

R^* and five loop calculations

Ben Ruijl*, Franz Herzog, Takahiro Ueda, Jos Vermaseren,
Andreas Vogt

*LCDS Leiden and Nikhef Amsterdam

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Universiteit
Leiden

Leiden Centre of Data Science



Physics of the pole parts

Interesting physics just from the pole parts of diagrams:

- Anomalous dimensions (beta function, etc.)
- Splitting functions
- Decay rates (Higgs decay, etc.)

Pole parts are easier to compute than the finite parts

Goal

Compute poles of five loop diagrams

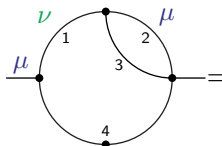
R^* operation

- The poles parts come from the divergent momentum configurations
- The recursive R^* operation takes care of combinatorics of subdivergences [Chetyrkin, Smirnov '83]
- We have extended R^* to Feynman integrals with arbitrary numerator structure [Herzog, Ruijl '17]

UV counterterm operation

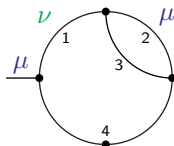
$\Delta(G)$ = poles of G when all momenta go to ∞ with all contributions from subdivergences subtracted

Identifying divergent (sub)diagrams



$$= \int d^D p_1 \int d^D p_2 \frac{Q \cdot p_2 p_1^\nu}{p_1^2 p_2^2 p_3^2 p_4^4}$$

Identifying divergent (sub)diagrams



$$= \int d^D p_1 \int d^D p_2 \frac{Q \cdot p_2 p_1^\nu}{p_1^2 p_2^2 p_3^2 p_4^4}$$

- Get degree of divergence through power counting
- Each loop contributes +4 due to the measure
- All momenta $\rightarrow \infty$: $8 + 1 + 1 - 2 - 2 - 2 - 4 = 0$ (log)
- $p_2, p_3 \rightarrow \infty$: $4 + 1 - 2 - 2 = 1$ (linear)
- $p_4 \rightarrow 0$: $4 - 4 = 0$ (log IR)

R^* -operation by example

$$K\left(\frac{2}{\epsilon^2} + 4 + 2\epsilon\right) \equiv \frac{2}{\epsilon^2}$$

$$K \left(\text{Diagram 1} \right) = \Delta \left(\text{Diagram 2} \right)$$

$$K \left(\text{Diagram 3} \right) = \Delta \left(\text{Diagram 4} \right) + \Delta \left(\text{Diagram 5} \right) \cdot \text{Diagram 6}$$

Consider all sets of non-overlapping divergent subdiagrams

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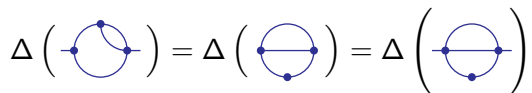
$$K \left(\text{Diagram 3} \right) = \Delta \left(\text{Diagram 4} \right) + \Delta \left(\text{Diagram 5} \right) \cdot \text{Diagram 6}$$

Consider all sets of non-overlapping divergent subdiagrams

$$\begin{aligned} K \left(\text{Diagram 7} \right) &= \Delta \left(\text{Diagram 8} \right) \\ &+ \Delta \left(\text{Diagram 9} \right) \cdot \text{Diagram 10} + \Delta \left(\text{Diagram 11} \right) \cdot \text{Diagram 12} \\ &- \Delta \left(\text{Diagram 13} \right) \Delta \left(\text{Diagram 14} \right) \cdot \text{Diagram 15} \end{aligned}$$

Counterterm operation Δ

For log diagrams, Δ does not depend on external momenta or masses!

$$\Delta \left(\text{Diagram 1} \right) = \Delta \left(\text{Diagram 2} \right) = \Delta \left(\text{Diagram 3} \right)$$


Infrared rearrangement (IRR)

Rearrange diagrams to simpler ones we can compute (always possible)

How to compute Δ ?

- We use its definition:

$$\Delta(G) \stackrel{\text{IRR}}{=} \underbrace{\Delta(G')}_{\text{Simpler than } G} = K(G') - \underbrace{\text{subdivergences}(G')}_{\text{Lower-loop diagrams}}$$

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- Recursive application:

$$\begin{aligned} \Delta \left(\text{circle with two external lines} \right) &= K \left(\text{circle with two external lines} \right) \\ \Delta \left(\text{circle with two external lines and a self-energy loop} \right) &= K \left(\text{circle with two external lines and a self-energy loop} \right) - \Delta \left(\text{circle with two external lines} \right) \cdot \text{circle with two external lines} \end{aligned}$$

- FORCER can compute all rearranged diagrams [Ruijl,Ueda,Vermaseren '17]

Five-loop example

$$K = \Delta + \text{subdiv}$$

Five-loop example

$$K = \Delta \text{ (red diagram) } + \text{ subdiv } \text{ (black diagram) }$$

The diagram illustrates the five-loop example in the R^* -operation. It shows the decomposition of a five-loop diagram K into a sum of two diagrams. The first diagram, labeled Δ , is a red diagram representing a five-loop diagram with external lines $\mu\nu$, ν , and μ . The second diagram, labeled "subdiv", is a black diagram representing a subdivergent diagram with the same external lines. The equation is $K = \Delta + \text{subdiv}$.

Five-loop example

$$\begin{aligned}
 K &= \Delta + \text{subdiv} \\
 \Delta &= K - \text{subdiv}
 \end{aligned}$$

The diagrams are Feynman diagrams for a five-loop process. The first diagram, labeled K , is a circle with three internal lines meeting at a central vertex, labeled $\mu\nu$, ν , and μ . The second diagram, labeled Δ , is a circle with three internal lines meeting at a central vertex, labeled $\mu\nu$, ν , and μ , with a red line connecting the top two vertices. The third diagram, labeled K , is a circle with three internal lines meeting at a central vertex, labeled $\mu\nu$, ν , and μ . The fourth diagram, labeled Δ , is a circle with three internal lines meeting at a central vertex, labeled $\mu\nu$, ν , and μ , with a red line connecting the top two vertices. The fifth diagram, labeled K , is a circle with three internal lines meeting at a central vertex, labeled $\mu\nu$, ν , and μ . The sixth diagram, labeled Δ , is a circle with three internal lines meeting at a central vertex, labeled $\mu\nu$, ν , and μ , with a red line connecting the top two vertices.

The equations show that the five-loop diagram K is equal to the four-loop diagram Δ plus a subdivergence, and the four-loop diagram Δ is equal to the five-loop diagram K minus a subdivergence. The subdivergence is labeled "subdiv" and is represented by a red line connecting the top two vertices.

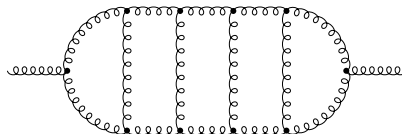
The diagrams are grouped into two categories: "computable!" (the first three diagrams) and "four loops or fewer" (the last three diagrams).

R^* -operation overview

- 1 Make input diagram logarithmic [creates a lot of terms]
- 2 Infrared rearrange diagrams
- 3 Identify counterterms
- 4 Make each counterterm log [**millions** of counterterms]
- 5 Tensor reduce the diagrams
- 6 Repeat process for new counterterms

Computational blow-up

- Troublesome five loop diagrams:



- Represents 12 029 521 scalar integrals!
- Computation may require a terabyte of disk space
- Time-consuming integral reductions
- R^* is slow (subgraph finding, tensor reduction)

Computational blow-up

- Every triple gluon vertex creates 6 terms:

$$v_{3g}(p_1^{\mu,a}, p_2^{\nu,b}, p_3^{\rho,c}) = -if^{abc} \left[(p_1 - p_2)^\rho g_{\mu\nu} + (2p_2 + p_1)^\mu g_{\nu\rho} \right. \\ \left. + (-2p_1 - p_2)^\nu g_{\mu\rho} \right]$$

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- Subgraph finding and Taylor expansions are invariant under momentum contraction, so rewrite the rule:

$$v_{3g}(p_1^{\mu,a}, p_2^{\nu,b}, p_3^{\rho,c}) = -if^{abc} [p_1^\sigma t_3^{\sigma\nu\rho\mu} - p_2^\sigma t_3^{\sigma\mu\rho\nu}]$$

$$t_3^{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2g_{\mu\nu}g_{\rho\sigma} .$$

- Only 1024 terms for time-consuming functions

Tensor reductions

- We encounter rank 10 tensor integrals
- Naive Passarino-Veltman reduction requires 843 908 625 terms
- Using group theory: 46 305 terms (still too much)

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- Naive Passarino-Veltman reduction requires 843 908 625 terms
- Using group theory: 46 305 terms (still too much)
- For rank 6 in general:

$$c_1 \cdot g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} g^{\mu_5 \mu_6} + c_2 \cdot g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} g^{\mu_5 \mu_6} + \dots (13 \text{ more})$$

- 15 coefficients and 15 tensor structures

Tensor reductions: symmetries

- In practice we have symmetries:

$$\Delta \left(\begin{array}{c} \mu_5 \mu_6 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \mu_1 \mu_2 \mu_3 \mu_4 \end{array} \right) \cdot \begin{array}{c} \mu_1 \mu_2 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \mu_3 \mu_4 \mu_5 \mu_6 \end{array}$$

- Symmetric in exchanges of μ_1, \dots, μ_4 and μ_5, μ_6 inside Δ
- Symmetric in μ_1, μ_2 and μ_3, \dots, μ_6 outside Δ

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- Symmetric in exchanges of μ_1, \dots, μ_4 and μ_5, μ_6 inside Δ
- Symmetric in μ_1, μ_2 and μ_3, \dots, μ_6 outside Δ
- We thus see:

$$c_1 \cdot (g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} g^{\mu_5 \mu_6} + 2g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} g^{\mu_5 \mu_6}) + \\ c_3 \cdot (2g^{\mu_1 \mu_2} g^{\mu_3 \mu_5} g^{\mu_4 \mu_6} + 10g^{\mu_1 \mu_5} g^{\mu_2 \mu_6} g^{\mu_3 \mu_4})$$

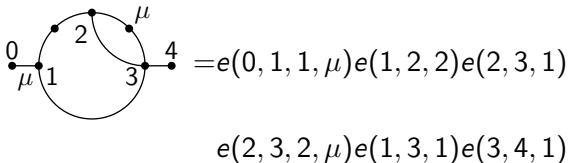
- Only 2 coefficients and 4 terms

Canonical labeling of counterterms

- Prevent double work by canonicalizing counterterms
- Extend the notion of an edge with additional information:

$e(\text{first vertex, second vertex, propagator power, indices})$

- For example:



$$= e(0, 1, 1, \mu) e(1, 2, 2) e(2, 3, 1) e(2, 3, 2, \mu) e(1, 3, 1) e(3, 4, 1)$$

- Use McKay's canonicalization algorithm

Results at five loops

- Computed five loop beta function for general colour group
 - Verified QCD result of [Baikov,Chetyrkin,Kühn '16]
 - Took 6 days on a pc with 32 cores
- Recomputed $H \rightarrow b\bar{b}$, R -ratio
 - Easy: took a few hours on one pc
- Computed $H \rightarrow gg$ [HRUVV '17]
 - Quartically divergent diagrams...
 - Hard: took two months

Acknowledgements

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