$ $=E\beta$ | $cp$ | $[T(2E_0+T)]^{1/2}$ $=T((\gamma+1)/(\gamma-1))^{1/2}$ | $\sqrt{E^2-E_0^2}=E\beta$ | $E_0\sqrt{\gamma^2-1}$ |
| $E_0 =$ | $cp/\beta\gamma$ $=E(1-\beta^2)^{1/2}$ | $cp(\gamma^2-1)^{-1/2}$ | $T/(\gamma-1)$ | $\sqrt{E^2-c^2p^2}$ | $E/\gamma$ |
| $T =$ | $\left[\frac{1}{\sqrt{1-\beta^2}}-1\right]E_0$ | $\sqrt{E_0^2+c^2p^2}-E_0$ $=cp\big((\gamma-1)/(\gamma+1)\big)^{1/2}$ | $T$ | $E-E_0$ | $E_0(\gamma-1)$ |
| $\gamma =$ | $(1-\beta^2)^{-1/2}$ | $\frac{cp}{E_0\beta}=\left[1-\left(\frac{cp}{E_0}\right)^2\right]^{1/2}$ | $1+T/E_0$ | $E/E_0$ | $\gamma$ |

Adapted from pp.7 Chao & Tigner, Handbook of Accelerator Physics & Engineering

# Magnetic Rigidity

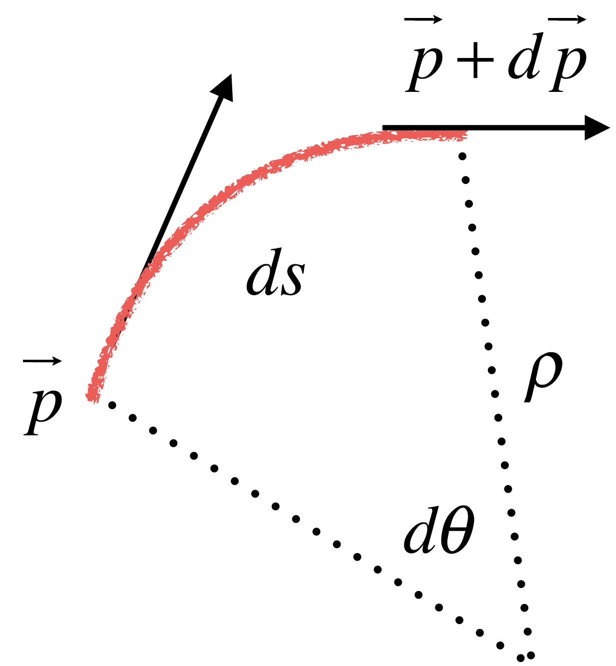
- A very useful quantity in accelerator physics, gives a measure of how hard it is to bend particles of a certain momentum

Lorentz force

$$F = q\vec{E} + q\vec{v} \times \vec{B}$$

In presence of perpendicular B field

$$F = q\frac{ds}{dt}B \qquad [1]$$




$$\frac{1}{\rho} = \frac{d\theta}{ds} \qquad [2]$$

$$\frac{d\vec{p}}{dt} = |p|\frac{d\theta}{dt}$$

$$\frac{d\vec{p}}{dt} = |p|\frac{d\theta}{ds}\frac{ds}{dt} \qquad \text{sub in [2]}$$

$$\frac{d\vec{p}}{dt} = \frac{|p|}{\rho}\frac{ds}{dt} \qquad [3]$$

Using [1] and [3] we get:

$$B\rho = \frac{pc}{qc} \qquad [4a]$$

In useful units:

$$B\rho[T.m] = 3.3356 \cdot pc[GeV] \qquad [4b]$$

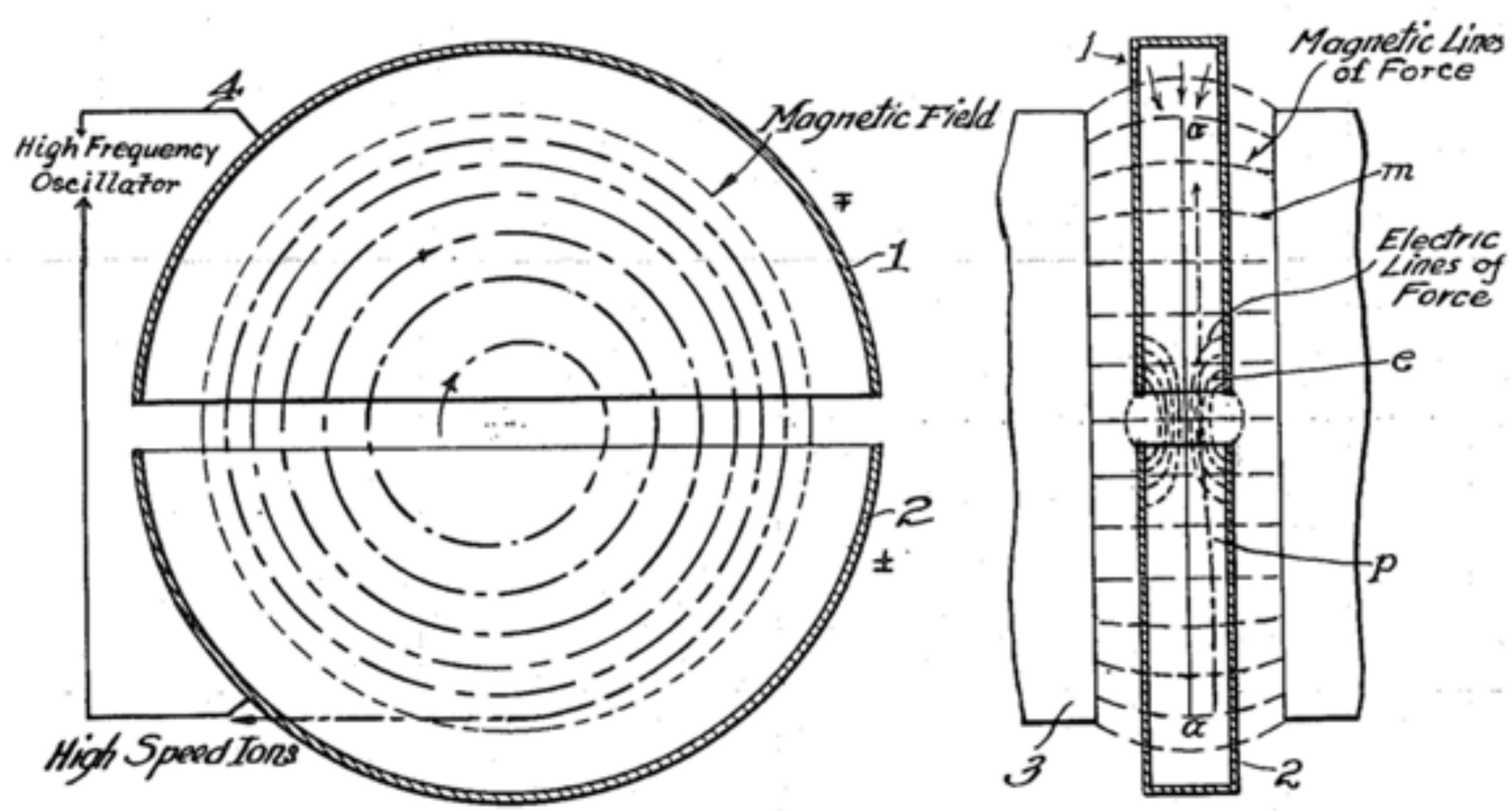
# Cyclotrons - transverse

In a constant field, a charged particle executes a circular orbit, with radius ρ and frequency ω

$$\omega_0 = qB_z / m$$

$$\rho = mv / qB_z$$

$$\omega_0 = v_\theta / \rho$$

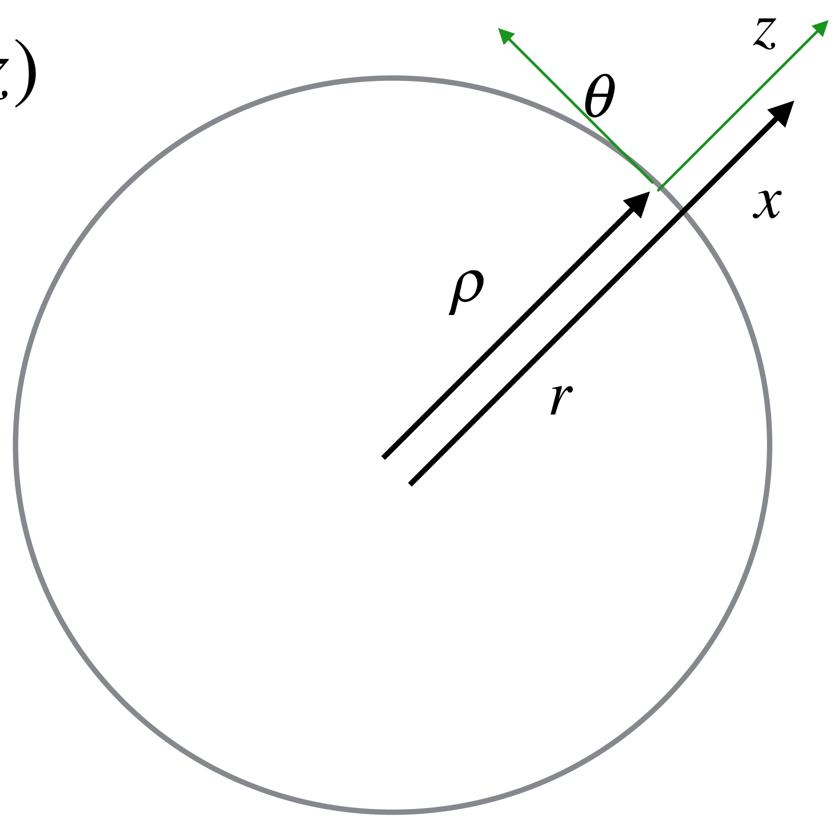



The Cyclotron, from E. Lawrence's 1934 patent

# Weak focusing in cyclotrons

Steenbeck 1935, Kerst and Serber 1941

$(r,\theta,z)$




Closed orbit in median plane

$x$ is a small orbit deviation

$$r = \rho + x = \rho(1 + x/\rho) \tag{5}$$

Expand B field around orbit:

$$B_z = B_{z,0} + \frac{\partial B_z}{\partial x} x$$

Define field index:

$$k = -\frac{1}{B_{z,0}\rho}\frac{\partial B_z}{\partial x} \tag{6}$$

Therefore:

$$B_z = B_{z,0}\left(1 - k\rho x\right) \tag{7}$$

$$n = -\frac{\rho}{B_{z,0}}\frac{\partial B_z}{\partial x}$$

nb. same derivation often uses n.

Looking at the horizontal restoring force

$$F_x = \frac{mv_\theta^2}{\rho} - qv_\theta B_z \qquad (8)$$

(centrifugal force - magnetic force)

And combining (5) and (7) we end up with (in the n-formulation):

$$F_x = -\frac{mv_\theta^2}{\rho}\frac{x}{\rho}(1-n)$$

We could just go from here to this equation of motion (after some expansions/ approximations).

$$\ddot{x} + \frac{v_\theta^2}{\rho^2}(1-n)x = 0 \qquad (9)$$

or $$\ddot{x} + \omega^2 x = 0$$

$$\omega = \omega_0\sqrt{1-n}$$

For horizontal stability, we require $n < 1$

For vertical stability (see later), we require $n > 0$

# Alternative (equivalent) formulation…

Alternatively (cf. Ted Wilson), let's start with the equation of motion in cylindrical co-ordinates (from Lorentz force) in theta…

$$\frac{d(mr\dot{\theta})}{dt} + m\dot{r}\dot{\theta} = q[\dot{z}B_r - \dot{r}B_z] \quad (10)$$

if particles have same velocity $\rho\dot{\theta} = v_0 = \dot{z}$

$$\frac{d}{dt}\left(m\frac{d\rho}{dt}\right) + \frac{m{v_0}^2}{\rho} + ev_0 B_z = 0$$

Substituting for small variations and changing from t to s:

$$\frac{d}{dt} = v_0 \frac{d}{ds} \qquad \Delta B_z = B_z - B_0 \qquad x = \rho - \rho_0$$

We get:

$$\frac{1}{mv_0}\frac{d}{ds}\left(p_0 \frac{dx}{ds}\right) + \frac{x}{{\rho_0}^2} + \frac{1}{\rho_0}\frac{\Delta B_z}{B_0} = 0 \quad (11)$$

# Alternative (equivalent) formulation…

Taylor expand field about the orbit…

$$B_z = B_0 + \frac{\partial B_z}{\partial x} x$$

Define field index as before

$$k = -\frac{1}{B_0 \rho} \frac{\partial B_z}{\partial x}$$

This gives horizontal focusing:

$$\boxed{\frac{1}{p_0} \frac{d}{ds}\left( p_0 \frac{dx}{ds} \right) + \left( \frac{1}{\rho^2} - k \right) x = 0}$$

Harmonic motion with oscillations per turn:

$$Q_x = \sqrt{\frac{1}{\rho^2} - k}, \qquad Q_z = \sqrt{k}$$

http://acceleratorinstitute.web.cern.ch/acceleratorinstitute/ACINST/

# Weak focusing in cyclotrons

- In reality, have a slightly decreasing field with radius

$$n = \rho^2 k \qquad\qquad 0 \le n \approx -\frac{\partial B_z}{\partial x} \le 1$$

- With relativity… for isochronicity we know we need:

$$B(r) = \gamma(r) B_0 \qquad \text{because} \qquad \omega_{rev} = \frac{qB(r)}{\gamma(r) m_0}$$

ie. need an increasing field (n<0)

which is not compatible with a decreasing field, n>0

# AVF cyclotron

## Thomas, 1938

Increase vertical focusing by introducing hills & valleys

This introduces a variation in $B_\theta$

We define the flutter factor

$$F = \frac{\langle B^2 \rangle - \langle B \rangle^2}{\langle B \rangle^2} \approx \frac{(B_{Hill} - B_{Valley})^2}{8 \langle B \rangle^2}$$

The betatron frequency turns out to be:

$$\nu_z^2 = n + \frac{N^2}{N^2 - 1} F + \ldots > 0$$

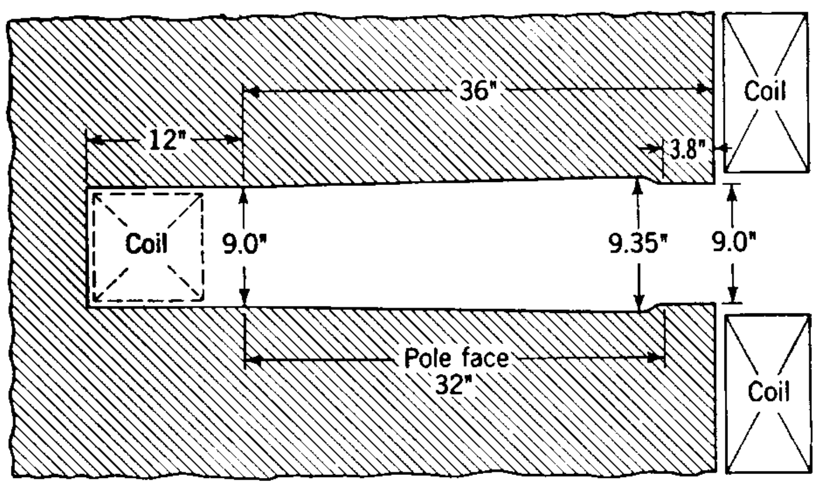
Focusing limit:

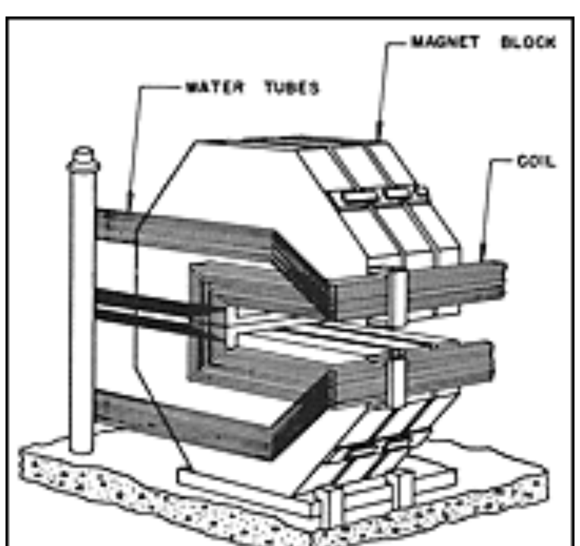
$$\frac{N^2}{N^2 - 1} F > -n = \gamma^2 - 1$$

Note: for high energies we want a large flutter factor, so B_valley = 0 -> separated sector cyclotron

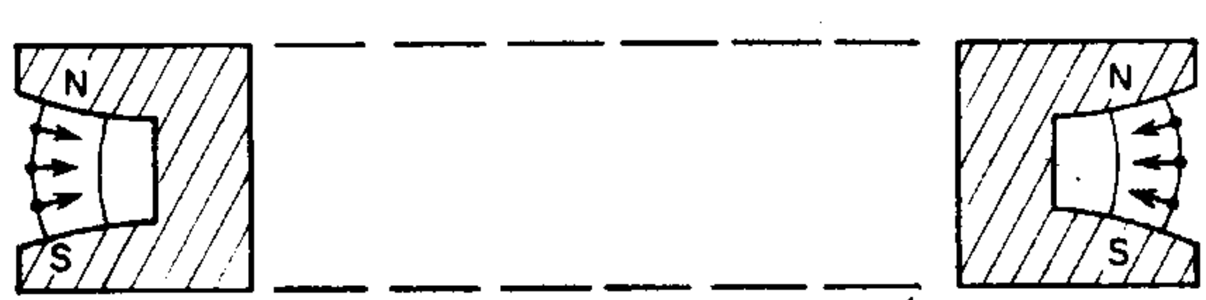
# Synchrotrons

"*Particles should be constrained to move in a circle of constant radius thus enabling the use of an annular ring of magnetic field … which would be varied in such a way that the radius of curvature remains constant as the particles gain energy through successive accelerations*" - Marcus Oliphant, 1943

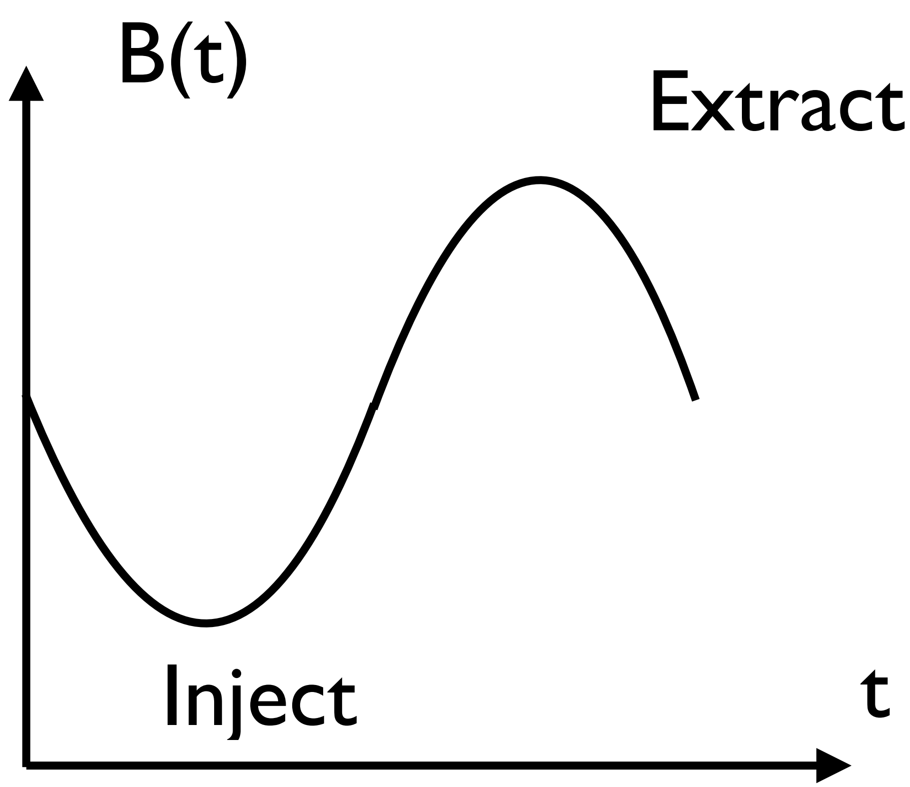



The Cosmotron, 3.3 GeV p+, BNL



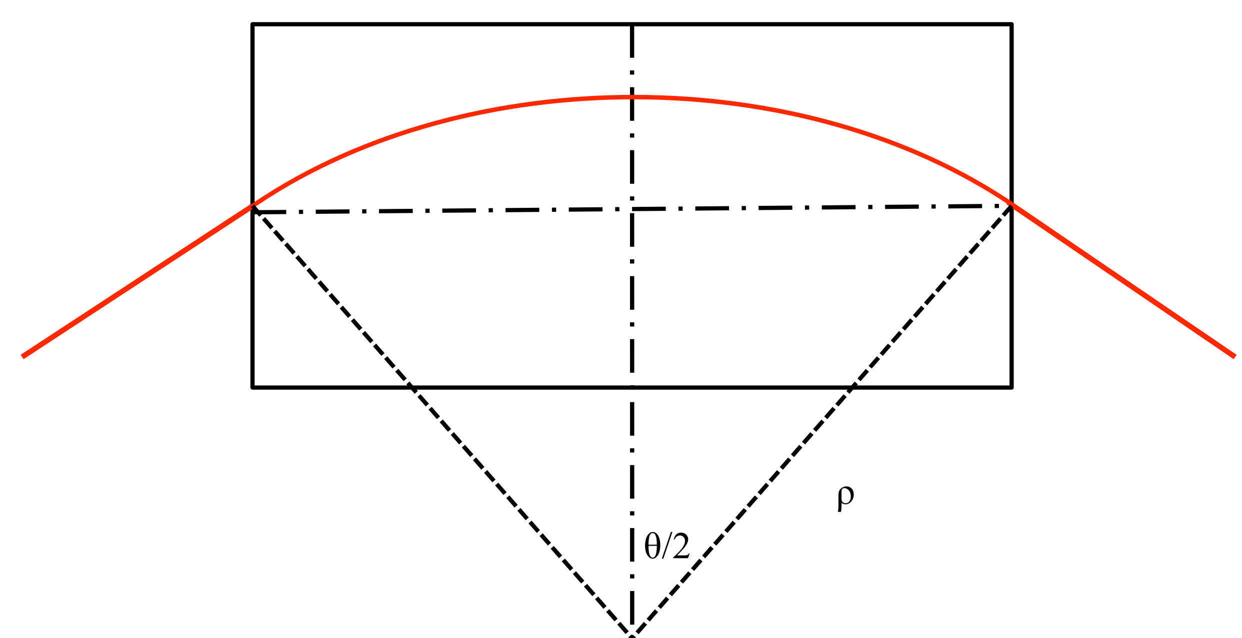
# Synchrotrons




Typical synchrotron magnet cycle

Bending angle in dipole magnet

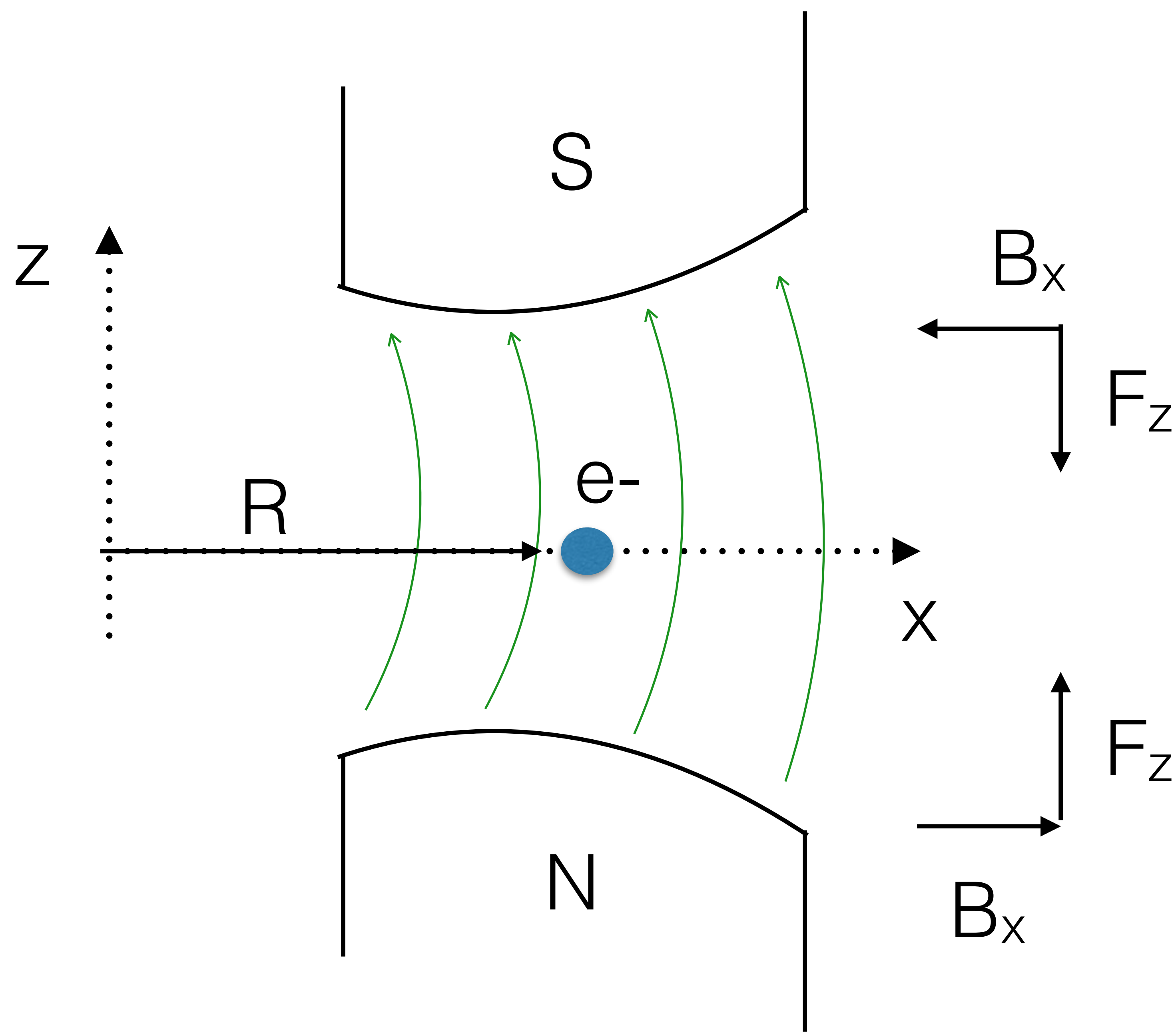
$$\sin(\theta/2) = \frac{B(t)L}{2(B(t)\rho)} \qquad \theta \approx \frac{B(t)L}{p(t)/q}$$



# Weak Focusing: Synchrotrons

- In vertical direction

For focusing in the vertical plane, we need a horizontal field component




$$\nabla \times \vec{B} = 0$$

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial r}$$

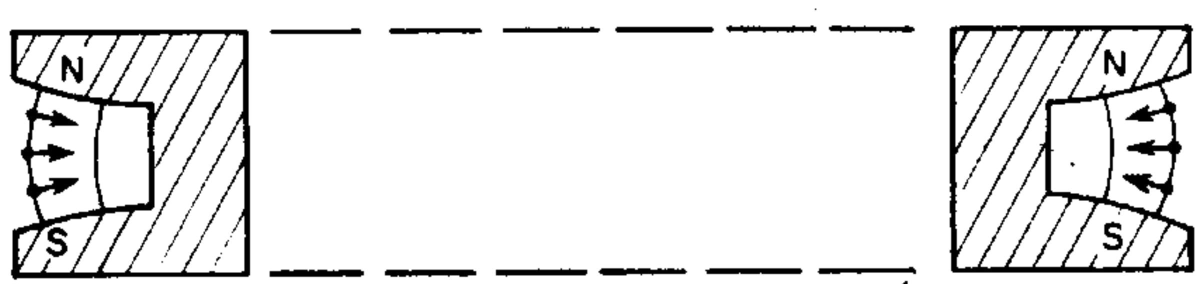
Require $\dfrac{\partial B_x}{\partial z} = \dfrac{\partial B_z}{\partial r} < 0$

so $n > 0$

i.e. decreasing field with radius

For both planes: $0 < n < 1$

The Cosmotron, 3.3 GeV p+, BNL




- Vertical focusing comes from the curvature of the field lines when the field falls off with radius ( positive n-value)

- Horizontal focussing from the curvature of the path, sometimes called ‘body focusing’

- The negative field gradient defocuses horizontally and must not be so strong as to cancel the path curvature effect

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# Magnetic Fields

- Maxwell's equations, time independent, no sources, so:

$$\vec{J} = 0$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\nabla \times \vec{B} = 0$$

$$\nabla \cdot \vec{B} = 0$$

- Consider a constant vertical field $B_z$, and

$$B_y + iB_x = C_n (x + iy)^{n-1}$$

- n is an integer > 0, C is a complex number
- (real part understood)

Now apply $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ to each side of $B_y + iB_x = C_n(x+iy)^{n-1}$

LHS:

$$= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} + i\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$= \left[\nabla \times \vec{B}\right]_z + i\nabla \cdot \vec{B}$$ Where we know $B_z$ is constant.

RHS:

$$= (n-1)(x+iy)^{n-2} + i^2(n-1)(x+iy)^{n-2} = 0$$

$$\therefore \quad \nabla \times \vec{B} = 0 \text{ and } \nabla \cdot \vec{B} = 0$$

So we find that as expected, the field $B_y + iB_x = C_n(x+iy)^{n-1}$ satisfies Maxwell's equations in free space

# Multipole fields

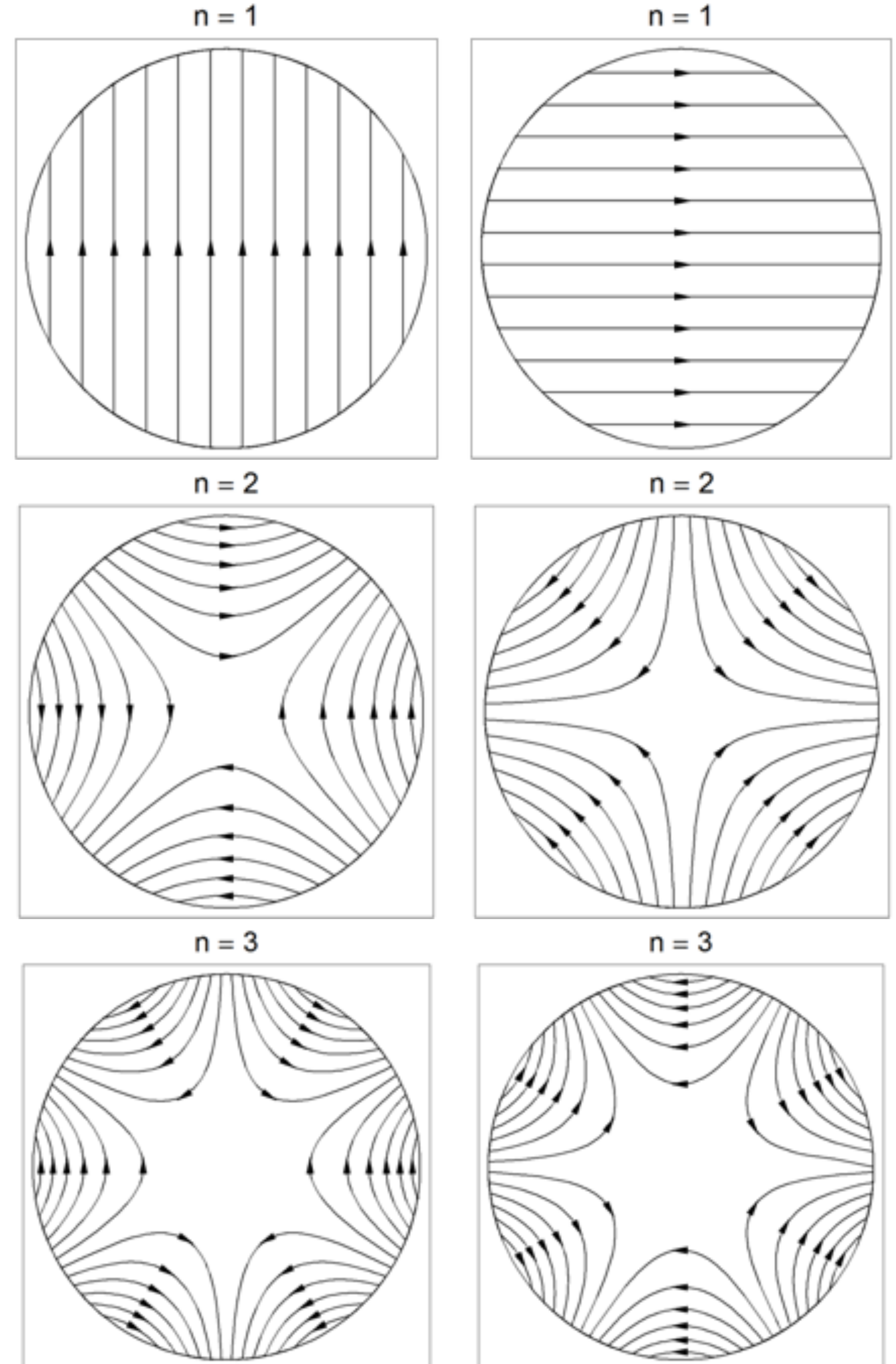
In the usual notation:

$$B_y + iB_x = B_{ref}\sum_{n=1}^{\infty}(b_n + ia_n)\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

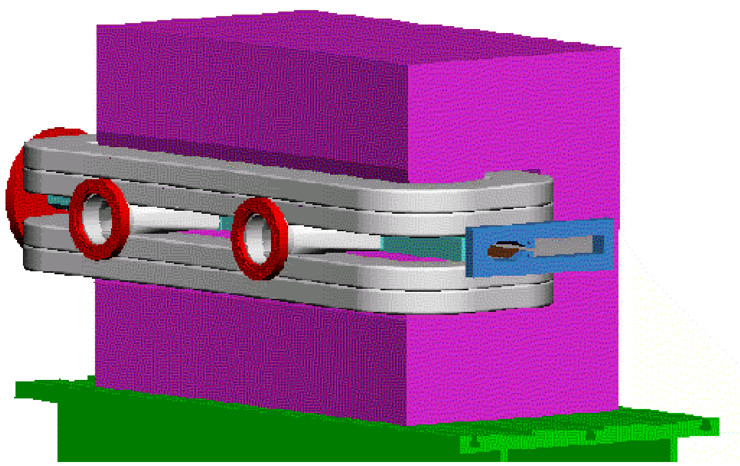
$b_n$ are "normal multipole coefficients" (LEFT)
and $a_n$ are "skew multipole coefficients" (RIGHT)
'ref' means some reference value

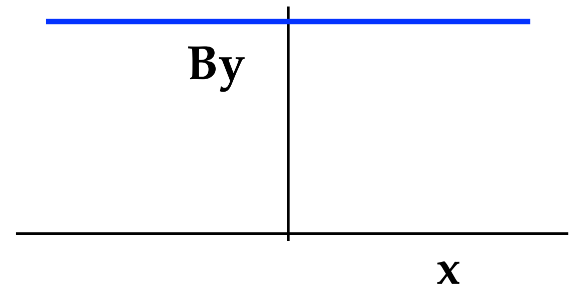
n=1, dipole field
n=2, quadrupole field
n=3, sextupole field

Images: A. Wolski, https://cds.cern.ch/record/1333874



# Multipole Magnets





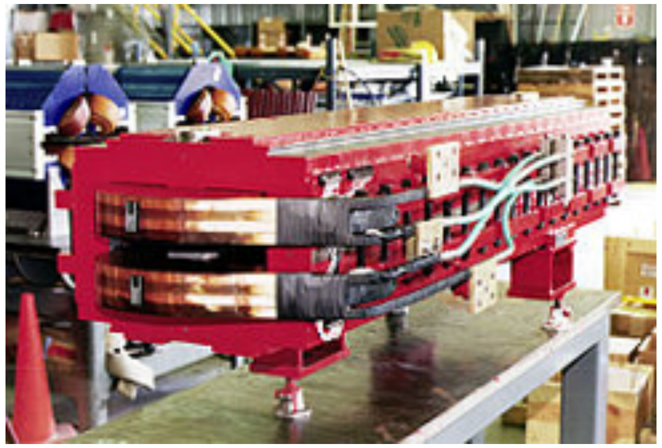
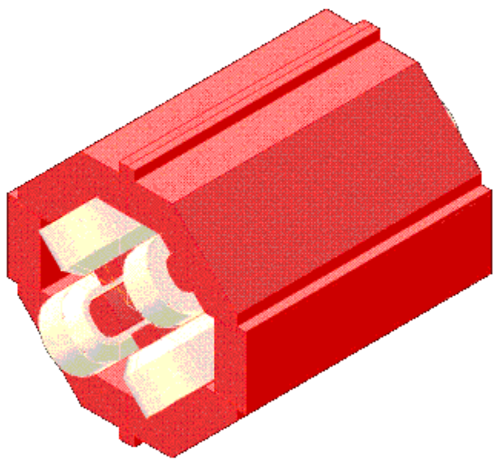
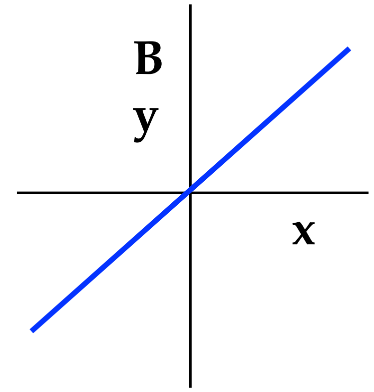



Image: Wikimedia commons





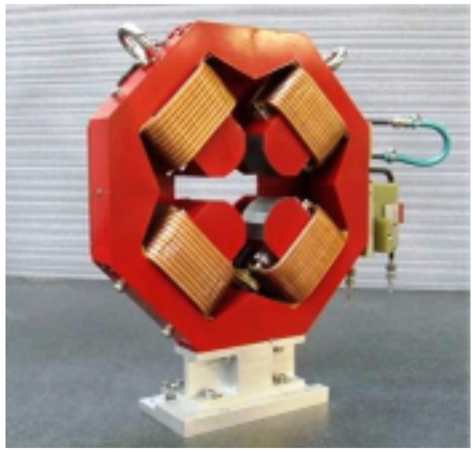
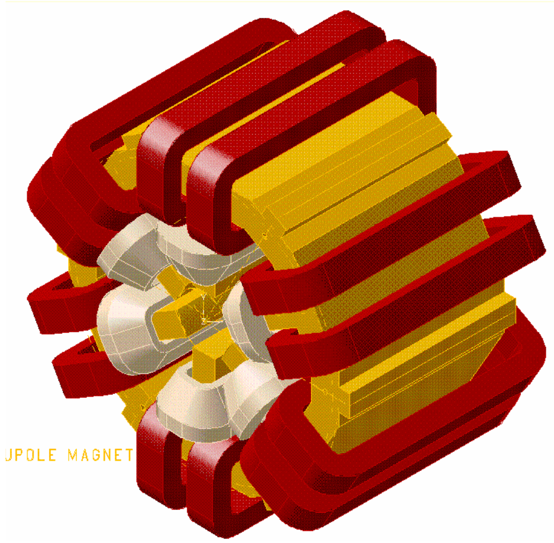
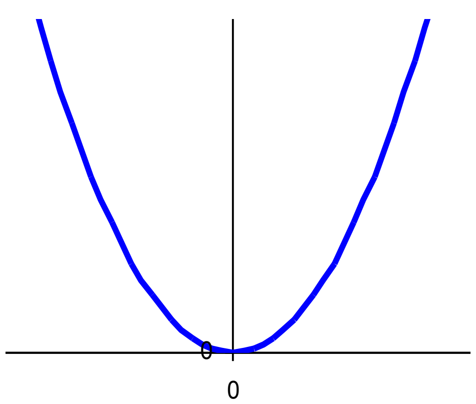



Image: STFC






Images: Ted Wilson, JAI Course 2012

Image: Danfysik

# Combined function magnets

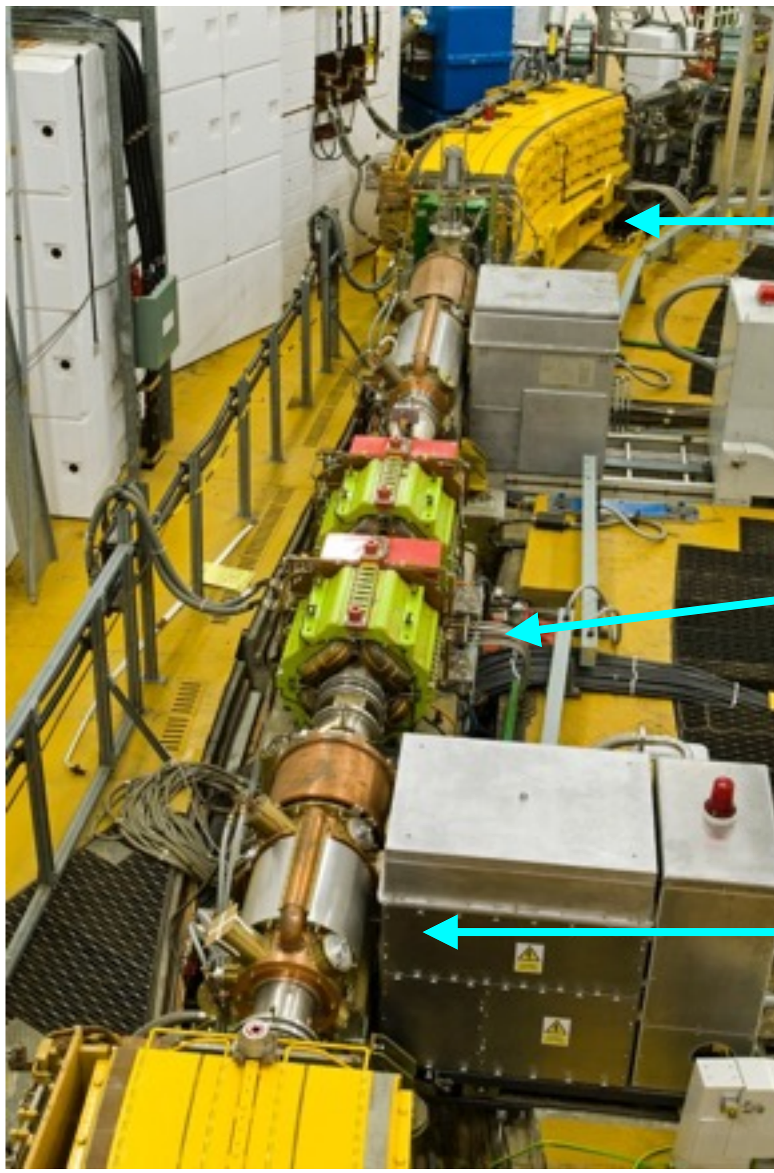
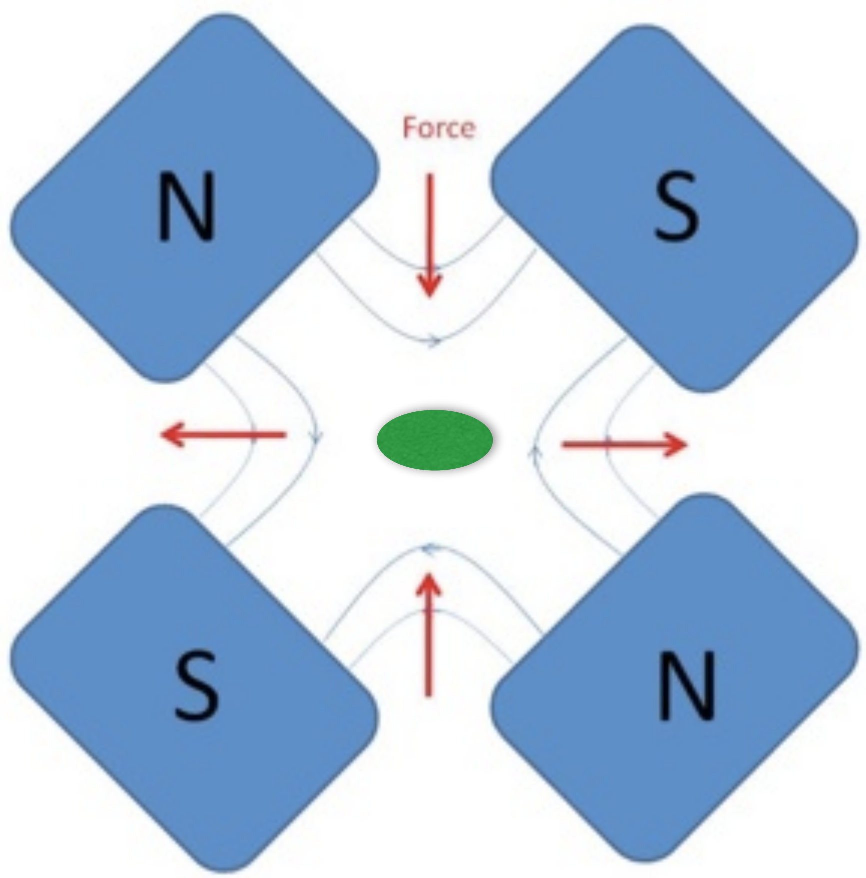
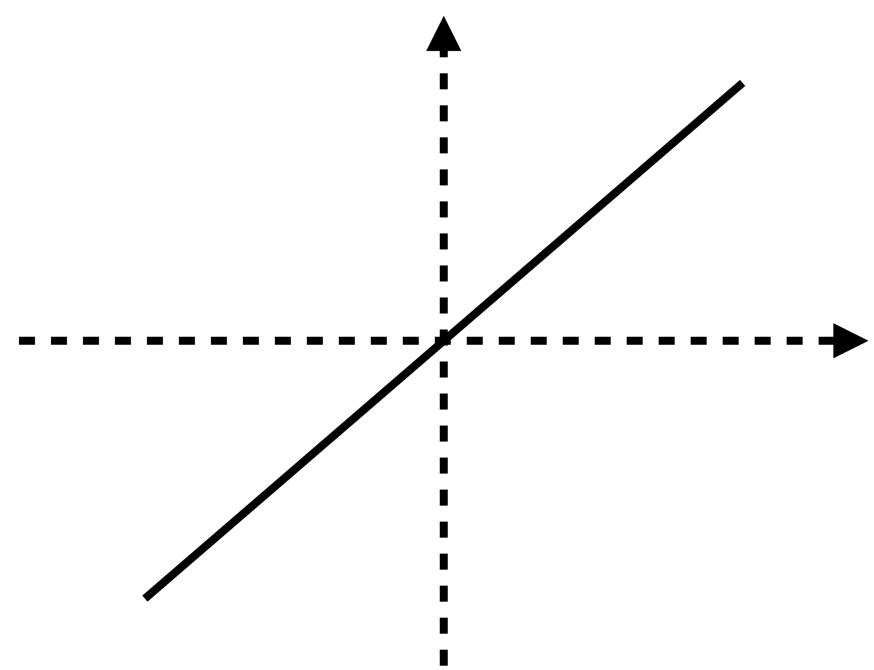



*Image courtesy of ISIS, STFC*

# Quadrupole focusing




$$B_y = gx$$



$$k = \frac{g}{p/q}$$

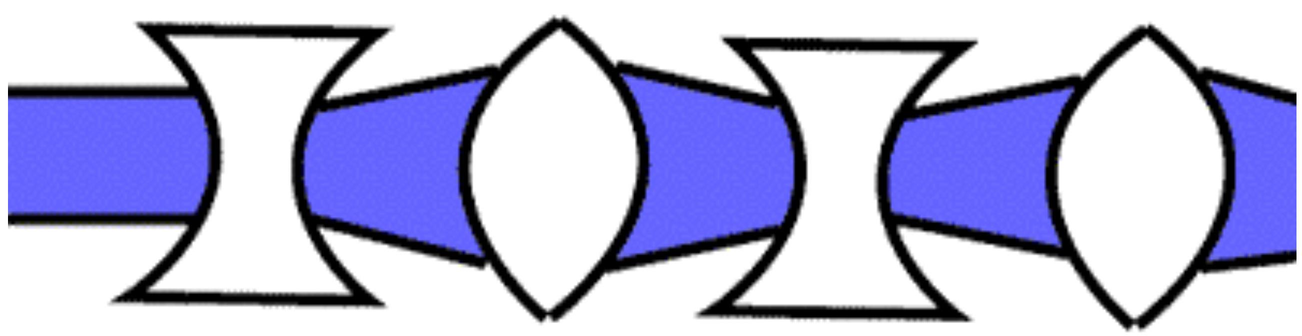
'normalised gradient' of quad

$$\frac{1}{f} = \frac{L(dB(t)/dx)}{p(t)/q}$$

'focal length'

# Synchrotrons - Alternating Gradient

- An issue: for large R the deviations from ideal orbit get very large. This meant large aperture and expensive magnets.
- Greater focusing was needed in both horizontal and vertical…
- *"What if some of the magnets in the cosmotron were reversed?"*



E. Courant realised that the focusing would be STRONGER & the magnets could be SMALLER!

- 1952: Courant, Livingston, Snyder publish about strong focusing

- 1954: Wilson et al. build first synchrotron with strong focusing for 1.1MeV electrons at Cornell, 4cm beam pipe height, only 16 Tons of magnets.

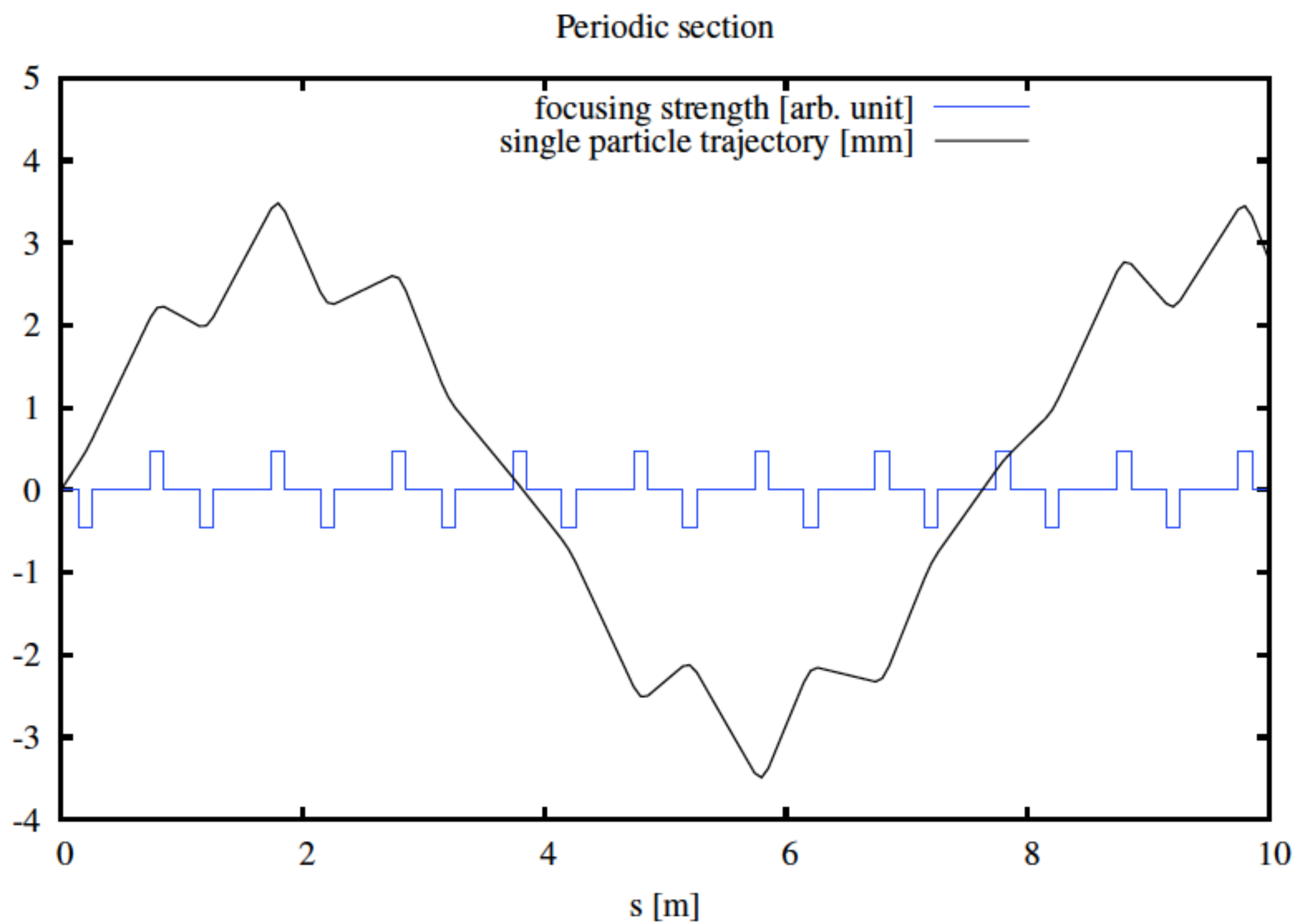
- 1959: CERN builds the PS for 28GeV after proposing a 5GeV weak focusing accelerator for the same cost (PS is still in use)

Historical note: Nicholas Christofilos

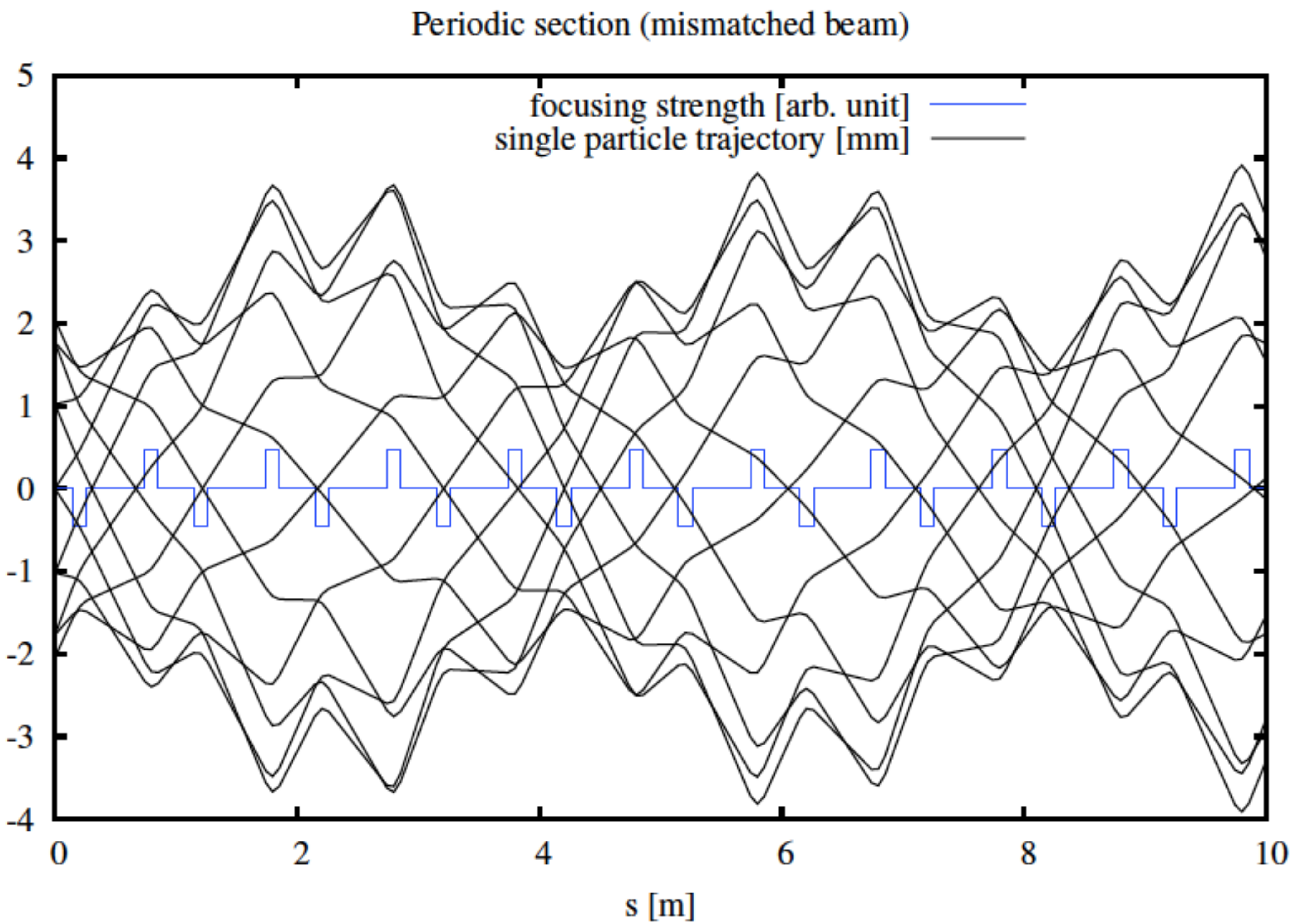
Greek physicist, had the AG idea in 1949, opted to patent it instead of publishing. He is often forgotten in physics books…

http://www.google.com/patents?vid=2736799

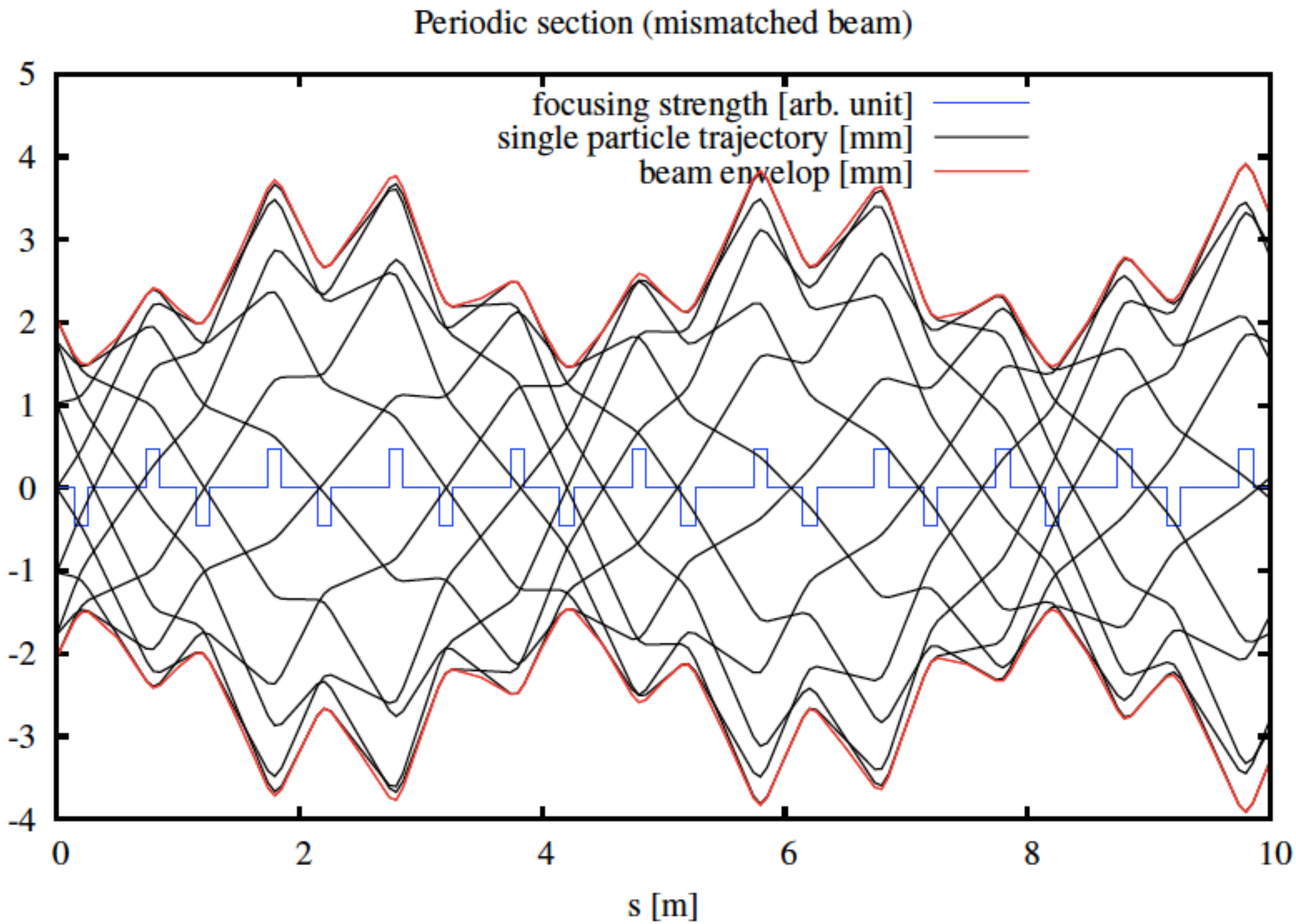
# Particle in AG focusing



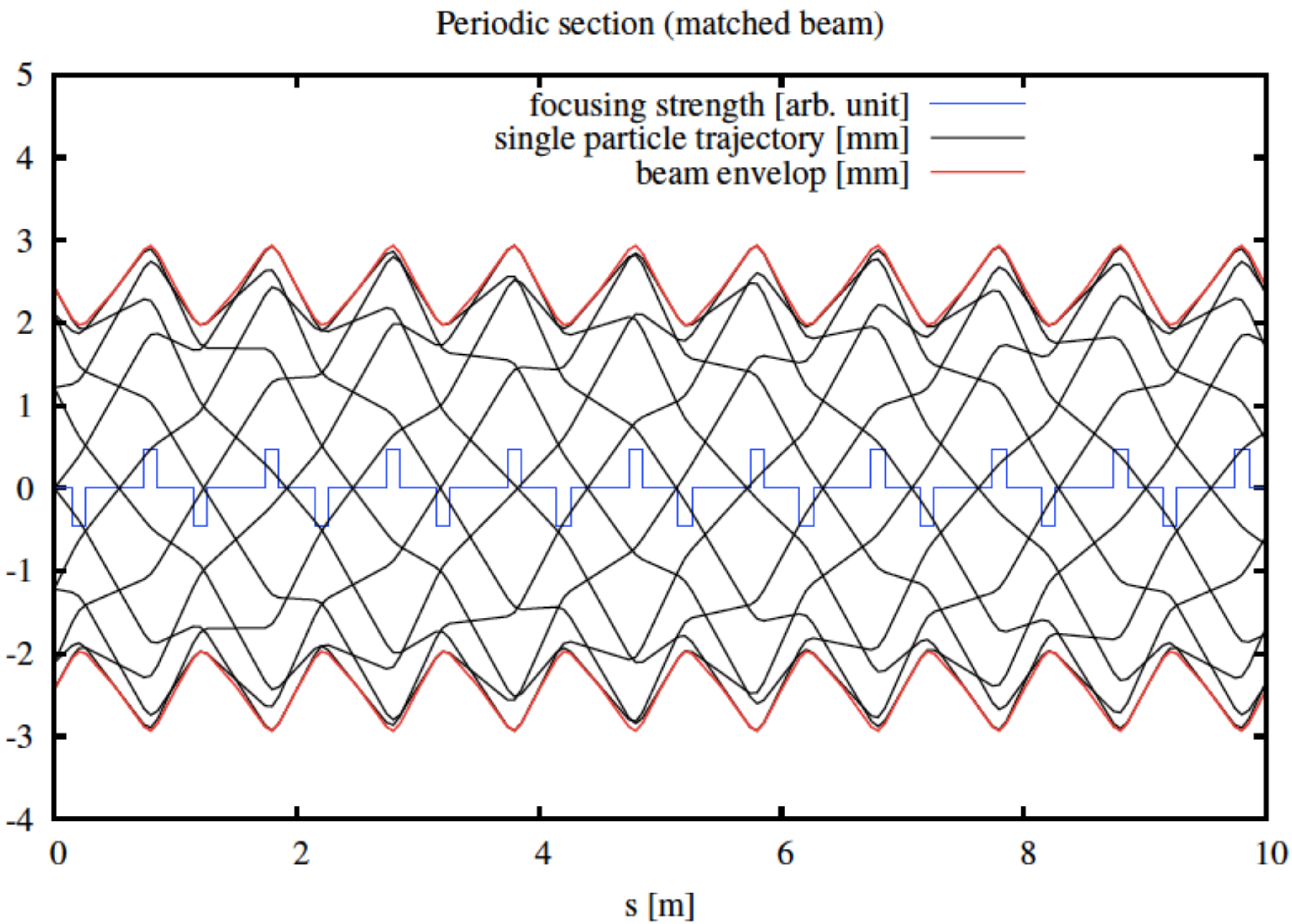
# Particle in AG focusing



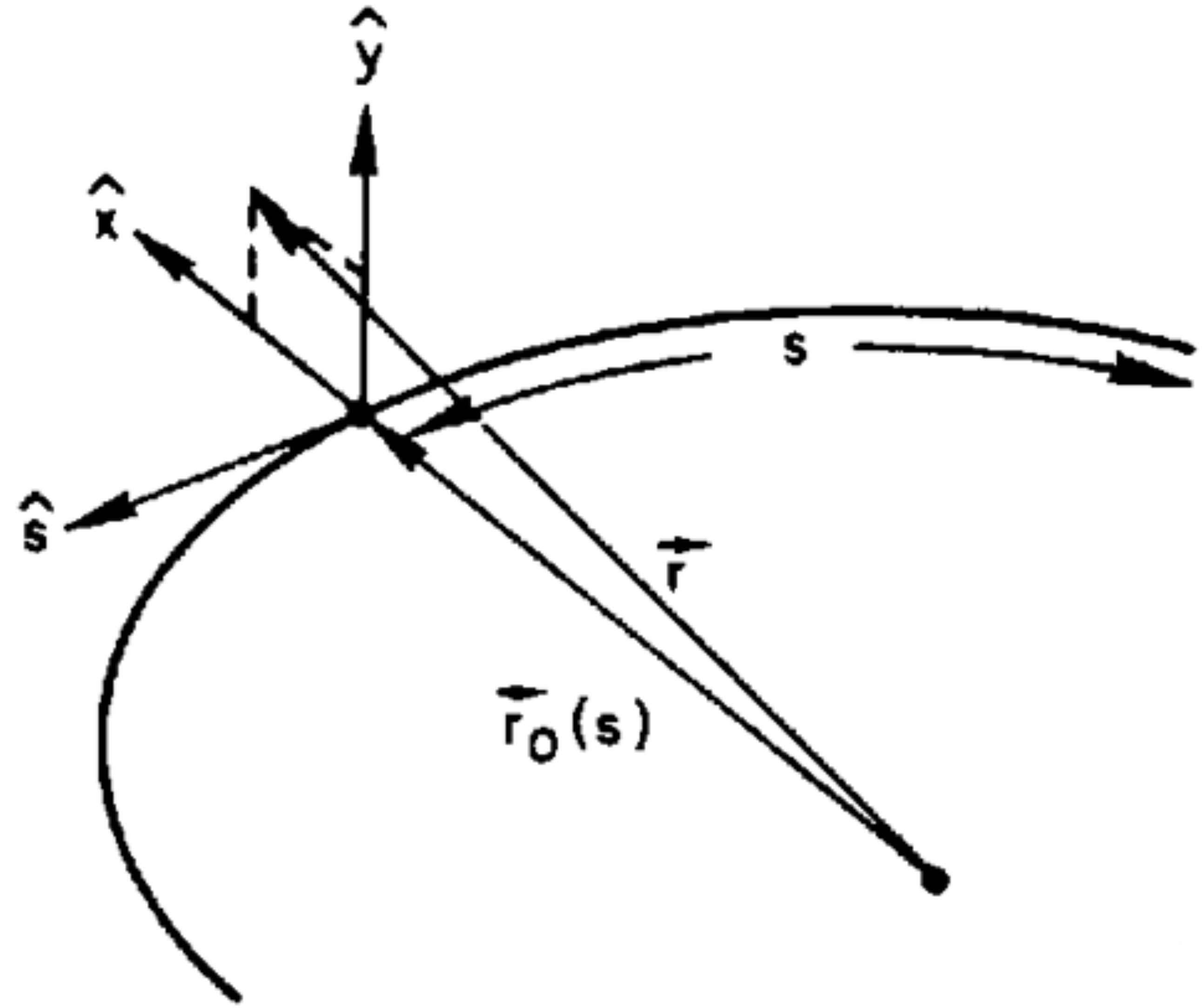
# Particle in AG focusing



# Particle in AG focusing



# Transverse co-ordinates




Particle motion is described with respect to a **reference orbit** in the non-inertial frame (x, y, s). This co-ordinate system is known as *Frenet-Serret*

# Hill's Equation (a first look)

**Hill's equation** is a linearised equation of motion describing particle oscillations:

$$\frac{d^2x}{ds^2}+k_x(s)x=0 \qquad \frac{d^2y}{ds^2}+k_y(s)y=0$$

Where k changes along the path, and

$$k_x(s)=\frac{1}{\rho^2}-\frac{B_1(s)}{B\rho} \qquad k_y(s)=\frac{B_1(s)}{B\rho} \qquad B_1(s)=\partial B_y/\partial x$$

evaluated at the closed orbit

Focusing functions are periodic over length L , ie. $K_{x,y}(s+L)=K_{x,y}(s)$

nb. In a quadrupole: $k_x(s)=-\frac{B_1(s)}{B\rho}$

Following similar notation to S. Y. Lee, Accelerator Physics, pp.41

E. D. Courant and H. S. Snyder, "Theory of the alternating-gradient synchrotron," Annals of Physics, vol. 3, no. 1, pp. 1–48, 1958.

# Solution of Hill's equation

(More next lecture…)

$$x = \sqrt{\beta(s)}\sqrt{\varepsilon}\,\sin[\phi(s) + \phi_0]$$

initial phase

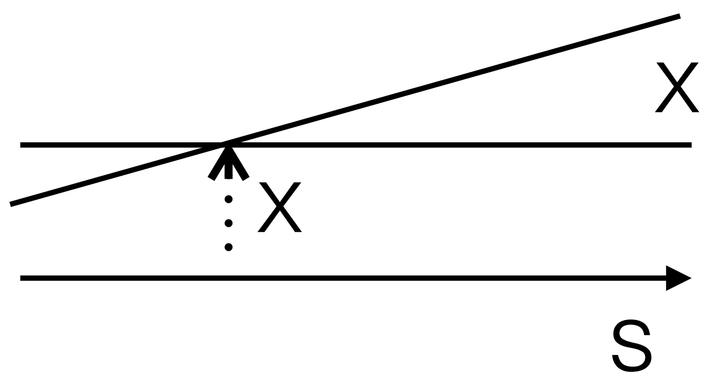
phase

betatron function
property of the machine
(not the beam)

emittance
(property of beam)

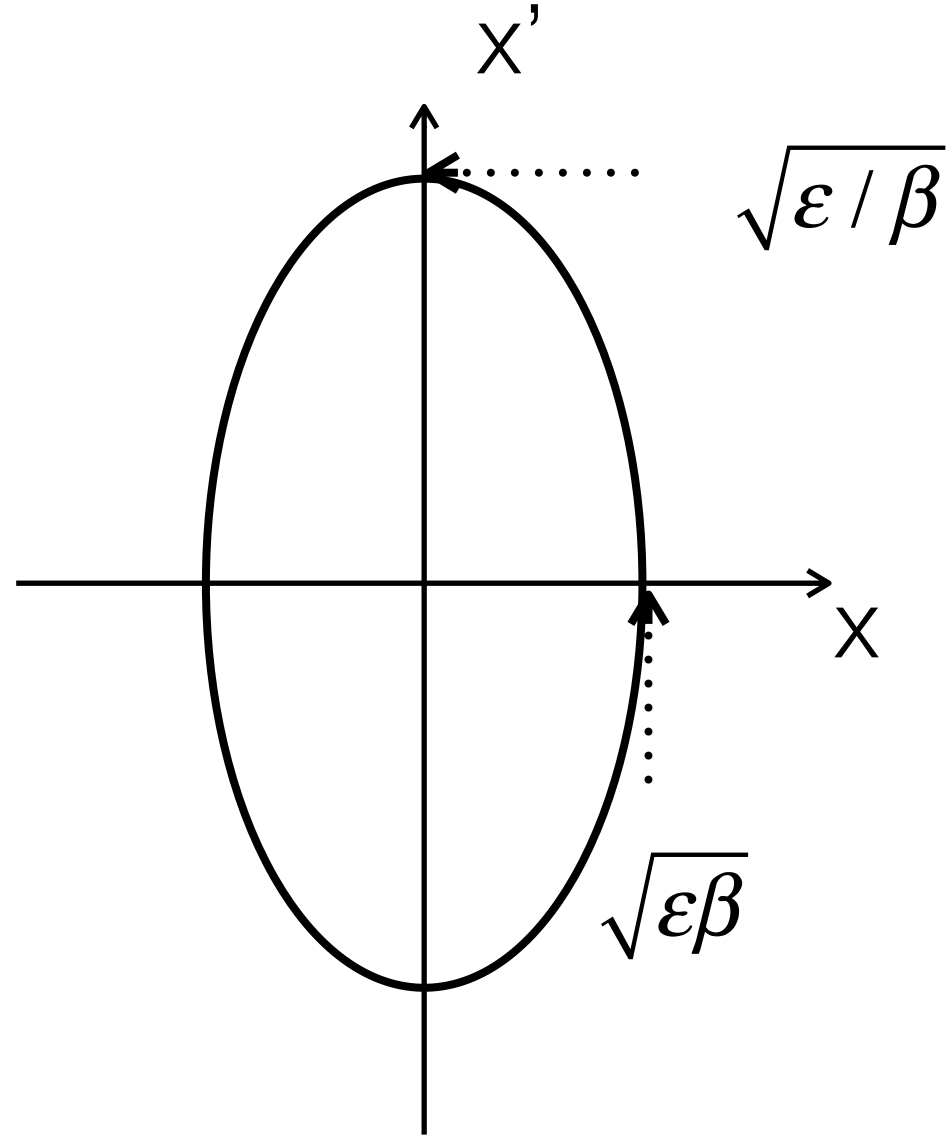
phase advance 'tune' $\phi = \int \frac{ds}{\beta(s)}$

# Transverse 'phase space' ellipse




Ellipse area:

$$A = \pi\sqrt{\varepsilon\beta}\sqrt{\varepsilon / \beta} = \pi\varepsilon$$




Ellipse can change shape but not area!
Emittance is conserved. (cf. 'Louiville's theorem')

# Topics covered (cf. Wilson)

Magnet types

Multipole field expansion

Taylor series expansion

Dipole bending magnet

Magnetic rigidity

Diamond quadrupole

Fields and force in a quadrupole

Transverse coordinates

Weak focussing in a synchrotron

Gutter

Transverse ellipse

Alternating gradients

Equation of motion in transverse coordinates