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Four-point function in general kinematics through geometrical splitting and reduction

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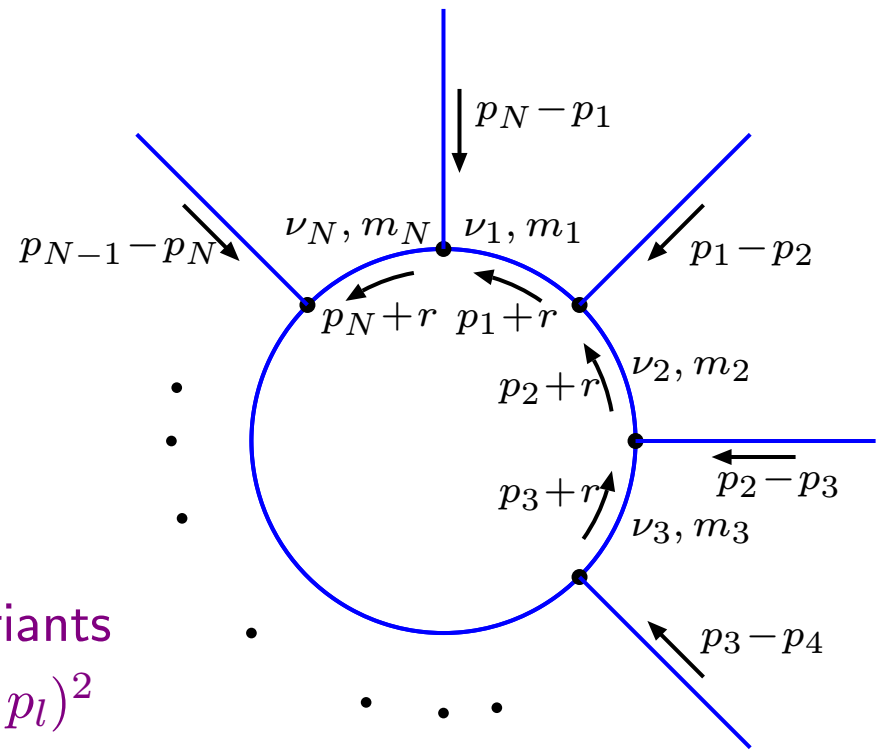
One-loop N-point function $J^{(N)}(n;\nu_1,\ldots,\nu_N)$



Depends on
$\frac{1}{2}N(N-1)$ invariants
$k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) \equiv \int \frac{\mathrm{d}^n k}{[(p_1+k)^2 - m_1^2]^{\nu_1} \cdots [(p_N+k)^2 - m_N^2]^{\nu_N}}$$

Feynman parameters

Parametric representation for the one-loop N-point function in n dimensions:

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-n}\pi^{n/2}\,\Gamma\left(N-\frac{n}{2}\right)\int\limits_0^1\ldots\int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left[\sum\sum\limits_{j<l}\alpha_j\alpha_l k_{jl}^2-\sum\alpha_i m_i^2\right]^{N-n/2}}$$

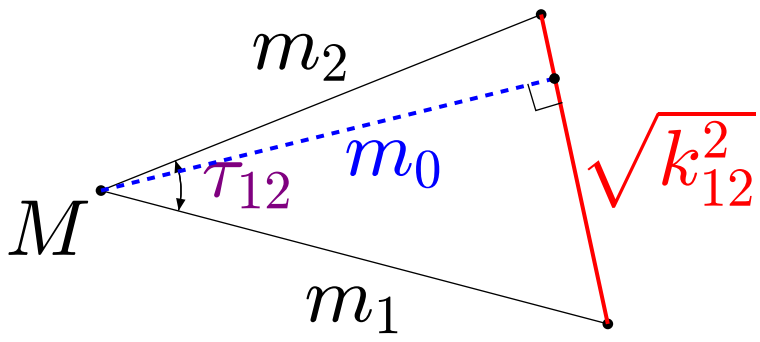
By using $\sum\alpha_i = 1$ we can make the quadratic form homogeneous in α_i:

$$\left[\sum\sum\limits_{j<l}\alpha_j\alpha_l k_{jl}^2-\left(\sum\alpha_i\right)\left(\sum\alpha_i m_i^2\right)\right] \Rightarrow -\left[\sum\alpha_i^2 m_i^2+2\sum\sum\limits_{j<l}\alpha_j\alpha_l m_j m_l c_{jl}\right],$$

$$c_{jl}\equiv\frac{m_j^2+m_l^2-k_{jl}^2}{2m_j m_l},\qquad c_{jl}=\cos\tau_{jl}=\begin{cases}1, & k_{jl}^2=(m_j-m_l)^2 \quad \text{pseudothreshold}\\ -1, & k_{jl}^2=(m_j+m_l)^2 \quad \text{threshold}\end{cases}$$

Direct geometrical interpretation: when $-1\le c_{jl}\le 1$ (i.e., angles τ_{jl} are real)

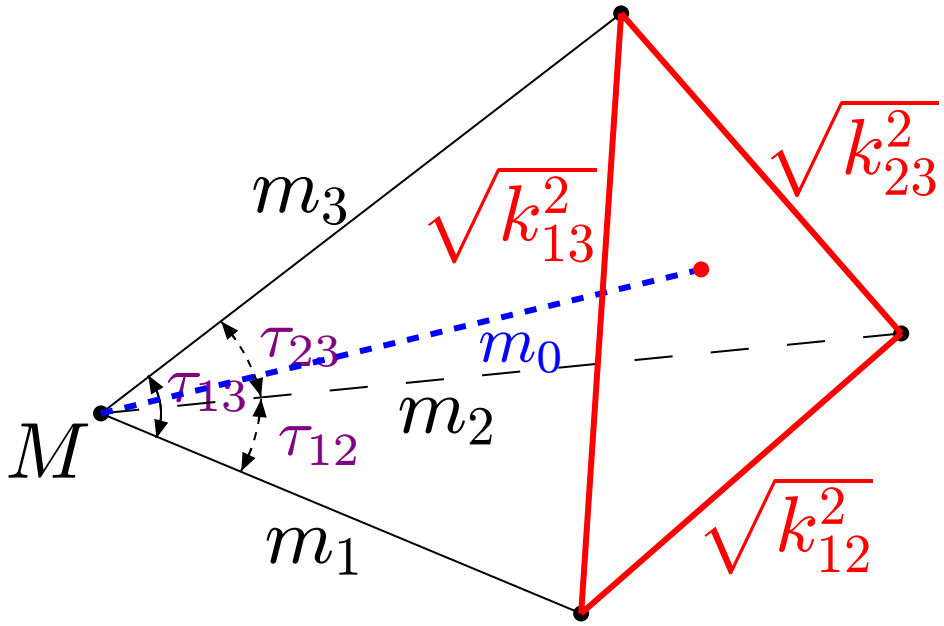
Two-point function: the basic triangle



$$\cos\tau_{12} = c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1m_2}$$

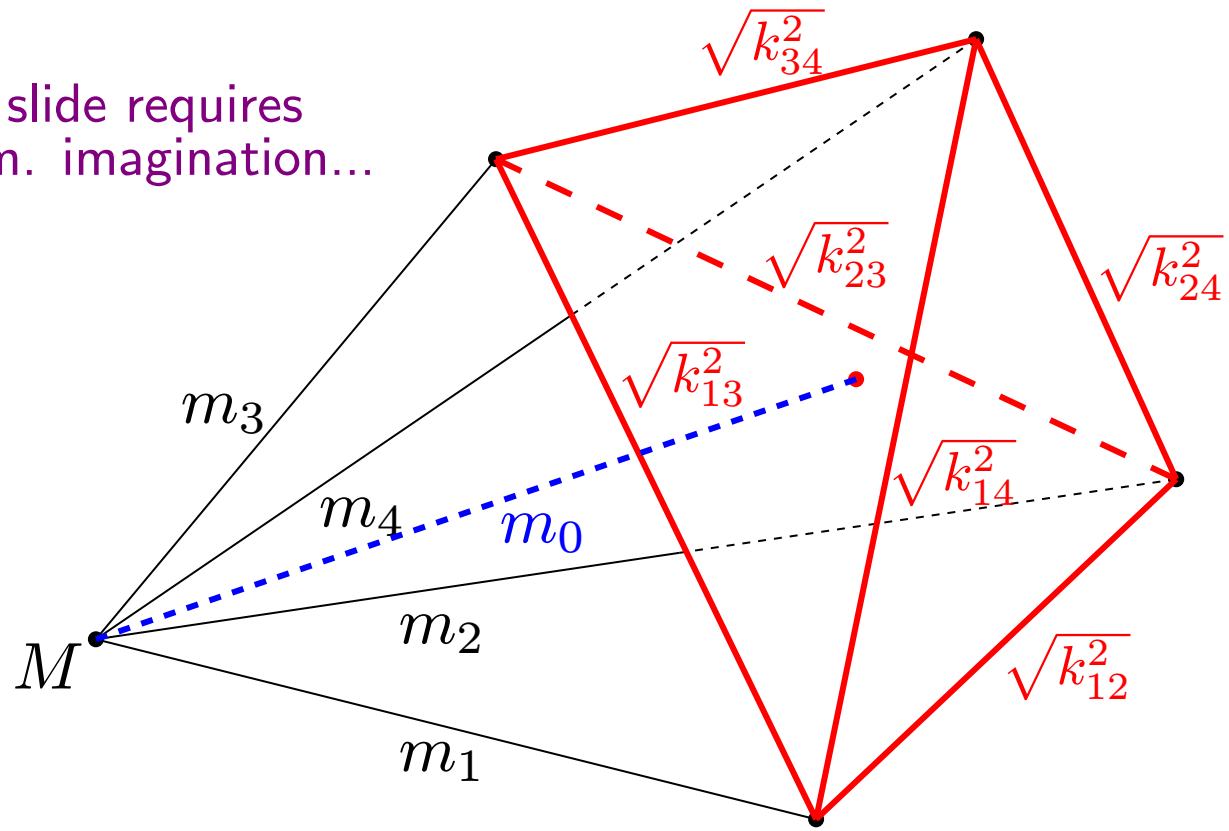
$$c_{12} = \cos\tau_{12} = \begin{cases} 1, & k_{12}^2 = (m_1 - m_2)^2 \quad \text{pseudothreshold} \quad (\tau_{12} = 0) \\ -1, & k_{12}^2 = (m_1 + m_2)^2 \quad \text{threshold} \quad (\tau_{12} = \pi) \end{cases}$$

Three-point function: the basic tetrahedron



Four-point function: the basic simplex for $N = 4$

This slide requires
4-dim. imagination...



$$D^{(4)} = \det \|c_{jl}\| \,, \qquad \Lambda^{(4)} = \det \|(k_{j4} \cdot k_{l4})\| \,,$$

$$V^{(4)} = \frac{m_1 m_2 m_3 m_4}{4!} \sqrt{D^{(4)}}, \qquad \overline{V}_0^{(3)} = \frac{1}{3!} \sqrt{\Lambda^{(4)}}, \qquad m_0 = m_1 m_2 m_3 m_4 \sqrt{\frac{D^{(4)}}{\Lambda^{(4)}}}$$

From Feynman parameters to the geometrical approach

Using linear and *quadratic* substitutions of α variables, we arrive at

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N-\frac{n}{2}\right)(\Pi f_i)\int\limits_0^\infty\ldots\int\limits_0^\infty\frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum\alpha_i f_i\right)^{n-N}}$$

Modified matrix: $C_{jl} = \left(\sqrt{F_j^{(N)}}c_{jl}\sqrt{F_l^{(N)}}\right)$, with $F_i^{(N)} = \frac{\partial}{\partial m_i^2}\left(m_i^2 D^{(N)}\right)$

obeying $\sum\limits_{l=1}^{N} c_{jl}F_l^{(N)}\frac{1}{m_l} = D^{(N)}\frac{1}{m_j} \Rightarrow \sum\limits_{l=1}^{N} C_{jl}\frac{\sqrt{F_l^{(N)}}}{m_l} = D^{(N)}\frac{\sqrt{F_j^{(N)}}}{m_j} \Rightarrow$

Eigenvector: $f_i = \frac{\sqrt{F_i^{(N)}}}{m_i}$, Eigenvalue: $D^{(N)} = \det\|c_{jl}\|$ (Gram determinant)

Feynman parameters: diagonalization

Whenever a quadratic form occurs, an obvious idea is to *diagonalize* it:
"rotate" variables $\alpha_i \to \beta_i$ so that $\alpha^T \|C\| \alpha = \sum_{i=1}^N \lambda_i \beta_i^2$
One of the β's (say β_N) is directed along f_i, so that $\lambda_N = D^{(N)}$ and denominator $(\sum \alpha_i f_i)$ is proportional to β_N.

Assume (for a moment) that all $\lambda_i > 0$ and rescale $\beta_i = \frac{\gamma_i}{\sqrt{\lambda_i}} \Rightarrow$

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int\!\!\ldots\!\!\int_{\Omega^{(N)}} \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\, \delta\left(\sum \gamma_i^2 - 1\right)$$

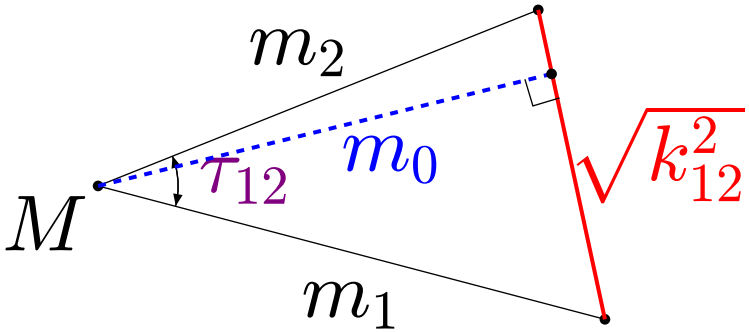
Remarkably: the same N-dim. solid angle $\Omega^{(N)}$ as in the *basic simplex*!

Special case: $N = n$ ($N = 2$ in 2d, $N = 3$ in 3d, $N = 4$ in 4d, etc.)

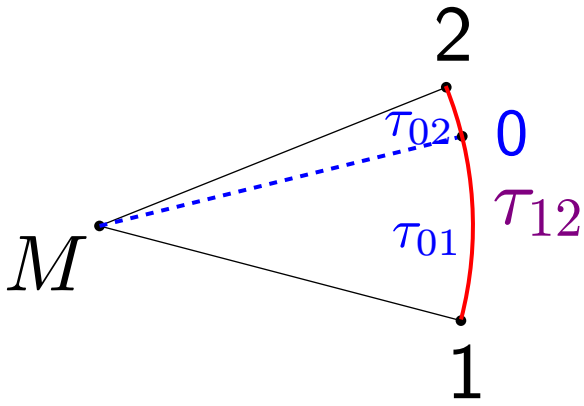
If some of λ_i are negative – *hyperbolic* surface (instead of *spherical*)
$\leftrightarrow$ analytical continuation!

A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299;
A.I.D., Phys. Rev. **D61** (2000) 087701;
A.I.D. and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266
A.I.D., AIHENP-99 Proceedings (hep-th/9908032); Nucl.Instr.Meth. **A559** (2006) 293
A.I.D., J. Phys. (Conf. Ser.) 762 (2016) 012068 (arXiv:1605.04828)

Two-point function, geometrical approach



the basic triangle

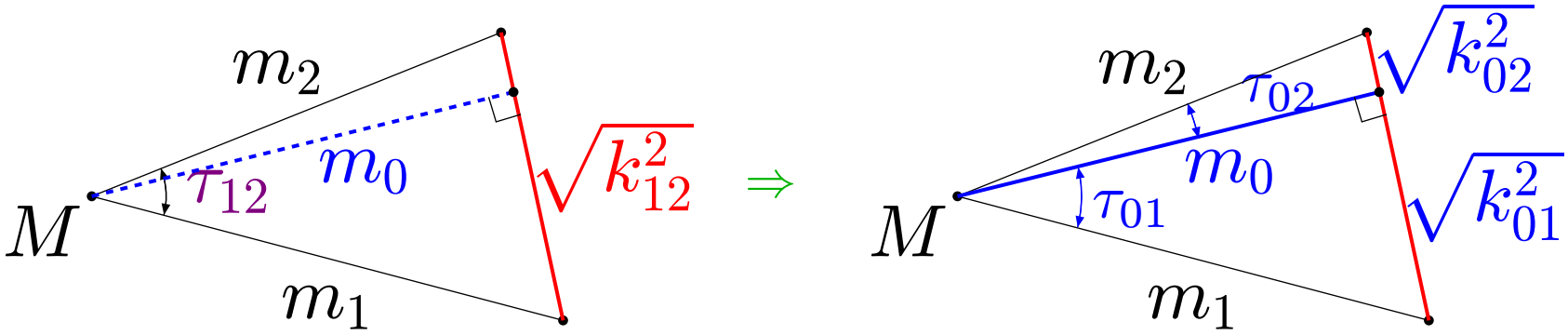


the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

$$m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1m_2\sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

Two-point function, splitting the basic triangle



$$k_{01}^2 = \frac{(k_{12}^2 + m_1^2 - m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2 = \frac{(k_{12}^2 - m_1^2 + m_2^2)^2}{4k_{12}^2}$$

$$J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2)J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right) + (k_{12}^2 - m_1^2 + m_2^2)J^{(2)}\left(n;1,1|k_{02}^2;m_2,m_0\right)\Big\}$$

This is an example of a functional relation between integrals with different momenta and massses, similar to those described in

O.V. Tarasov, Phys.Lett. **B670** (2008) 67

Two-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $\quad 3-1(\text{dimension})=2$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $\quad 3-1(k_{01}^2=m_1^2-m_0^2)-1(\text{dimension})=1$

Quadratic form in Feynman parametric integral:

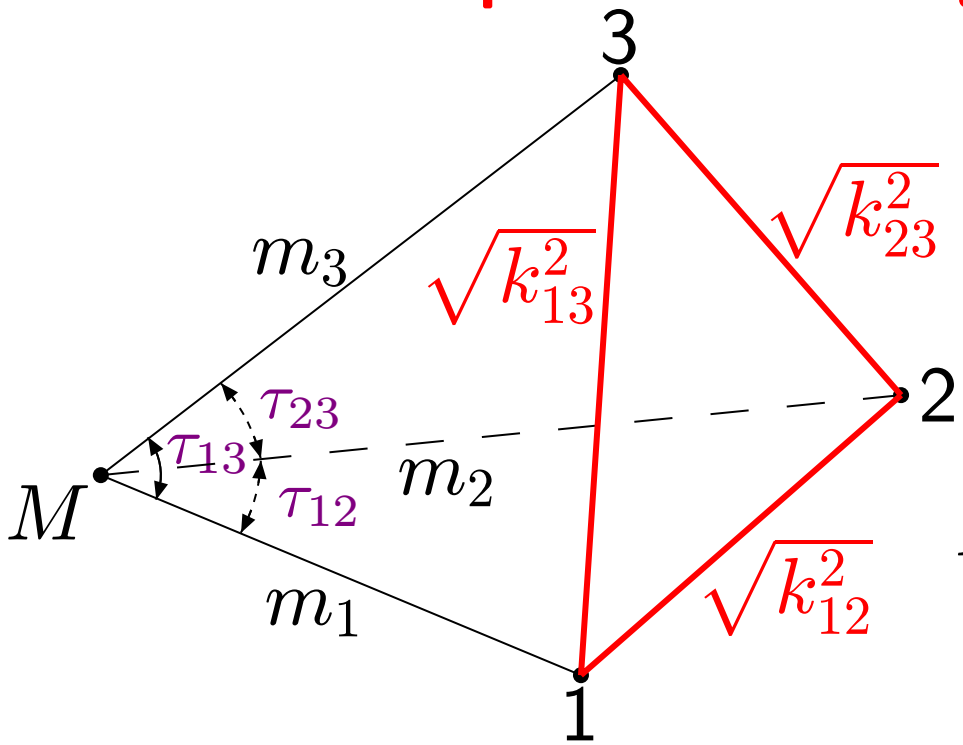
in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $\quad [\alpha_1\alpha_2k_{12}^2-\alpha_1m_1^2-\alpha_2m_2^2]$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $\quad [\alpha_1\alpha_2k_{01}^2-\alpha_1m_1^2-\alpha_2m_0^2]=-[\alpha_1^2k_{01}^2+m_0^2]$

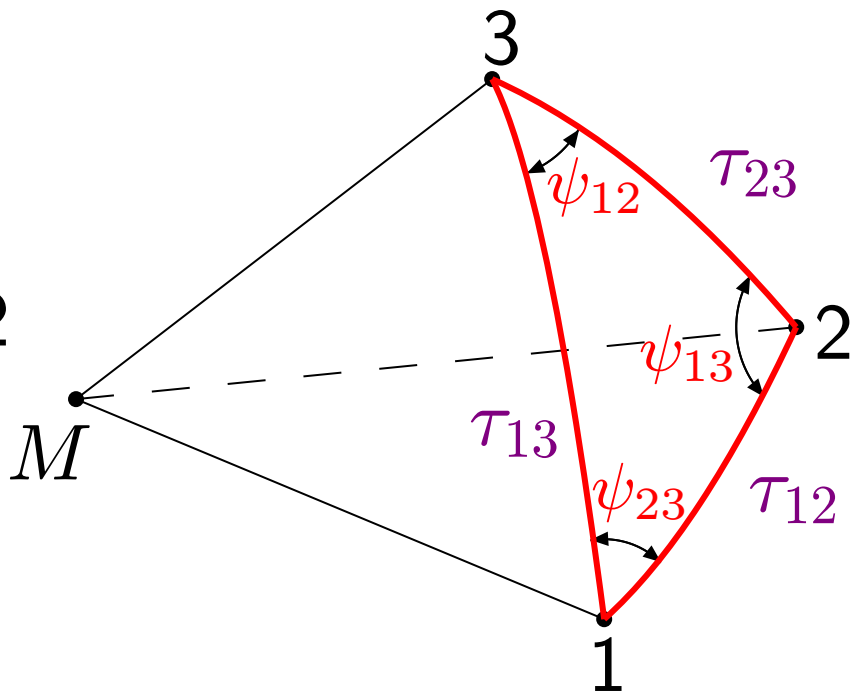
Result in arbitrary dimension:

$$
\begin{aligned}
J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right) &= \mathrm{i}\pi^{n/2}\Gamma(2-n/2)\int\limits_0^1\int\limits_0^1\frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\delta(\alpha_1+\alpha_2-1)}{[\alpha_1^2k_{01}^2+m_0^2]^{2-n/2}}\\
&= \mathrm{i}\pi^{n/2}\frac{\Gamma(2-n/2)}{(m_0^2)^{2-n/2}}\,{}_2F_1\left(\begin{array}{c}1/2,\ 2-n/2\\ 3/2\end{array}\middle|-\frac{k_{01}^2}{m_0^2}\right)
\end{aligned}
$$

Three-point function: geometrical approach



the basic tetrahedron



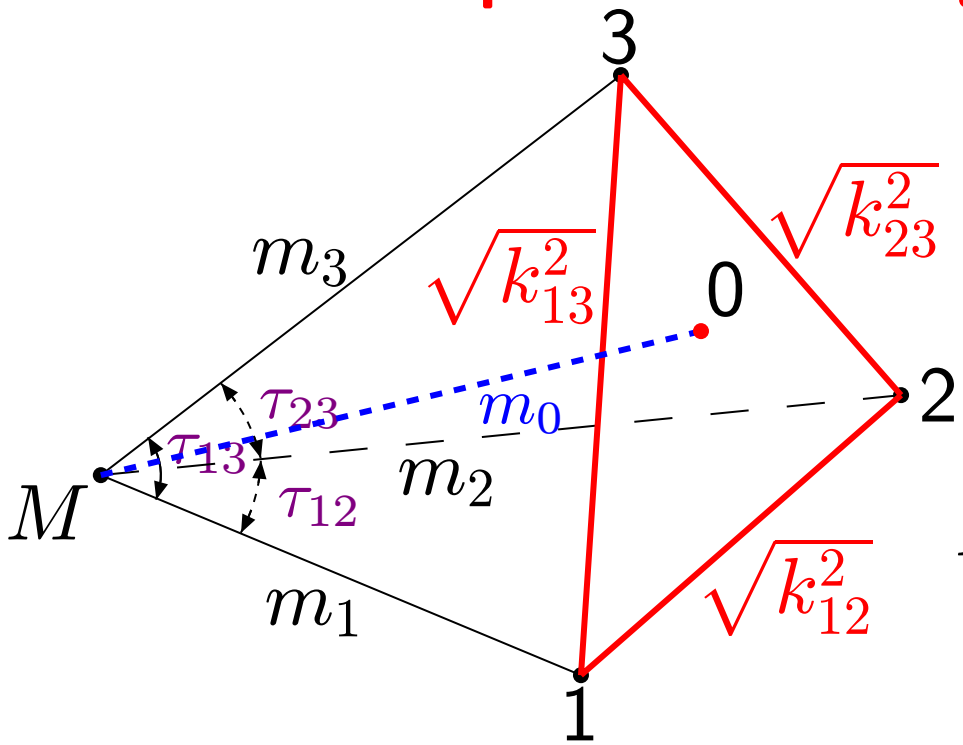
the solid angle

Special case $n = 3 \Rightarrow$ the area of spherical triangle ("spherical excess"):

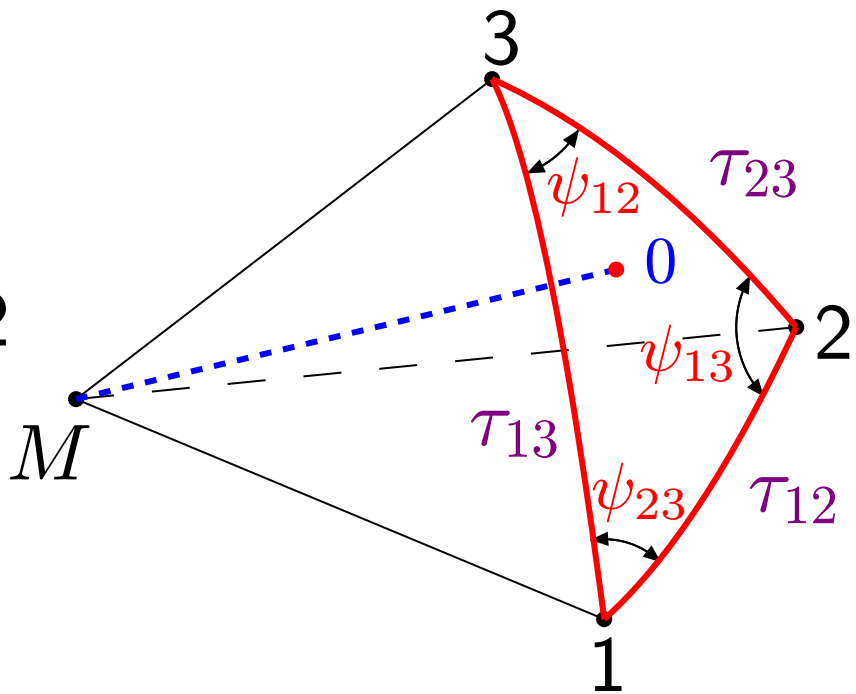
$$\Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \ .$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Three-point function: geometrical approach

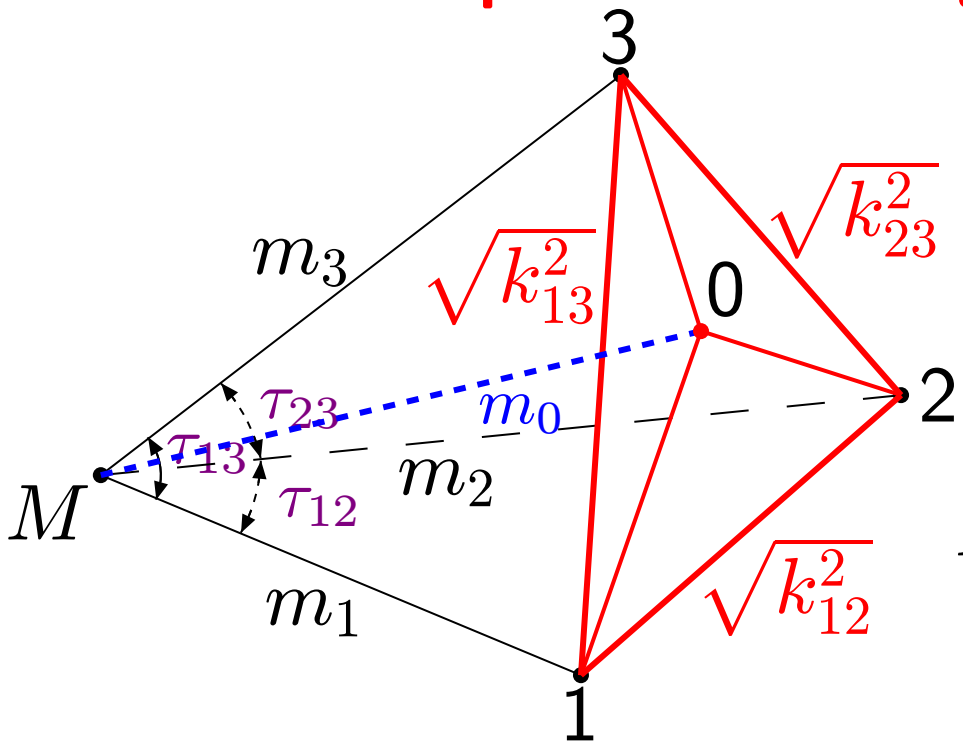


the basic tetrahedron

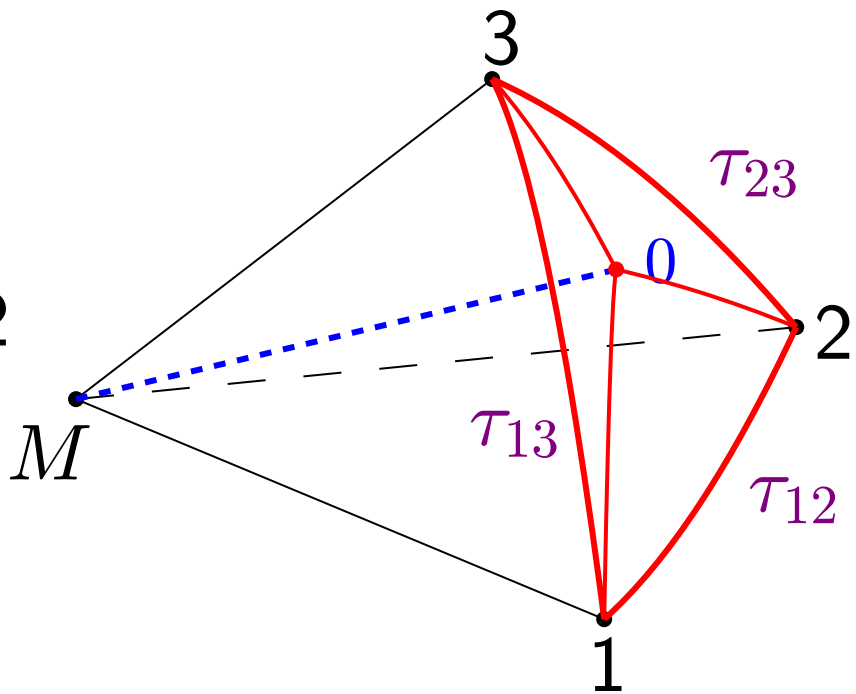


the solid angle

Three-point function: geometrical approach

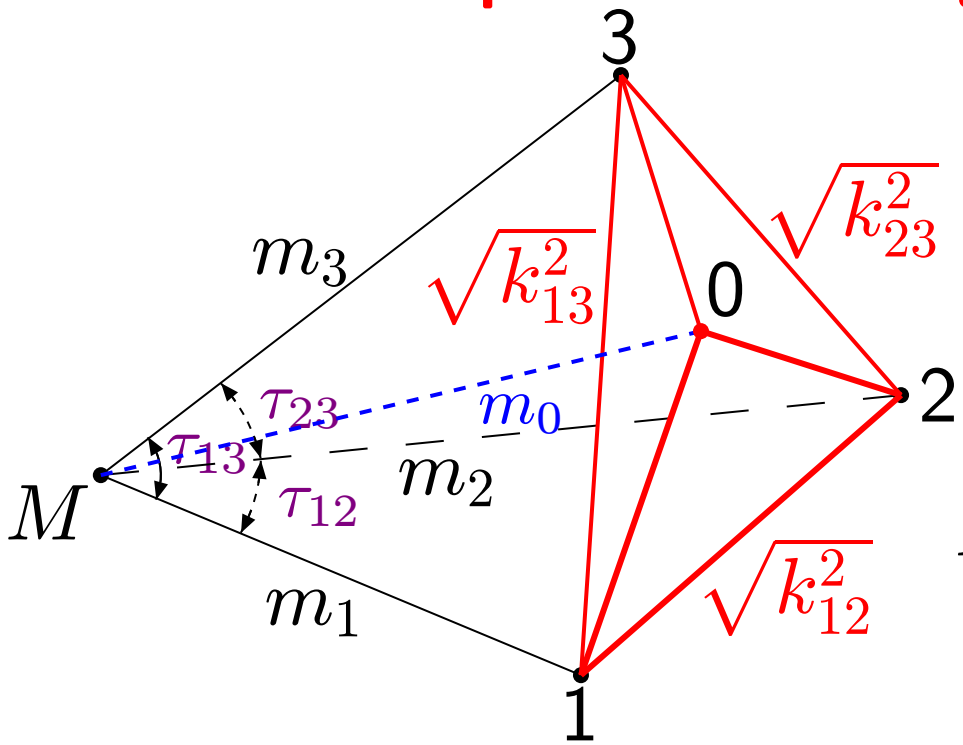


the basic tetrahedron

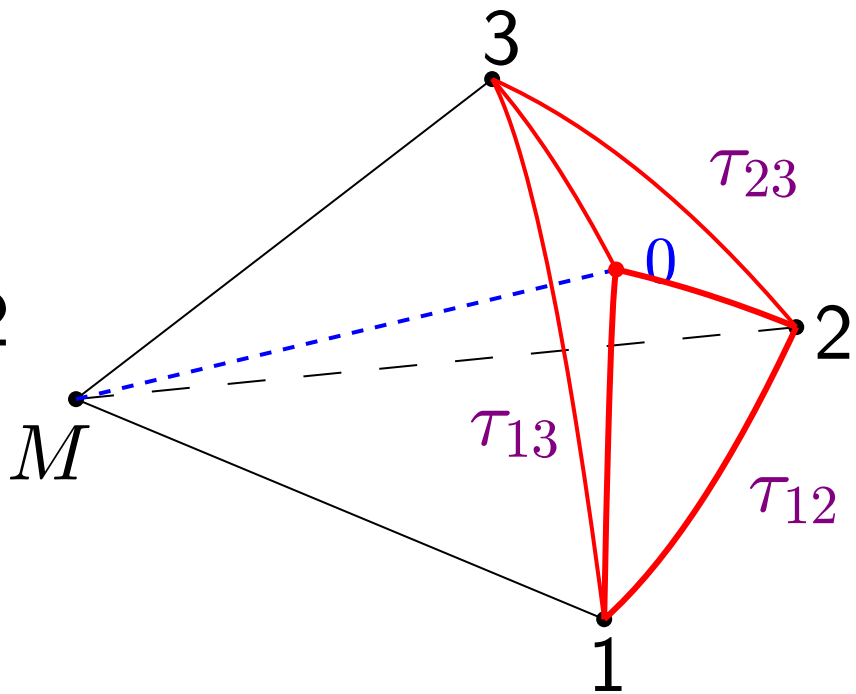


the solid angle

Three-point function: geometrical approach

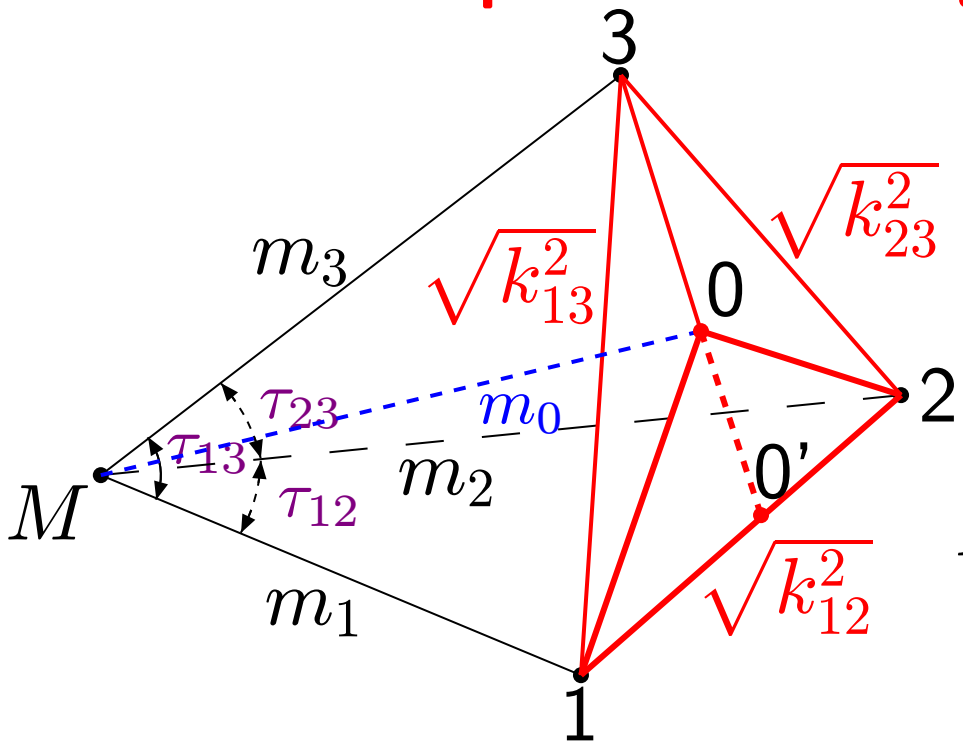


the basic tetrahedron

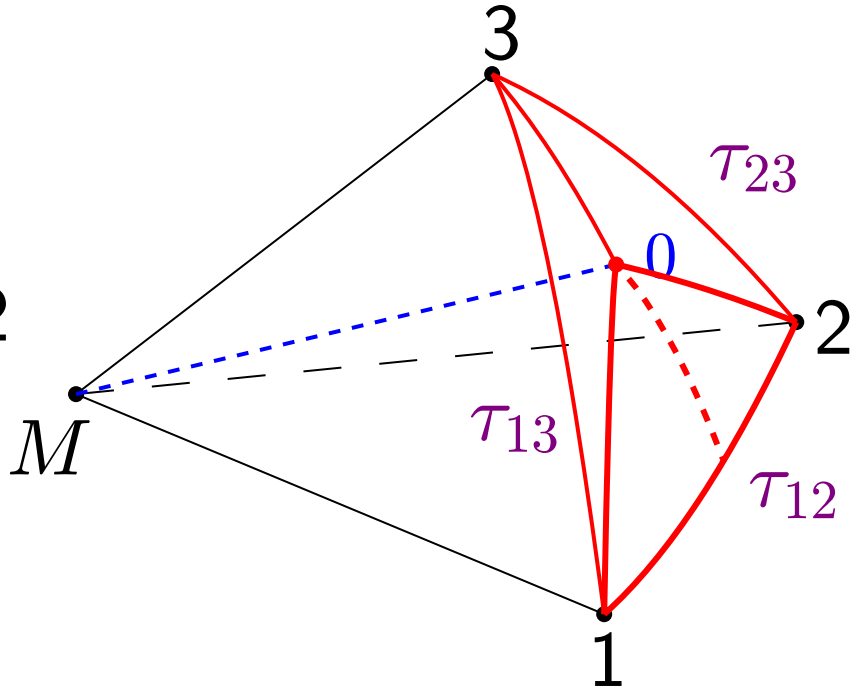


the solid angle

Three-point function: geometrical approach



the basic tetrahedron



the solid angle

Three-point function: splitting the basic tetrahedron

$$J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$$
$$= \frac{m_1^2m_2^2m_3^2}{\Lambda^{(3)}}\left\{\frac{F_1^{(3)}}{m_1^2}J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{03}^2,k_{02}^2;m_0,m_2,m_3\right)\right.$$
$$+\frac{F_2^{(3)}}{m_2^2}J^{(3)}\left(n;1,1,1\middle|k_{03}^2,k_{13}^2,k_{01}^2;m_1,m_0,m_3\right)$$
$$\left.+\frac{F_3^{(3)}}{m_3^2}J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)\right\}$$

with $k_{01}^2=m_1^2-m_0^2,\qquad k_{02}^2=m_2^2-m_0^2,\qquad k_{03}^2=m_3^2-m_0^2,$

$$F_3^{(3)}=\frac{1}{4m_1^2m_2^2}\left[k_{12}^2\left(k_{13}^2+k_{23}^2-k_{12}^2+m_1^2+m_2^2-2m_3^2\right)-(m_1^2-m_2^2)\left(k_{13}^2-k_{23}^2\right)\right],\ \text{etc.}$$

$$\frac{F_1^{(3)}}{m_1^2}+\frac{F_2^{(3)}}{m_2^2}+\frac{F_3^{(3)}}{m_3^2}=\frac{\Lambda^{(3)}}{m_1^2m_2^2m_3^2}$$

$$\Lambda^{(3)}=\frac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right]$$

Three-point function: further reduction of the integrals

$$J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$$
$$=\frac{1}{2k_{12}^2}\Bigg\{(k_{12}^2+m_1^2-m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$$
$$+(k_{12}^2-m_1^2+m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{00'}^2,k_{20'}^2;m_{0'},m_2,m_0\right)\Bigg\}$$

Note that

$$k_{10'}^2=\frac{(k_{12}^2+m_1^2-m_2^2)^2}{4k_{12}^2},\qquad k_{20'}^2=\frac{(k_{12}^2-m_1^2+m_2^2)^2}{4k_{12}^2},$$

— similarly to the reduction of the two-point function

Three-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$: $6-1(\text{dimension})=5$

in $J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$: $6-2(\text{relations})-1(\text{dimension})=3$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$: $6-3(\text{relations})-1(\text{dimension})=2$

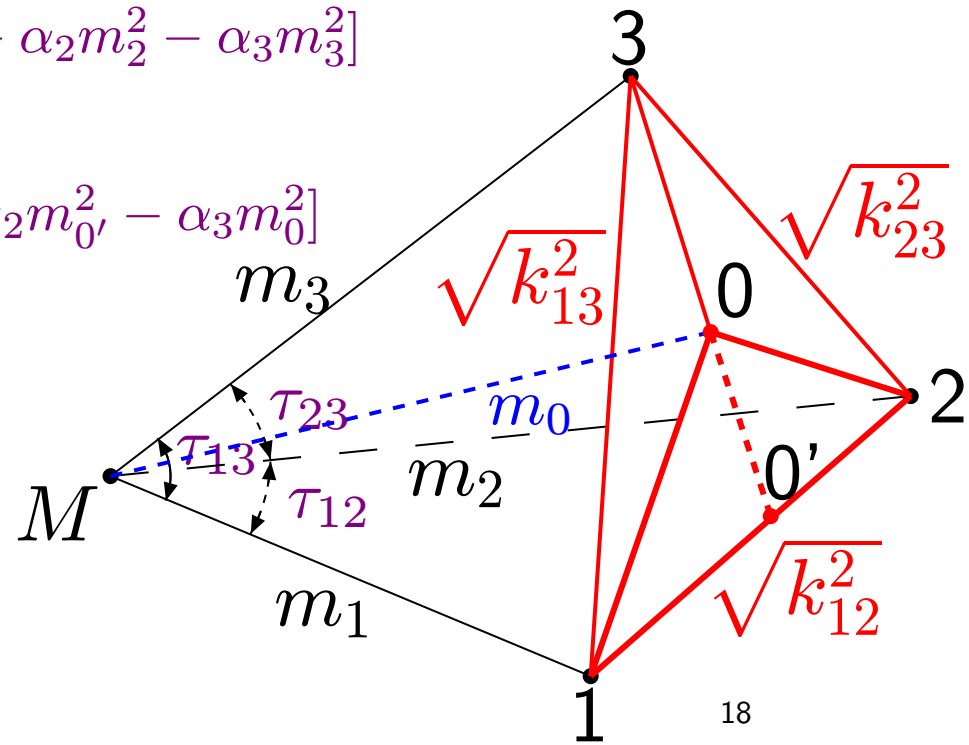
Quadratic form in Feynman parametric integral:

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_2\alpha_3k_{23}^2-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2]$$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$:

$$[\alpha_1\alpha_2k_{10'}^2+\alpha_1\alpha_3k_{01}^2+\alpha_2\alpha_3k_{00'}^2-\alpha_1m_1^2-\alpha_2m_{0'}^2-\alpha_3m_0^2]$$
$$=-[\alpha_1^2k_{10'}^2+(\alpha_1+\alpha_2)^2k_{00'}^2+m_0^2]$$



Three-point function: result in arbitrary dimension

$$J^{(3)}\left(n;1,1,1\middle|k^2_{00'},k^2_{01},k^2_{10'};m_1,m_{0'},m_0\right)$$

$$= -\mathrm{i}\pi^{n/2}\Gamma(3-n/2)\int\limits_0^1\int\limits_0^1\int\limits_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{[\alpha_1^2k^2_{10'}+(\alpha_1+\alpha_2)^2k^2_{00'}+m_0^2]^{3-n/2}}$$

$$= -\frac{\mathrm{i}\pi^{n/2}\Gamma(2-n/2)}{2(m_0^2)^{2-n/2}k^2_{00'}}\left\{\sqrt{\frac{k^2_{00'}}{k^2_{10'}}}\arctan\sqrt{\frac{k^2_{10'}}{k^2_{00'}}}\right.$$

$$\left.-\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2}F_1\left(1/2,1,2-n/2;3/2\middle|-\frac{k^2_{10'}}{k^2_{00'}},-\frac{k^2_{10'}}{m^2_{0'}}\right)\right\}$$

where F_1 is Appell hypergeometric function of two variables,

$$F_1\left(a,b_1,b_2;c|x,y\right)=\sum_{j_1,j_2}\frac{(a)_{j_1+j_2}\,(b_1)_{j_1}\,(b_2)_{j_2}}{(c)_{j_1+j_2}}\,\frac{x^{j_1}\,y^{j_2}}{j_1!\,j_2!}$$

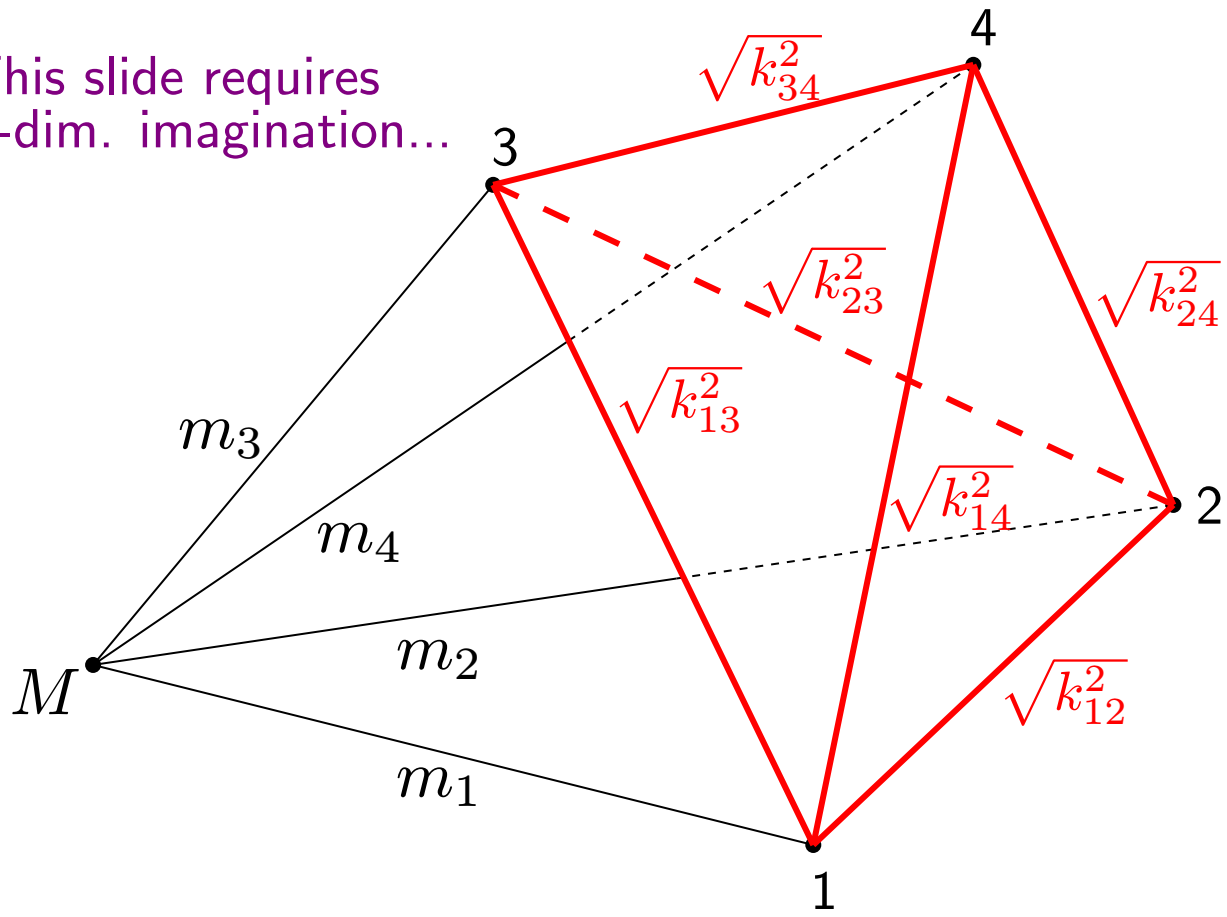
A.I.D., hep-th/9908032; Nucl.Instr.Meth. A559 (2006) 293

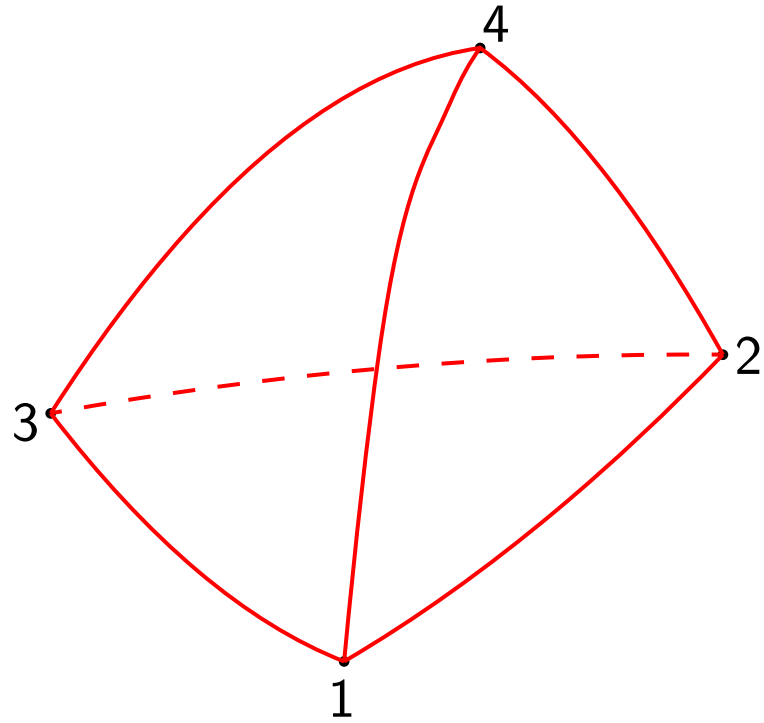
See also: O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

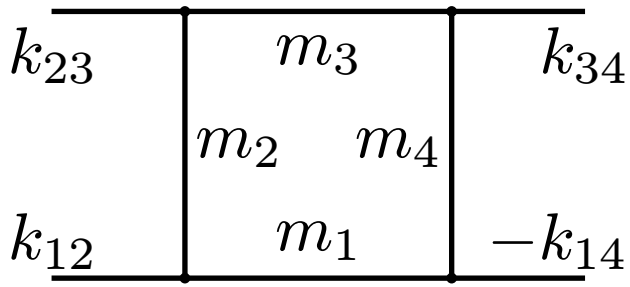
J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...





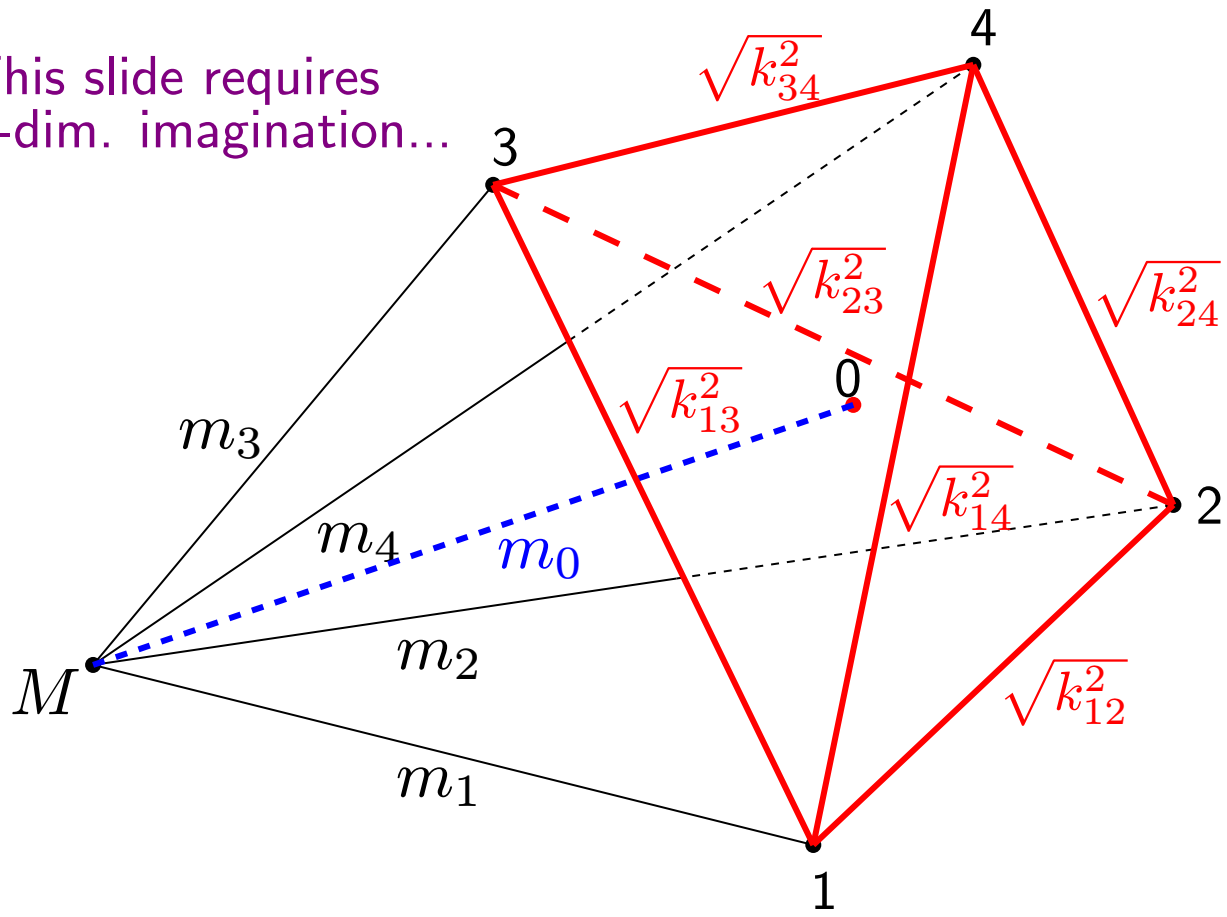


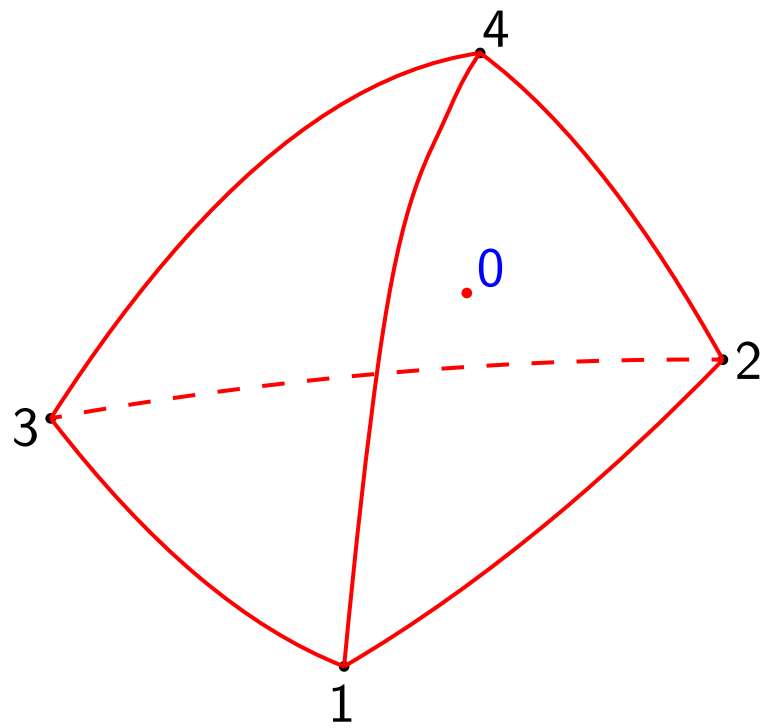
$k_{12}^2, k_{23}^2, k_{34}^2, k_{14}^2$ – external momenta squared

k_{13}^2, k_{24}^2 – Mandelstam variables s and t

Four-point function: basic simplex and non-Euclidean tetrahedron

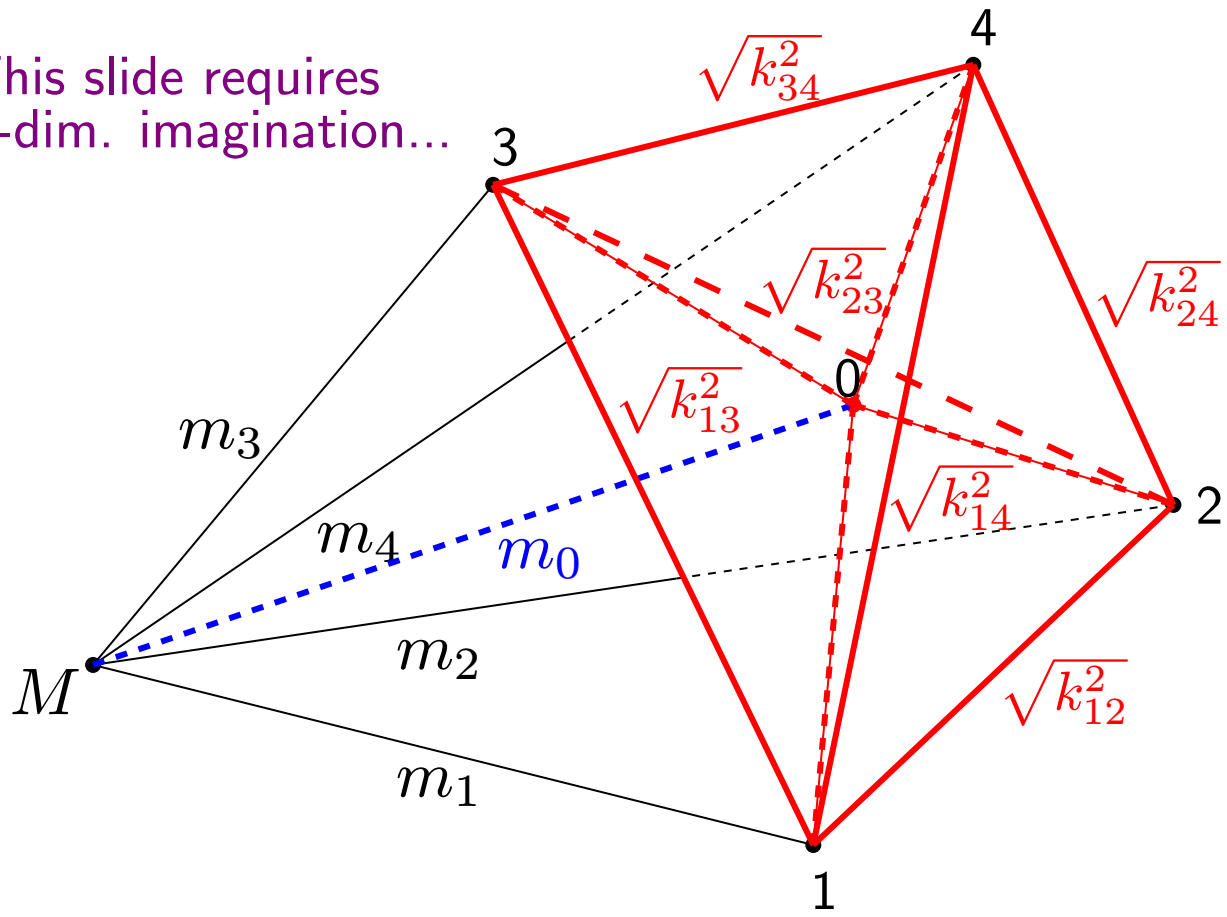
This slide requires
4-dim. imagination...

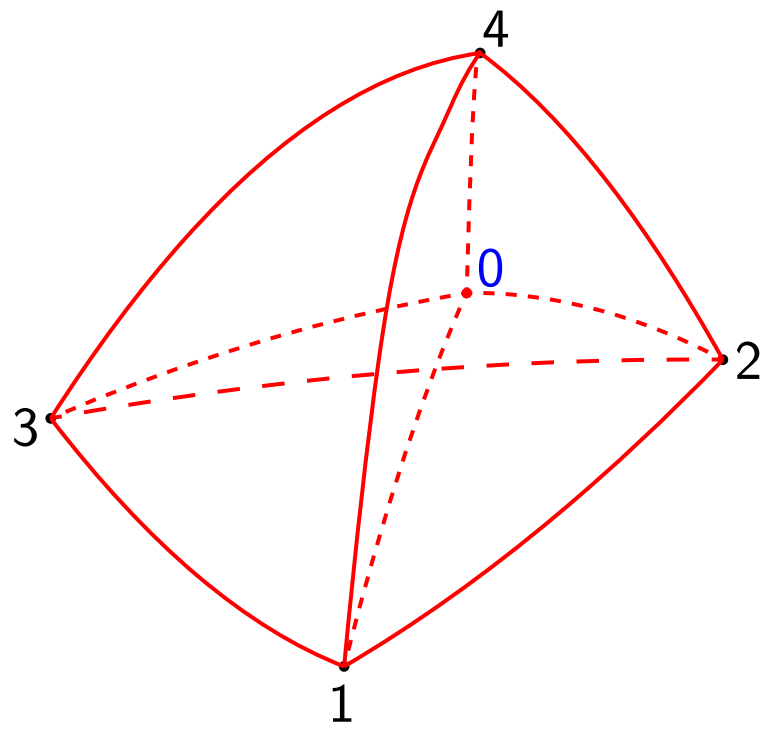




Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...





Four-point function: basic simplex and non-Euclidean tetrahedron

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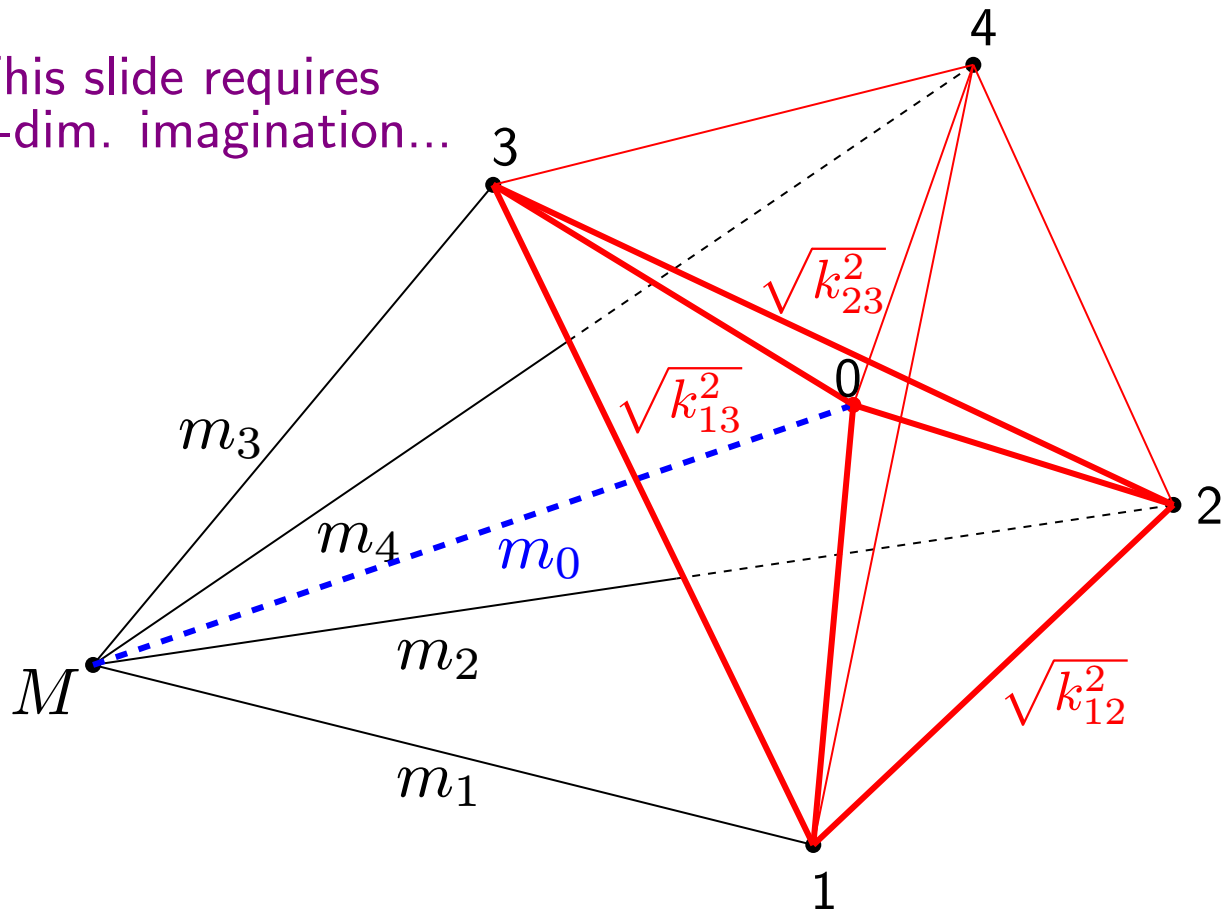

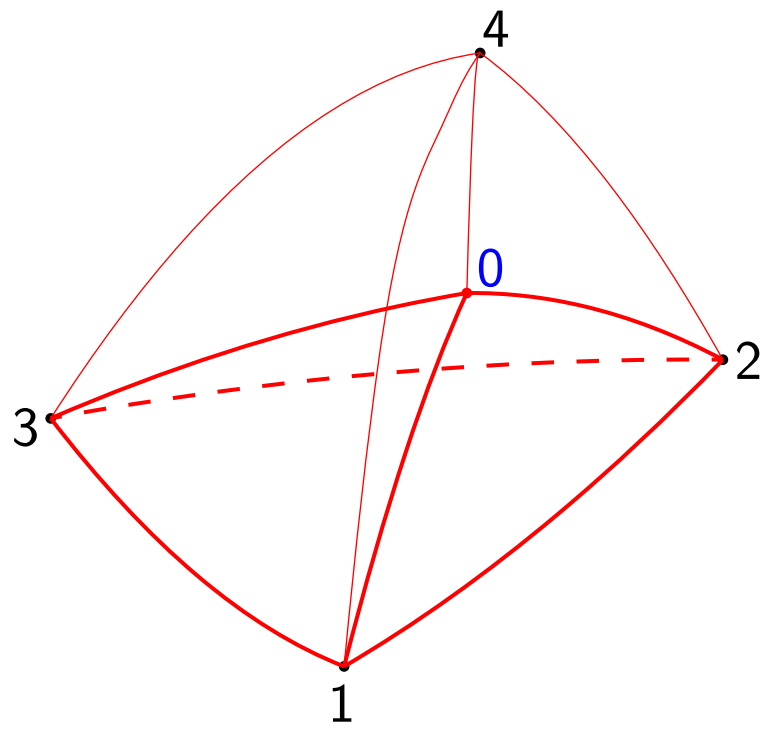

Four-point function: basic simplex and non-Euclidean tetrahedron

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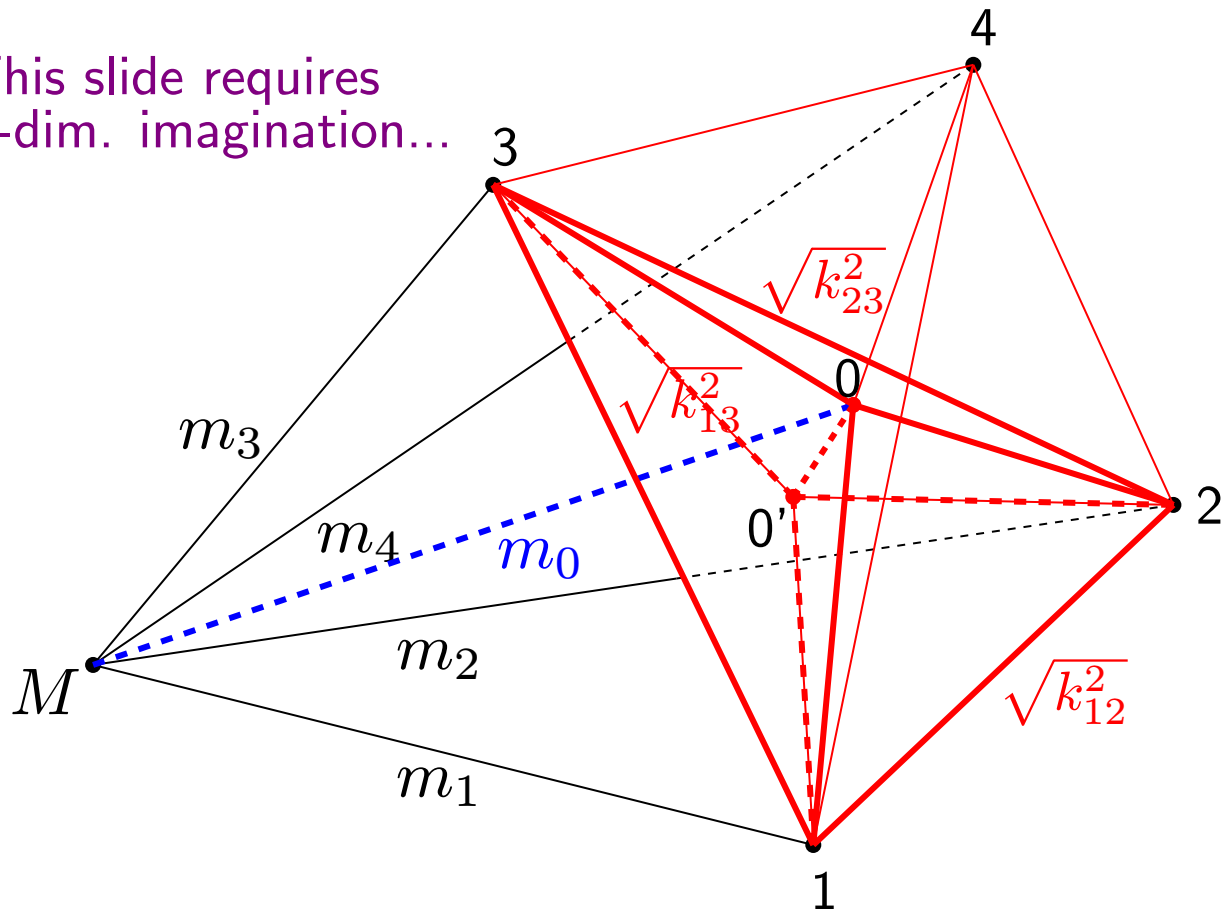

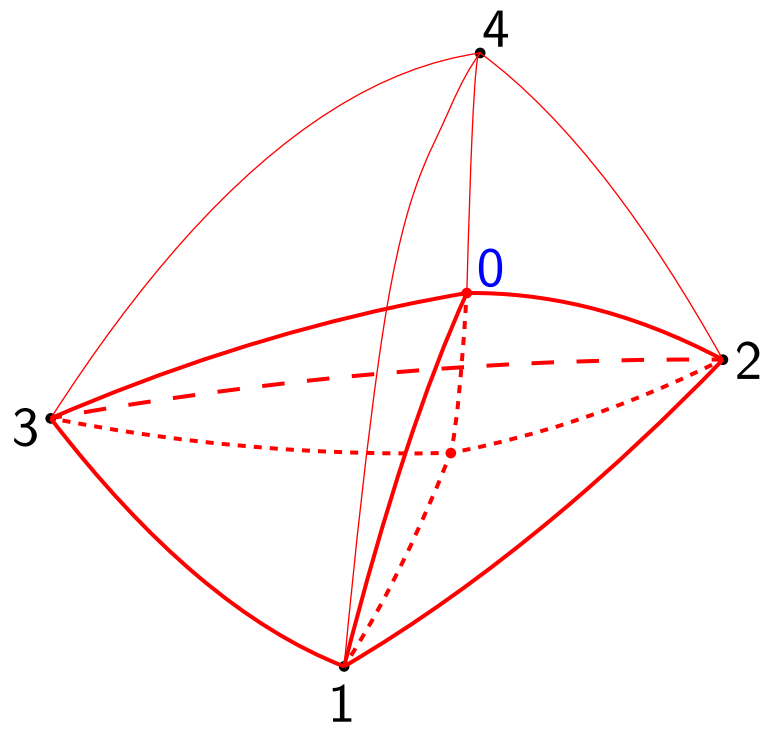

Four-point function: basic simplex and non-Euclidean tetrahedron

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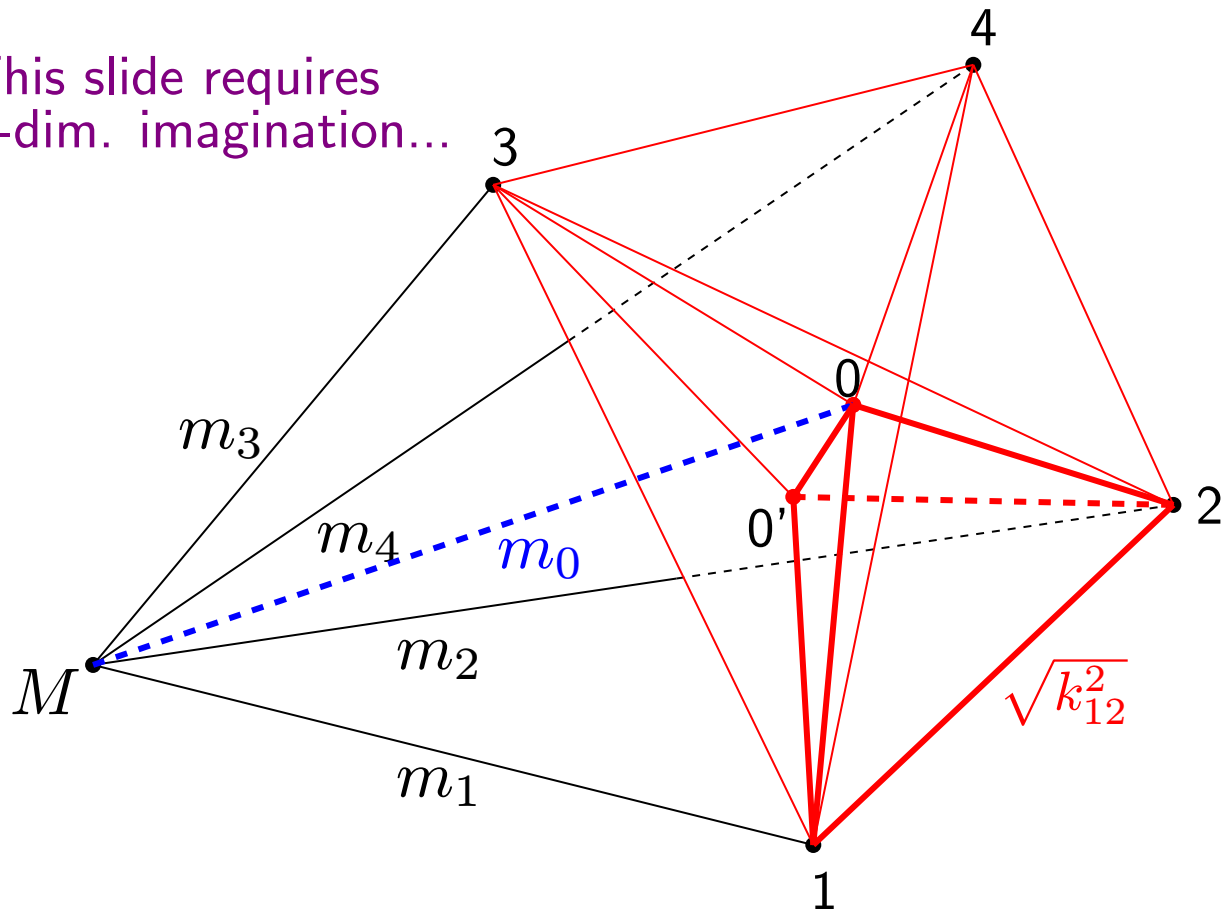
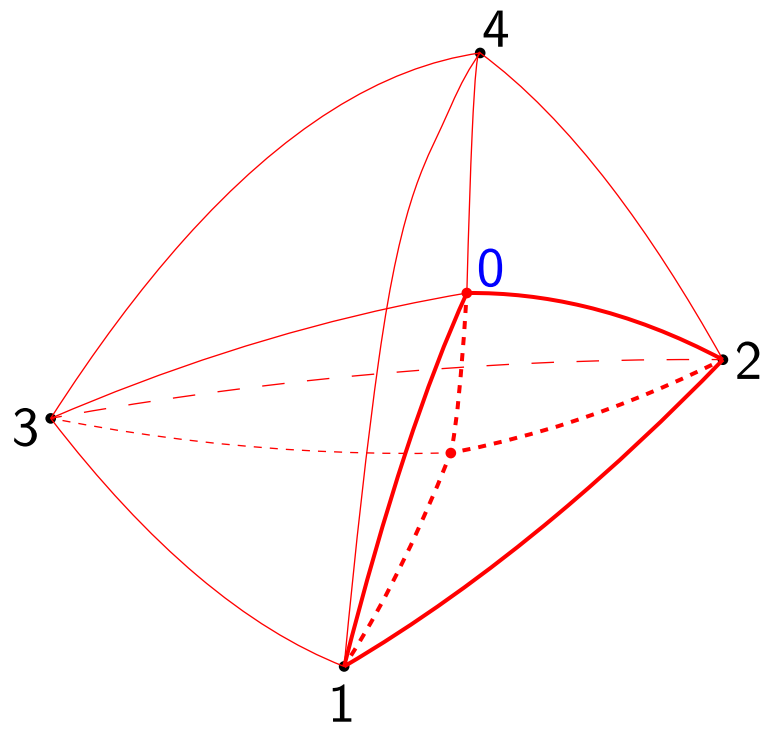
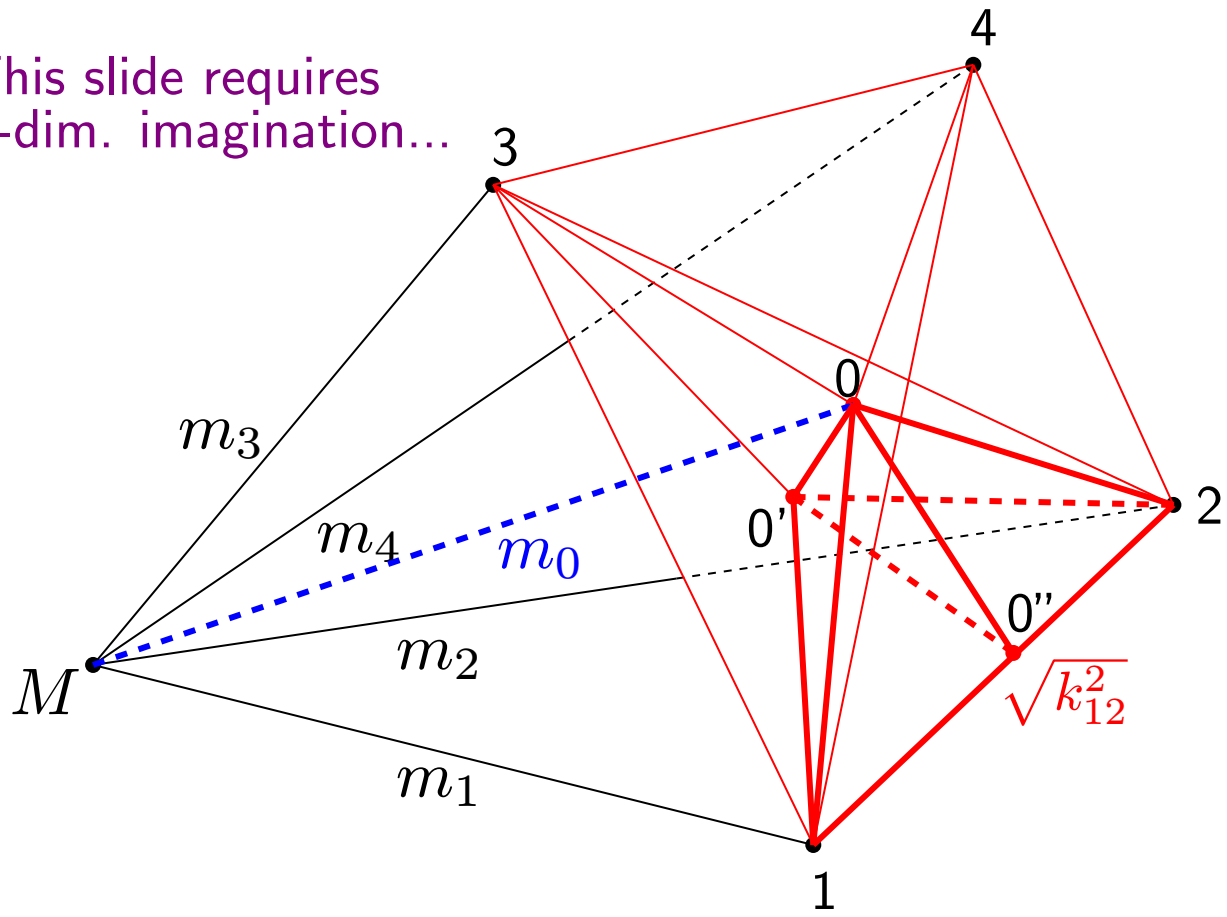
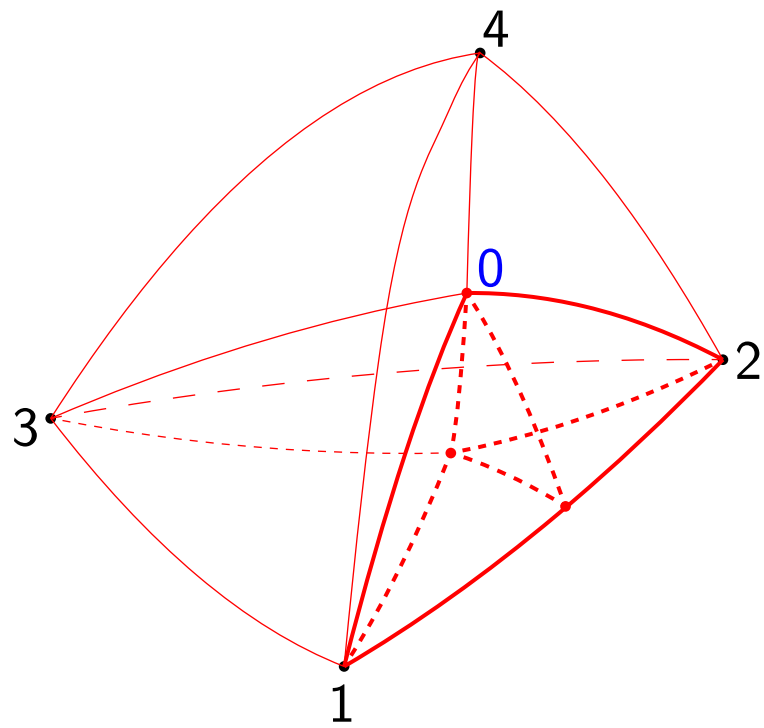




Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...





Four-point function: number of dimensionless variables

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$$10-1(\text{dimension})=9$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{03}^2,k_{01}^2,k_{13}^2,k_{02}^2\};\{m_1,m_2,m_3,m_0\}\right)$
(after splitting the tetrahedron 1234 into four tetrahedra):

$$10-3(\text{relations})-1(\text{dimension})=6$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{20'}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{02}^2\};\{m_1,m_2,m_{0'},m_0\}\right)$
(after splitting the tetrahedron 0123 into three tetrahedra):

$$10-5(\text{relations})-1(\text{dimension})=4$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$
(after splitting each of the resulting tetrahedra into two):

$$10-6(\text{relations})-1(\text{dimension})=3$$

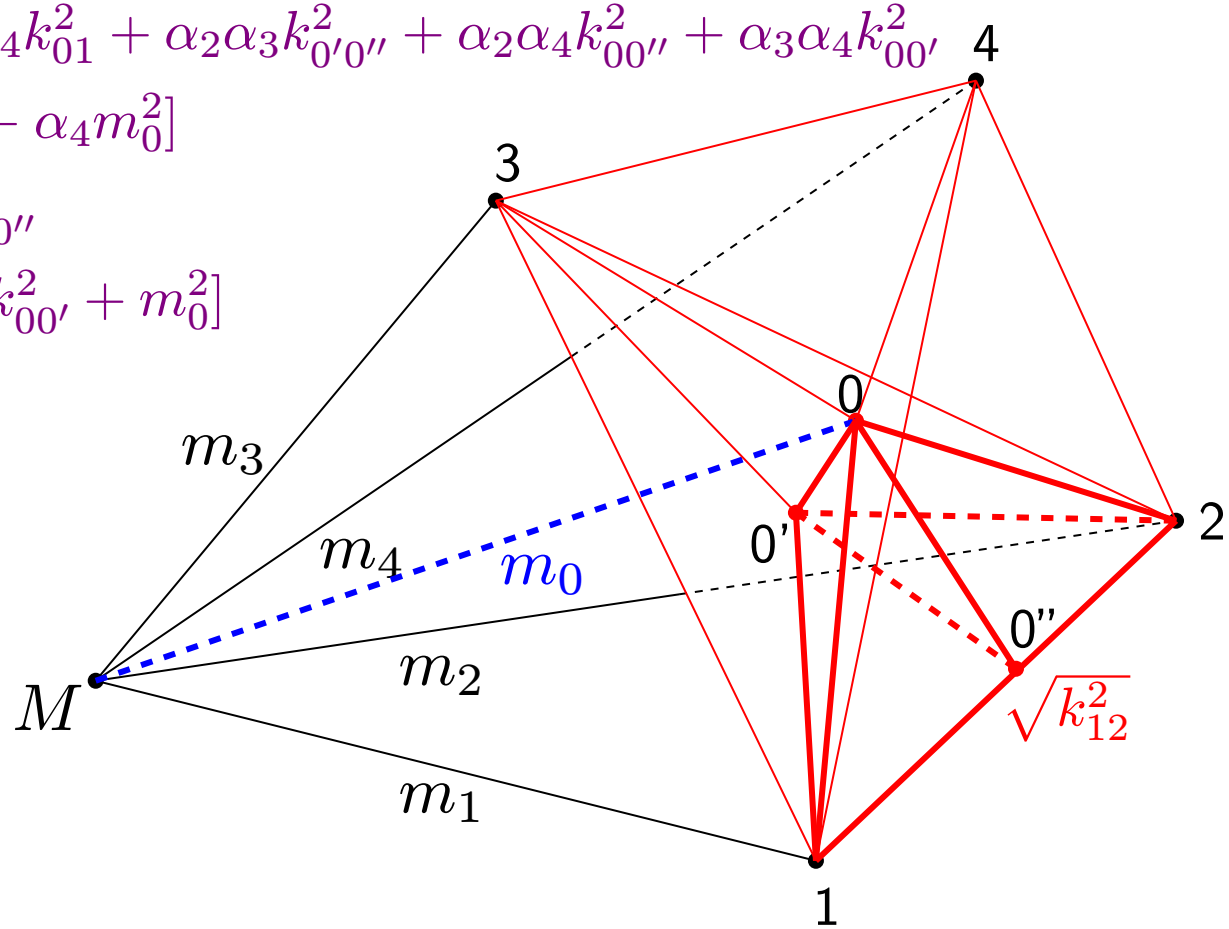
Four-point function: quadratic form in Feynman parametric integral

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_1\alpha_4k_{14}^2+\alpha_2\alpha_3k_{23}^2+\alpha_2\alpha_4k_{24}^2+\alpha_3\alpha_4k_{34}^2$$
$$-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2-\alpha_4m_4^2]$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$:

$$[\alpha_1\alpha_2k_{10''}^2+\alpha_1\alpha_3k_{10'}^2+\alpha_1\alpha_4k_{01}^2+\alpha_2\alpha_3k_{0'0''}^2+\alpha_2\alpha_4k_{00''}^2+\alpha_3\alpha_4k_{00'}^2$$
$$-\alpha_1m_1^2-\alpha_2m_{0''}^2-\alpha_3m_{0'}^2-\alpha_4m_0^2]$$
$$=-[\alpha_1^2k_{10''}^2+(\alpha_1+\alpha_2)^2k_{0'0''}^2$$
$$+(\alpha_1+\alpha_2+\alpha_3)^2k_{00'}^2+m_0^2]$$



Four-point function: result in arbitrary dimension

$$J^{(4)}\left(n;1,1,1,1\Big|\{k^2_{10''},k^2_{0'0''},k^2_{00'},k^2_{01},k^2_{10'},k^2_{00''}\};\{m_1,m_{0''},m_{0'},m_0\}\right)$$

$$= \mathrm{i}\pi^{n/2}\Gamma(4-n/2)\int\limits_0^1\int\limits_0^1\int\limits_0^1\int\limits_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{[\alpha_1^2k^2_{10''}+(\alpha_1+\alpha_2)^2k^2_{0'0''}+(\alpha_1+\alpha_2+\alpha_3)^2k^2_{00'}+m_0^2]^{4-n/2}}$$

$$= \frac{\mathrm{i}\pi^{n/2}\Gamma(3-n/2)}{2k^2_{0'0''}(m_0^2)^{3-n/2}}\left\{\sqrt{\frac{k^2_{0'0''}}{k^2_{10''}}}\arctan\sqrt{\frac{k^2_{10''}}{k^2_{0'0''}}}\;{}_2F_1\left(\begin{array}{c}1/2,3-n/2\\3/2\end{array}\Bigg|-\frac{k^2_{00'}}{m_0^2}\right)\right.$$

$$\left.-\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2}F_N\left(1,1,3-n/2,1/2,(n-3)/2,1/2;3/2,3/2,3/2\Big|-\frac{k^2_{10''}}{k^2_{0'0''}},-\frac{k^2_{00'}}{m_0^2},-\frac{k^2_{10'}}{m_{0'}^2}\right)\right\}$$

where F_N is one of the Lauricella-Saran functions,

$$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2|x,y,z) = \sum_{j_1,j_2,j_3}\frac{(a_1)_{j_1}(a_2)_{j_2}(a_3)_{j_3}(b_1)_{j_1+j_3}(b_2)_{j_2}}{(c_1)_{j_1}(c_2)_{j_2+j_3}}\,\frac{x^{j_1}y^{j_2}z^{j_3}}{j_1!j_2!j_3!}$$

See also in: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303 (F_S can be transformed into F_N)

Reduced number of variables and simplified quadratic forms

	total # of dimensionless variables	# of splitting pieces	reduced # of variables
$N=2$	$3-1=2$	2	1
$N=3$	$6-1=5$	6	2
$N=4$	$10-1=9$	24	3
arbitrary N	$\frac{1}{2}(N-1)(N+2)$	$N!$	$N-1$

$J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$:

$$\Rightarrow -[\alpha_1^2 k_{01}^2 + m_0^2]$$

$J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$:

$$\Rightarrow -[\alpha_1^2 k_{10'}^2 + (\alpha_1+\alpha_2)^2 k_{00'}^2 + m_0^2]$$

$J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$:

$$\Rightarrow -[\alpha_1^2 k_{10''}^2 + (\alpha_1+\alpha_2)^2 k_{0'0''}^2 + (\alpha_1+\alpha_2+\alpha_3)^2 k_{00'}^2 + m_0^2]$$

$\Rightarrow$ for $N>4$ we should also expect squares of sums of partial sums of α's

Summary

- Geometrical splitting provides straightforward way to reduce general integrals to those with lesser number of independent variables and predict the set and the number of these variables in the resulting integrals; it also allows to derive functional relations between integrals with different momenta and masses.

- Resulting integrals (after splitting) can be calculated either within geometrical approach (by integrating over non-Euclidean simplices), or by going back to the Feynman parametric representation, which becomes greatly simplified due to right-triangle connections between the invariants.

- Explicit results for general N-point integrals in arbitrary dimension can be presented in terms of hypergeometric functions of $(N-1)$ variables, in particular:
  - for the 2-point diagram we get the hypergeometric function ${}_2F_1$;
  - for the 3-point diagram we get Appell hypergeometric function F_1;
  - for the 4-point diagram we get the Lauricella-Saran function F_N (which can be transformed into F_S).