High-precision calculation of the 4-loop contribution to the electron g-2 in QED

Stefano Laporta

Dipartimento di Fisica, Università di Bologna, Italy

stefano.laporta@bo.infn.it

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Summary

- The anomalous magnetic moment of a lepton
- Experimental value of electron g-2
- Theoretical value of electron g-2
- High-precision numerical result of the mass-independent QED 4-loop coefficient
- Analytical fits
- How to calculate the g-2
- The program `SYS`

Anomalous magnetic moment of a lepton

$$\vec{\mu} = g \frac{e\hbar}{2mc} \vec{s} \qquad \text{magnetic moment } \mu \ , \quad \text{spin } s$$

g giromagnetic ratio

$$g = \begin{cases} 1 & \text{classical result} \\ 2 & \text{from Dirac equation} \\ 2.002319\ldots & \text{Quantum ElectroDynamics} \end{cases}$$

$$g = 2(1 + a)$$

$$a = \frac{g-2}{2} \quad \text{anomaly}$$

Storage of a single electron in a Penning trap (electrical quadrupole + axial B-field)

$$\omega_c = 2\frac{eB}{mc} \qquad \text{cyclotron frequency}$$

$$\omega_s = g\frac{eB}{mc} \qquad \text{spin precession frequency}$$

$$\omega_a = \omega_s - \omega_c \qquad \text{spin flip frequency}$$

$$a = \frac{g-2}{2} = \frac{\omega_a}{\omega_c} \qquad \text{frequencies ratio}$$

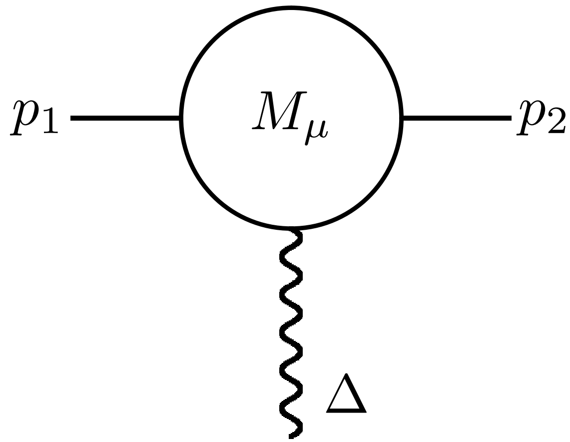
Experimental values of a_e

$a^{exp}_{e^-} = 1\,159\,652\,188.4(4.3) \times 10^{-12}$ 4.3 ppb UW,Dehmelt et al 1987 (Nobel Prize 1989)

$a^{exp}_{e^+} = 1\,159\,652\,187.9(4.3) \times 10^{-12}$ 4.3 ppb UW,Dehmelt et al 1987 (Nobel Prize 1989)

$a^{exp}_{e^-} = 1\,159\,652\,180.85(.76) \times 10^{-12}$ 0.66 ppb Harvard, Gabrielse 2006

$a^{exp}_{e^-} = 1\,159\,652\,180.73(.28) \times 10^{-12}$ 0.24 ppb Harvard, Gabrielse 2008



electron-photon vertex

$$\bar{u}(p_2) M_\mu u(p_1) = \bar{u}(p_2) \left[F_1(-\Delta^2)\gamma_\mu - \frac{i}{4m} F_2(-\Delta^2)(\gamma_\mu \Delta_\mu - \gamma_\nu \Delta_\mu) \right] u(p_1) \ ,$$

Form factors $F_1(t)$, $F_2(t)$

$$F_1(0) = 1 \ , \qquad F_2(0) = a_e$$

Contributions to a_e

dominant small terms (i.e. $\leq 3 \times 10^{-12}$)

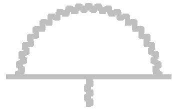
$$a_e^{SM} = a_e^{QED} \overbrace{+ a_e^{QED}(\mu) + a_e^{QED}(\tau) + a_e^{QED}(\mu,\tau) + a_e(\text{hadr}) + a_e(\text{weak})}$$

- a_e^{QED} mass-independent:
  - 1-loop analytical, (Schwinger 1948)
  - 2-loop analytical (Petermann,Sommerfeld 1956)
  - 3-loop analytical (S.L.,Remiddi 1996)
  - 4-loop semi-analytical (S.L. 2017)
  - 5-loop numerical (3% precision) (Kinoshita 2017)
- $a_e^{QED}(X)$ mass-dependent:
  - 2-loop analytical (Elend 1966)
  - 3-loop analytical (S.L.,Remiddi 1992; S.L. 1994)
  - 4-loop analytical expansion in small mass ratio m_e/m_X, (Kurz, Liu, Marquard, Steinhauser 2013)
  - 5-loop numerical (10% precision) (Kinoshita 2014)
- $a_e^{QED}(X)$ small because scales as $(m_e/X)^2$

Mass-independent QED contribution, 1-3 loop

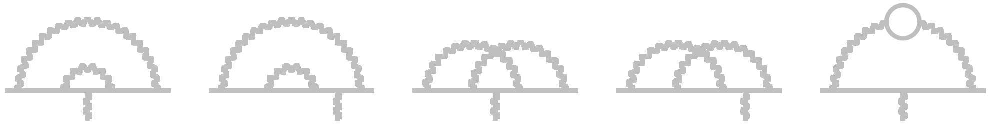
$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

$$C_1 = \frac{1}{2}$$ (Schwinger 1948) *1 diagram*



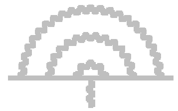
$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$$ fine structure cons

$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

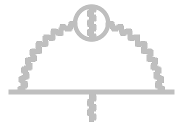


$$= -0.328\ 478\ 965\ 579\ \dots,$$ (Petermann, Sommerfield 1957) *7 diagrams*

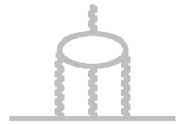
$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right]$$



$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

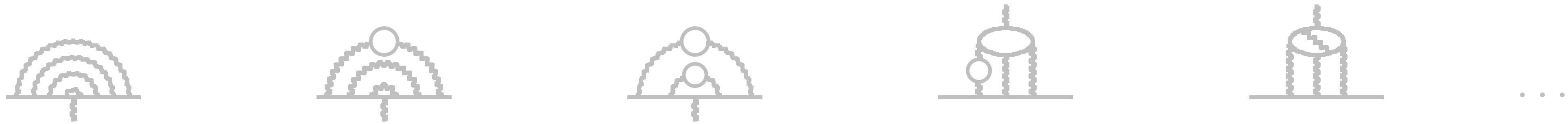


$$= 1.181\ 241\ 456\ \dots,$$ (S.L., E.Remiddi 1996) *72 diagrams*



$\zeta(p)=\sum_{n=0}^{\infty} \frac{1}{n^p}$, $a_4=\sum_{n=0}^{\infty} \frac{1}{2^n n^4}$, ...

Mass-independent QED contribution, 4-loop coefficient



$C_4 \rightarrow$ **891 diagrams** obtained by inserting a photon in 104 self-mass diagrams

A few diagrams containing vacuum polarizations known in analytical form. (Caffo, Turrini, Remiddi 1984)

Previous numerical values obtained by using MonteCarlo integration

$C_4 = -1.434(138)$ (Kinoshita, 1990)
$C_4 = -1.5098(384)$ (Kinoshita, 1999)
$C_4 = -1.7283(35)$ (Kinoshita, 2003) -0.24 shift due to the discovery of one error
$C_4 = -1.9144(35)$ (Kinoshita, 2007) -0.22 shift due to the discovery of another error
$C_4 = -1.91298(84)$ (Kinoshita, 2014)

My new high-precision result is (S.L, 2017)

$$C_4 = -1.912245764926445574152647167439830054060873390658725345171329\ldots$$

good agreement with Kinoshita 2014 (0.9σ) $\longrightarrow$ important independent check!

Mass-independent QED contribution, 5-loop coefficient

5 loop coefficient → **12672 diagrams**

$C_5 = 7.795(336)$ (Kinoshita et al., 2014) -1.2 shift due to the discovery of an error

$C_5 = 6.599(223)$ (Kinoshita et al., June 2017) 3% precision

The electron anomaly is dominated by QED terms. The other interactions contributes to the 10^{-12} level.

$$\alpha^{-1}(Rubidium : 2016) = 137.035\,998\,996(85) \qquad (0.62\ ppb)$$

$$\begin{aligned}
C_1(\alpha/\pi) &= 1\,161\,409\,733.631(720) \times 10^{-12} \\
C_2(\alpha/\pi)^2 &= -\,1\,772\,305.065(3) \times 10^{-12} \\
C_3(\alpha/\pi)^3 &= 14\,804.203 \times 10^{-12} \\
C_4(\alpha/\pi)^4 &= -\,55.667 \times 10^{-12} \\
C_5(\alpha/\pi)^5 &= 0.446(15) \times 10^{-12} \\
a_e^{QED}(\mu) &= 2.738 \times 10^{-12} \\
a_e^{QED}(\tau) &= 0.009 \times 10^{-12} \\
a_e(\text{hadronic v.p.,2-loop}) &= 1.866(11) \times 10^{-12} \\
a_e(\text{hadronic v.p.,3-loop}) &= -\,0.223(1) \times 10^{-12} \\
a_e(\text{hadronic l-l }) &= 0.035(10) \times 10^{-12} \\
a_e(\text{weak}) &= 0.028(1) \times 10^{-12}
\end{aligned}$$

Comparison of a_e the determination of fine structure constant

$$\alpha^{-1}(Rubidium : 2016) = 137.035\,998\,996(85) \qquad (0.62\ ppb)$$

$$a_e^{SM}(\alpha) = 1\,159\,652\,182.031\overset{C_5}{(15)}\overset{had}{(15)}\overset{\alpha}{(720)} \times 10^{-12}$$

$$a_e^{exp} = 1\,159\,652\,180.730(280) \times 10^{-12} \qquad 0.25\ ppb$$

$$a_e^{SM}(\alpha) - a_e^{exp} = 1.30(77) \times 10^{-12} \quad 1.6\sigma \text{ agreement}$$

same order of magnitude of the hadronic contribution and muon-loop vacuum polarization

Assuming QED is correct

$$\alpha^{-1}(a_e) = 137.035\,999\,1500\overset{C_5}{(18)}\overset{had}{(18)}\overset{a_e}{(330)}$$

$$= 137.035\,999\,1500(332) \qquad (0.25\ ppb)$$

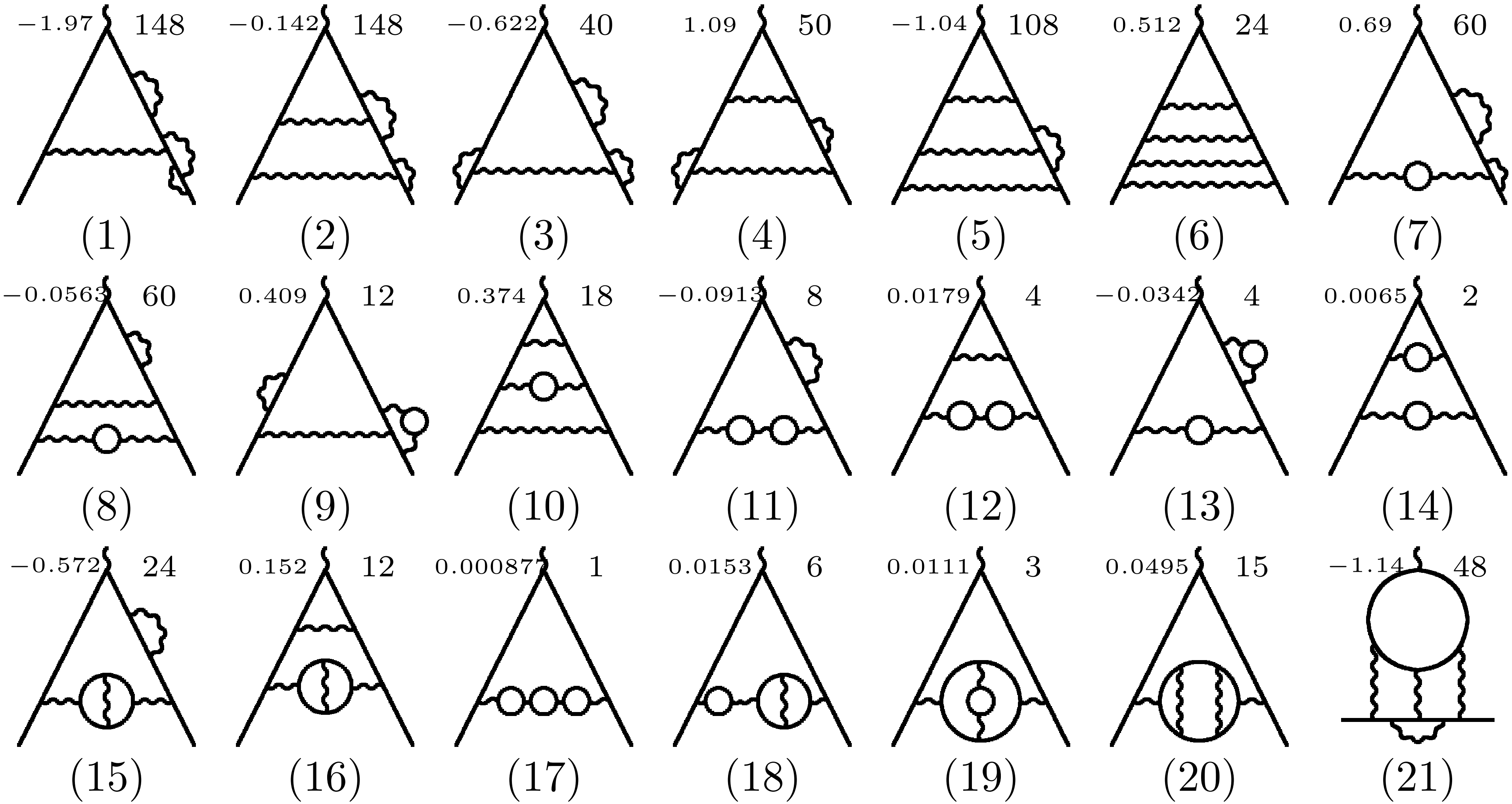
$\alpha(a_e)$ more precise than $\alpha^{exp}(Rubidium) \rightarrow \alpha(a_e)$ used in CODATA least-square adjustment of fundamental constants.

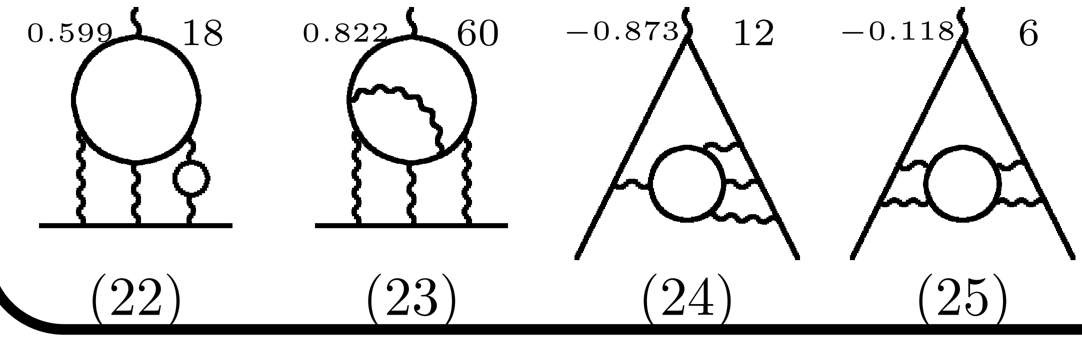
- Contributions of single diagrams may be I.R. or U.V. divergent, and depending on gauge transformation.
- It is convenient to regroup the diagrams in gauge-invariant sets
- The contribution of gauge-invariant sets are I.R. finite and invariant under gauge transformations of the internal photons.

- 2 loops: 7 diagrams arranged into 3 gauge-invariant sets
- 3 loops: 72 diagrams arranged into 9 gauge-invariant sets (Cvitanovic,1977)
- 4 loops: 891 diagrams arranged into 25 gauge-invariant sets

Calculation was performed in the Feynman gauge.

891 diagrams arranged into 25 gauge-invariant sets

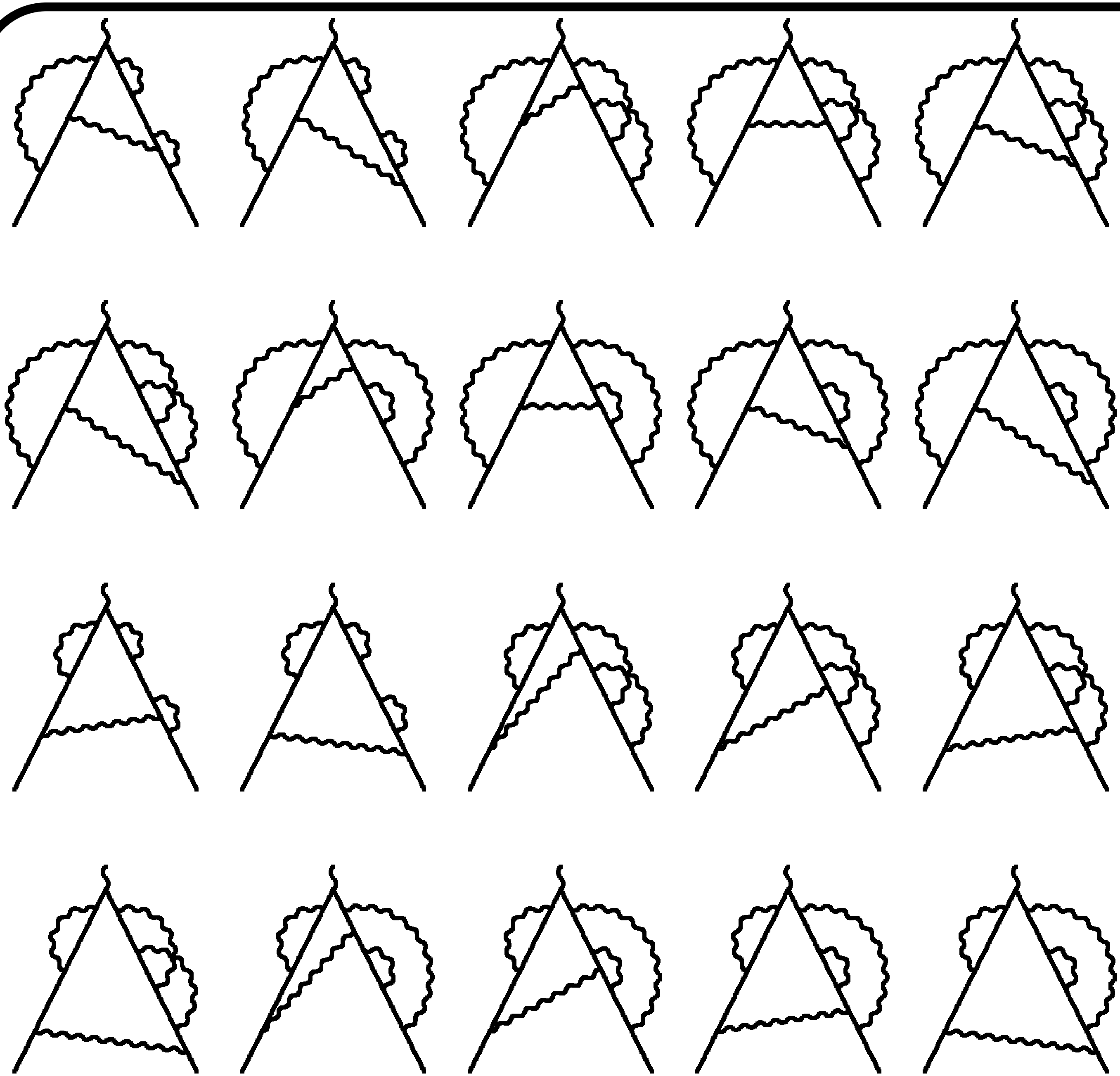




Typical representative diagrams of gauge-invariant sets

Left: contribution to g-2 right: number of diagrams

example: all the diagrams of gauge-invariant set 3 (40 diagrams)



+ mirror images

Partial sums of gauge-invariant sets

Partial sums of gauge-invariant sets:

$$\text{Sum of 13 positive sets} = +\,4.7475040110940016166772\ldots$$

$$\text{Sum of 12 negative sets} = -\,6.6597497760204471908298\ldots$$

$$\text{No closed electron loops} = -\,2.1768660277395400774432\ldots$$

$$\text{Closed electron loops only} = +\,0.2646202628130945032906\ldots$$

$$C_4 = -\,1.9122457649264455741526\ldots$$

C_4: full precision result (1100+ digits)

$C_4 =$

-1.9122457649264455741526471674398300540608733906587253451713298480060
3844398065170614276089270000363158375584153314732700563785149128545391
9028043270502738223043455789570455627293099412966997602777822115784720
3390641519081665270979708674381150121551479722743221642734319279759586
0740500578373849607018743283140248380251922494607422985589304635061404
9225266343109442400023563568812806206454940132249775943004292888367617
4889923691518087808698970526357853375377696411702453619601349757449436
1268486175162606832387186747303831505962741878015305514879400536977798
3694642786843269184311758895811597435669504330483490736134265864995311
6387811743475385423488364085584441882237217456706871041823307430517443
0557394596117155085896114899526126606124699407311840392747234002346496
9531735482584817998224097373710773657404645135211230912425281111372153
0215445372101481112115984897088422327987972048420144512282845151658523
6561786594592600991733031721302865467212345340500349104700728924487200
6160442613254490690004319151982300474881814943110384953782994062967586
7875385249781946989793132162197975750676701142904897962085 05...

Analytical PSLQ fits

1100 digits? why such an high precision?

Due to the expected analytical complexity of C_4, a completely analytical calculation (similar to that of C_3) seemed out of reach. So I decided to:

1. compute an extremely high-precision valuer of C_4
2. make a (right) analytical ansatz (not easy!)
3. fit an analytical expression by using the PSLQ algorithm

- PSLQ algorithm (Ferguson and Bailey 1992)
- find integer relation between real numbers or bounds on size of coefficients.
- requires high precision; at least number of digits of coefficients * number of real numbers
- a parallel version of the algorithm exists (Bailey and Broadhurst 1999)

4-loop analytical result very complicated! Analytic terms divided in 5 groups:

1. usual constants (harmonic) polylogarithms of 1 and 1/2.
2. harmonic polylogarithms of arguments $e^{i\pi/3}$ and $e^{2i\pi/3}$ new
3. harmonic polylogarithms of arguments $e^{i\pi/2}$ new
4. elliptic constants with semi-analytic representation new
5. unknown elliptic constants

- 202 diagrams $\rightarrow H_n(1), H_n(1/2), \ldots,\ n = 2, \ldots, 7$
- 339 diagrams $\rightarrow H_n(e^{ix\pi/3}),\ n = 1, \ldots, 7$
- 6 diagrams $\rightarrow H_n(e^{ix\pi/3}) + H_n(e^{ix\pi/3})$
- 32 diagrams $\rightarrow$ elliptic
- 300 diagrams $\rightarrow$ elliptic $+ H_n(e^{ix\pi/3})$
- 12 diagrams $\rightarrow$ elliptic $+ H_n(e^{ix\pi/3}) + H_n(e^{ix\pi/2})$

4-loop 891 diagrams can be obtained by insertion of one photon in 104 self-mass diagrams

the 104 4-loop electron self-masses

analytical fit "usual constants"

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \qquad \text{121 terms}$$

$$\begin{aligned}
T = & \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) \\
& + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) \\
& - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 \\
& - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) \\
& + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 \\
& + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 \\
& + \frac{407771}{432}\zeta^2(3)\ln 2 - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 \\
& - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2
\end{aligned}$$

$a_n = \mathrm{Li_n}(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

terms of weight 5 cancel out

$$\begin{aligned}
V_a =& -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) \\
&+\frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&-\frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi \\
&-\frac{223}{243}\zeta(4)\pi\ln 2 + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&-\frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

$$\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(\mathrm{e}^{\mathrm{i}\theta})$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{13487}{60}\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13487}{60}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) \\
&+ \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&- \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi \\
&+ \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&+ \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&+ \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)
\end{aligned}$$

analytical fit $HPL(e^{i\pi/2})$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned} W_b =& -\frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 \\ & +104\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\ & -2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg) \end{aligned}$$

$\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ is the Catalan's constant $\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$

analytical fit (elliptic constants)

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
E_a =& \pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) \\
&+\pi\left(\frac{4715}{1944}\ln 2\; f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
&+\pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\; f_2(0,0,2) + \frac{5525}{162}\ln 2\; f_2(0,1,1)\right. \\
&-\frac{5525}{243}\ln 2\; f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
&-\frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) \\
&\left.+\frac{69245}{124416}f_2(2,1,0)\right)
\end{aligned}$$

$$\begin{aligned}
E_b =& -\frac{4715}{1458}\zeta(2)f_1(0,0,1) \\
&+\zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2) - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\right) .
\end{aligned}$$

$$A_3 = \int_0^1 dx \frac{K_c(x) K_c(1\text{-}x)}{\sqrt{1-x}} = \frac{2\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3 \left({\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}} ; 1 \right) - \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3 \left({\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}} ; 1 \right) \right)$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{4\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3 \left({\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}} ; 1 \right) + \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3 \left({\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}} ; 1 \right) \right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{486\pi^2}{1925}\ {}_7F_6 \left({\frac{7}{4}\ -\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ \frac{4}{3}\ \frac{3}{2}\ \frac{3}{2} \atop \frac{3}{4}\ 1\ \frac{7}{6}\ \frac{11}{6}\ \frac{13}{6}\ \frac{17}{6}} ; 1 \right) ,$$

$$K_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1 \left({\frac{1}{3}\ \frac{2}{3} \atop 1} ; x \right) , \qquad E_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1 \left({\frac{1}{3}\ -\frac{1}{3} \atop 1} ; x \right) .$$

A_3 cancels out in the diagram contributions

analytical fit (family of semi-analytical elliptic constants)

f_j are defined as follows:

$$f_1(i,j,k) = \int_1^9 ds\, D_1^2(s) \left(s - \frac{9}{5}\right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) \;,$$

$$f_2(i,j,k) = \int_1^9 ds\, D_1(s) \mathrm{Re}\left(\sqrt{3} D_2(s)\right) \left(s - \frac{9}{5}\right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) \;,$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \;,$$

$$D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \;;$$

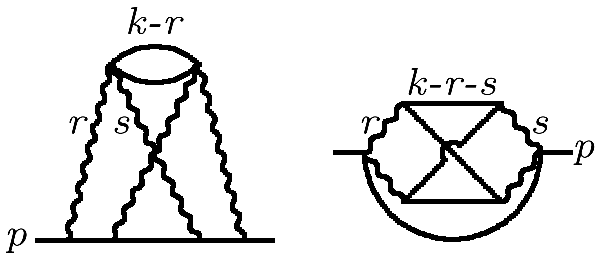
$K(x)$ is the complete elliptic integral of the first kind.

$D_1(s) \sim$ discontinuity of the 2-loop sunrise diagram with equal masses in $D = 2$ dimensions.

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

The term containing the ϵ^0 coefficients of the ϵ−expansion of six master integrals (see f, f', f'', g, g', g''):

$$U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}\ .$$



(f, f', f'') $\quad$ (g, g', g'')

(f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$

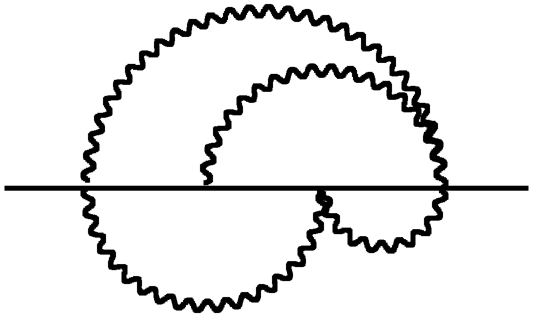
These master integrals appear in topologies 81 and 83 (gauge-invariant sets 24 and 25, vacuum polarization diagrams containing a light-light scattering).

$$C_4 = \sum_{i=1}^{334} p_i(D) M_i(D) \; - \text{ren.count.}$$

334 master integrals

$$M_i(D = 4-2\epsilon) = \frac{c_{i,1}}{\epsilon^4} + \frac{c_{i,2}}{\epsilon^3} + \frac{c_{i,3}}{\epsilon^2} + \frac{c_{i,4}}{\epsilon} + c_{i,5} + c_{i,6}\epsilon + c_{i,7}\epsilon^2 + c_{i,8}\epsilon^3 + c_{i,9}\epsilon^4 + \ldots$$

PSLQ example: analytical fit of the constants of a simple Feynman master integral



$$M = G_1\epsilon^{-4} + G_2\epsilon^{-3} + G_3\epsilon^{-2} + G_4\epsilon^{-1} + G_5 + G_6\epsilon + G_7\epsilon^2 + G_8\epsilon^3 + \ldots$$
$$\epsilon = (4-D)/2$$

G_i calculated numerically

$$G_1 = -0.125 = -\frac{1}{8}$$
$$G_2 = -1.0208333333333333333333333333333333333333333333333\ldots$$
$$= -\frac{49}{48}$$

PSLQ example: analytical fit of the constants of a simple Feynman master integral

$$
\begin{aligned}
G_3 = -\,& 6.92304585176135691250561758073508351793477638071038121196 5\\
& 0273403742599066296907522601901707577596677104372311933008 83\\
& 7222080560196749827281566346982681496908920375341757759717 88\\
& 1801279665946352521256740830557458021906431436568471148502 55\\
& 7715229006714242899539741558924048835784950659811389470098 63\\
& 9893841125240788540758490814881847786222960311370319940157 87\\
& 3557094358420426841440317445474545987282459679309471053376 35\\
& 4785143357498033138668937081465899387397891572388565607386 77\\
& 0372686471384237168290806493123605415111390097376323550069 25\\
& 8854841971673105540019163639177528860542207366989926442538 27\\
& 4910223441048619340927332587572404877536112433092197630242 10\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
= & -\frac{449}{96} - \frac{1}{6}\pi^2 - \frac{1}{2}\zeta(3)
\end{aligned}
$$

PSLQ example: analytical fit of the constants of a simple Feynman master integral

$$G_4 = -37.34590476123416646296258857716878202975412845114983929350$$

975379101494569986522447286295965498175193008755049407617318

241541230755384984012771693284382137786671238383193278215911

699745531610241557815326518069833637529200263227759084846413

891908555842241268629613734515267877304185820333971736014274

870837672418857267208975059165822228755252807452394315786825

977396123268312697403717987494727482026759704834143753522289

356366953699461348143553177120930854756026079409386426269634

939970863752335142927754017415741426672268787548791316415626

354788806525645923920039447209933591118993917501594536291837

....

$$= -\frac{2429}{192} - \frac{7}{4}\pi^2 - \frac{11}{2}\zeta(3) - \frac{1}{120}\pi^4$$

PSLQ example: analytical fit of the constants of a simple Feynman master integral

$$G_5 = -143.47667143390176837023343649555889412613383285716961326464298325679292473455106703850978281693651964137907412200263538025361361239319963824321514424204788062148835099055742173911428722265718783591831768768958827443810483492617807476951269881273876711301821017658409399734696701406855998264688225272127755625878113739536035140838781174979446599245583374771865649618037985630458850201101201670058683600307758880583977551059458017236861981872962895923905661504533595803110981399424241291445970205296408268497263501085248266025318207296387320468394841970136688774723668896495701527265876225522921623501996225012249135019274123015912429620404066565985205430342800\ldots$$

$$= \frac{2687}{384} - \frac{277}{24}\pi^2 - \frac{125}{3}\zeta(3) + \frac{1}{24}\pi^4 + \frac{3}{2}\zeta(5) + \frac{2}{3}\pi^2\zeta(3)$$

PSLQ example: analytical fit of the constants of a simple Feynman master integral

$$
\begin{aligned}
G_6 = -\,&759.4931219244805531750027062565091039800739025248110753812\\
&535030783175454276732950075929236453708605626801145370849082\\
&255534947151867951949458895745617956028279728229330613197811\\
&005565630190911782627506827079524253744740621041739130359580\\
&396421188172103050058536274356882955796037739819050629546700\\
&624555062338054840118285721965054834801057864007530232777424\\
&163014869814929392743200585991919690018429850597819939478973\\
&901106830219199405493921430338252888933871882176410104701353\\
&379896656372984976108900161124311605029088620444720092933626\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
=&\frac{95689}{256} - \frac{2377}{48}\pi^2 - \frac{2999}{12}\zeta(3) - \frac{46}{45}\pi^4 + \frac{149}{2}\zeta(5)\\
&+\frac{15}{2}\zeta(3)^2 - \frac{58}{3}\pi^2\zeta(3) - \frac{29}{270}\pi^6
\end{aligned}
$$

PSLQ example: analytical fit of the constants of a simple Feynman master integral

$$G_7 = -2342.20751410602307542352254059079270988532873205655947080 7$$

$$359481483571384691680645591697318599261483194890419734356986$$

$$640536482839180927737599376306979737829110608311707671767935$$

$$983139125960766918329923883871930584868496516072868729243183$$

$$317800519694759939914751761141283435810030791136838793708071$$

$$157346099787020302357526852412095436287332846448926242430503$$

$$236449547474407307581291123637921078586418676517549877972867$$

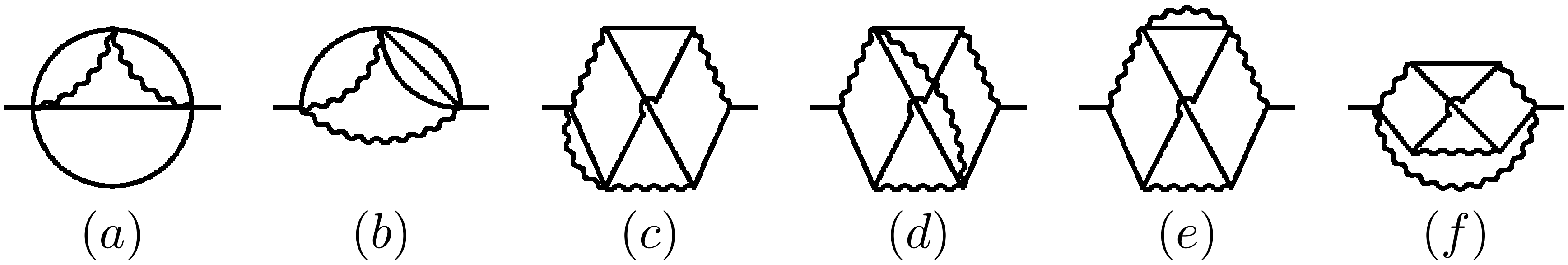
$$914941194048119667003119789862197556984062402874182297886028$$

....

$$= \frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5)$$

$$-\frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2$$

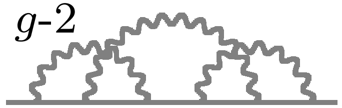
$$+504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)$$



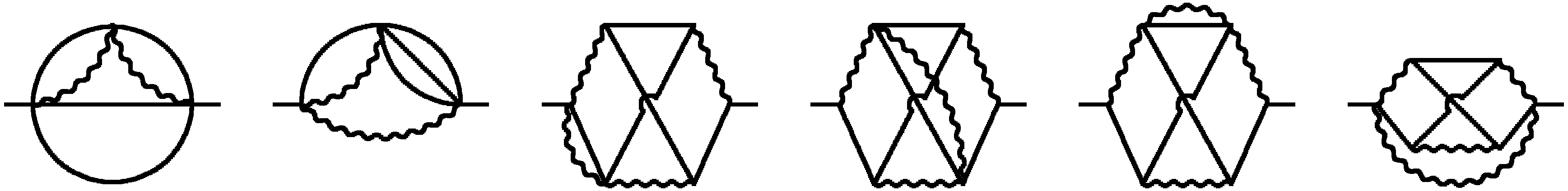
$$I(a) = \frac{7}{12\epsilon^4} + \frac{10}{3\epsilon^3} + \frac{121}{12\epsilon^2} + \left(\frac{1541}{72} + \frac{7}{6}\zeta(3)\right)\epsilon^{-1} + \frac{42155}{432} - \frac{380}{3}\zeta(2) + \frac{14}{3}\zeta(3) + \frac{3}{2}\zeta(4)$$
$$+\sqrt{3}\left(6\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - 10\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) + O(\epsilon^2)$$ self-mass 60

$$I(b) = \frac{5}{8\epsilon^4} + \frac{59}{16\epsilon^3} + \left(\frac{1099}{36} + 3\zeta(2)\right)\epsilon^{-2} + \left(\frac{3781}{192} + \frac{33}{2}\zeta(2) + 6\zeta(3)\right)\epsilon^{-1} + \frac{25033}{1152} - \frac{47}{4}\zeta(2)$$
$$+\frac{69}{2}\zeta(3) + \frac{411}{8}\zeta(4) - \sqrt{3}\left(9\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + 9\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) + O(\epsilon^2)$$ self-mass 63

(S.L. 1993) 3-loop g-2



$\pi\left(9\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + 5\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)$ appeared in intermediate results, cancelled out in final results.

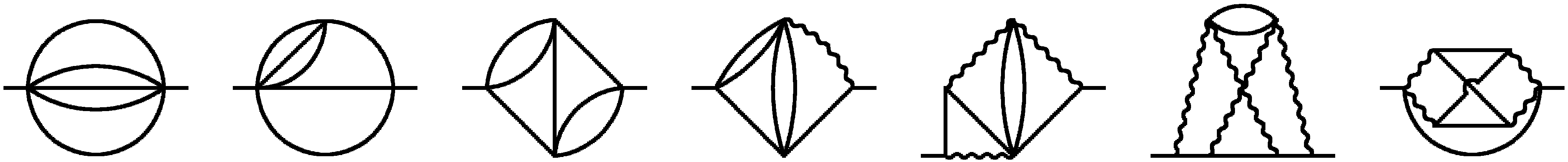


(a) (b) (c) (d) (e) (f)

$$I(c) = \left(-\frac{45}{4}\zeta(5) + \frac{17}{2}\zeta(2)\zeta(3)\right)\epsilon^{-1} - \frac{14897}{96}\zeta(6) - 45\zeta(5) - \frac{7}{16}\zeta^2(3) + 84\zeta(2)a_4 + 34\zeta(2)\zeta(3)$$
$$+\frac{147}{2}\ln 2\zeta(2)\zeta(3) - \frac{105}{2}\ln^2 2\zeta(4) + \frac{7}{2}\ln^4 2\zeta(2) + \sqrt{3}\Big(4\ \mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right)$$
$$+4\ \mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + 4\ \mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{54}{13}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+\frac{207}{26}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + 7\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{2}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+\frac{36}{5}\zeta(2)\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{24}{5}\zeta(2)\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - 12\zeta(2)\ln 2\ \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)$$
$$-18\zeta(2)\ln 2\ \mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{4787}{143}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{40}{3}\mathrm{Im}H_4\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{5}{9}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2$$
$$+\frac{1765}{12}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{3469}{27}\zeta(2)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{20}{3}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{27413}{16848}\pi\zeta(5)$$
$$+\frac{1}{9}\pi\zeta(4)\ln 2 - \frac{97}{90}\pi\zeta(2)\zeta(3)\Big) + O(\epsilon);$$

$I(d), I(e), I(f)$ similar expressions

Elliptic master integrals



(a) (b) (c) (d) (e) (f) (g)

$$I(a, D{=}4\text{-}2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + \frac{\pi\sqrt{3}}{240}(297B_3 - 1477C_3) - \frac{2320981}{20736} + O(\epsilon)$$

$$I(a, D{=}2\text{-}2\epsilon) = \sqrt{3}\pi B_3 + M_{501}\epsilon + M_{602}\epsilon^2 - M_{702}\epsilon^3 + \ldots$$

These M.I. were calculated with 5000-10000 digits.

PSLQ fits

- The presence of harmonic polylogarithms of complex arguments increases considerably the size of the basis needed to fit the numerical values.
- The general basis of real and imaginary parts of harmonic polylogarithms of argument of $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$ up to weight 7 has dimension $F_{17} - 1 = 1596$.
- To fit successfully quantities at weight seven, it is necessary to increase considerably the precision of the calculations, and to try various selections of the general basis.
- The fit of V_a, V_b and the master integrals involved has needed thousands of PSLQ runs with basis of ~ 500 elements calculated with 9600 digits of precision.
- The multi-pair parallel version of PSLQ algorithm has been essential to work out these difficult analytical fits in reasonable times (each one 2d on 8-core cpu).

- Generation of diagrams and extraction of the contribution to g-2
- Reduction to master integrals
- Numerical calculation of master integrals
- Renormalization: generation and calculation of the counterterms
- Checks

generation of diagrams and extraction of contribution

The approach has been devised in order to maximize the number of possible consistency checks and minimize error sources, even at the expense of big increasing of computer time.

- All the 4-loop self-mass diagrams are generated with a C program
- For each of the 104 self-mass diagrams, vertex diagrams are built by inserting a photon in all possible ways. One keeps also the non-contributing diagrams because of Furry's theorem for the sake of subsequent checks.
- The contribution to g-2 is extracted from the unrenormalized amplitude of each vertex diagram using suitable projectors (FORM). The results are expressions which typically contain 100-30000 different Feynman integrals.

Reduction to master integrals

- The total number of topologically distinct master integrals is 334.
- 82 topologies have number of master integrals $n > 1$
- For topologies with $n > 1$, one chooses scalar products as numerators, which depend on the momentum flow of the diagram.
- For the sake of subsequent checks, one processes separately the contributions from each one of the 104 self-mass diagrams.
- The master integrals are calculated separately for each self-mass diagram. The total number of of master integrals to be calculated increases to 4607.
- Because of differences in the numerators, topologies with $n > 1$ will have a slightly different set of master integrals in different self-mass diagrams. This provides a lot of useful checks.

Reduction to master integrals

- The reduction (exact in D) is performed by generating a large system of integration-by-parts identities and solving it with the algorithm (L.S.,2001) implemented in SYS.
- The number of necessary identities is $4 \times 10^6 - 50 \times 10^6$. Disk sizes of systems are $4GB - 100GB$.
- Reduction was repeated on two different machines, with 32bit and 64bit versions of SYS. In addition, the principal self-mass diagrams were reprocessed using a different momentum flow, checking that reduction to master integrals remained the same (after converting different sets of master integrals).

Numerical calculation of master integrals

- I used combinations of the difference methods (Laporta,2001) and differential equations methods (Kotikov,1991), (Remiddi,1997), (Remiddi,Gehrmann 2000).
- An approach consists in calculating with difference equations (inserting the exponent n of the first electron propagator) the integral obtained putting the photon mass λ equal to the electron mass m, and integrating a differential equation in λ from $\lambda = m$ to $\lambda = 0$.
- An alternative approach consists in calculating with difference equations the diagram obtained by putting the the external momentum $p = 0$, and integrating a differential equation in p^2 from $p^2 = 0$ to $p^2 = -m^2$.
- The systems difference and differential equations for master integrals are obtained by building systems of suitable integration-by-parts identities and solving them by using the algorithm (Laporta,2001), using rational arithmetic in D.
- The sizes of the systems of difference or differential equations to be numerically solved are in the range $1MB - 3GB$.

Numerical calculation of master integrals

- Due to the difference in n,p or ,m, there are "master integrals" of the systems of difference and differential equations which actually are not master integrals. Nevertheless for the sake of subsequent checks, I calculated also them. The number of master integrals calculated increases to 10083.
- Difference equations are solved using the Laplace transformation method (integral representation of solutions and differential equation of the integrand).
- Differential equations are solved numerically, by using series expansion with truncated expansions in $\epsilon = (4 - D)/2$ as coefficients.
- The minimum number of terms of the expansion in ϵ in 9 ($e^{-4} \ldots\ e^{4}$).
- There are cancellations of ϵ terms in intermediate steps; no care is used avoiding cancelations, as the corresponding numerical zeroes are extremely useful checks. In the worst cases 37 terms of expansions are needed.
- The standard precision of calculations is 4096bit (1232 digits). About 130 digits are lost due to cancellations, so $1232 - 130 = \ 1100$.

- The fit of some selected master integrals required a precision much higher, up to 16kbits (9864 digits).

Renormalization

- Renormalization counterterms are generated with two procedures C and FORM.
- Their expression are reduced to master integrals, already known in analytical form.
- The calculation was performed in the Feynman gauge
- I checked explicitly the (internal) gauge invariance of the expressions for arbitrary gauge of the photon line going into 1-,2- or 3-loop vacuum polarization diagrams.

Calculations

In order to perform this calculation, in 1995 I begun writing a C program, `SYS`, containing all the necessary ingredients:

- a symplified fast algebraic (invoking repeatedly FORM, that I had successfully used for C_3, has a not negligible time cost)
- a numerical solver of systems of difference and differential equations
- a library of arbitrary precision mathematical routines, integer and floating point (in mid-1990 the GMP library was still in its infancy).

The program SYS

- C program, about 23000 lines.
- The program automatically determines the master integrals of a diagram, it builds and solves the systems of difference or differential equations.
- Input: description of the diagram, number of terms of the expansion in $D-4$.
- The program contains a simplified algebraic manipulator, used to solve systems of identities among integrals with this kind of coefficients: arbitrary precision integers, rationals, ratios of polynomials in one and two variables (for example D and x) with integer coefficients.
- Efficient management of systems of identities of size up to the limit of disk space (tested up to 500 million of identities).
- Numerical solution of systems of difference and differential equations up to 900 equations, using arbitrary precision floating point complex numbers and truncated series in ϵ.
- All the coefficients of the expansions in ϵ are worked out in numerical form,

even those of divergent terms.

- Floating number precision: up to 9800 digits (essentially one sums expansions in *one* variable).
- Arithmetic libraries which deal with operations on arbitrary precision integers, polynomials, rationals, arbitrary precision floating point numbers and truncated series in ϵ were written on purpose by the author. *Independent* of all other available libraries.
- Several Multicore/multinode parallel versions of the program were written on purpose.
- Sistematic protection of large buffers, I/O with crc/checksums. Found several subtle corruptions in the years, like marginal coupling of non-ECC RAM modules (1 bit changed per week), failing RAID systems (corrupted blocks of 64KBytes), etc. . . .)

  This has permitted to obtain an high reliability result.

Observations

- a_e is dominated by the QED contribution
- hadronic and muon-loop vacuum polarizations contributes to 10^{-12} level
- 1,2,3-loop QED coefficient are known exactly
- 4-loop QED coefficient is now known "near-exactly" (more than 1100 digits)
- the ultimate limit is the error in the hadronic contribution $\approx 10^{-14}$
- that corresponds to $0.15\left(\frac{\alpha}{\pi}\right)^5$ or $64\left(\frac{\alpha}{\pi}\right)^6$
- historically checks with the experiment or independent theoretical results have often highlighted inconsistences in QED contributions
- for this reason an independent calculation of 5-loop coefficient would be important

Possible new results

There are some quantities which can be calculated using the results of the QED C_4 calculation (table of master integrals):

- the high-precision calculation of the slope $F_2'(0)$ (in progress)
- the high-precision calculation of renormalization constants in QED at 4 loop (in progress)
- the high-precision calculation of the on-shell renormalization constants in QCD at 4 loop; due to the presence of the gluon-gluon vertex in QCD, this will require time-consuming high-precision calculations of some (~ 50) additional master integrals. Adequate computing resources are necessary.
- High-precision C_5? Surely a many-years task.

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The End