Numerical techniques for 2- and 3-loop integrals

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I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, J. Usovitsch, arXiv:1607.08375, arXiv:17mm.nnnnn

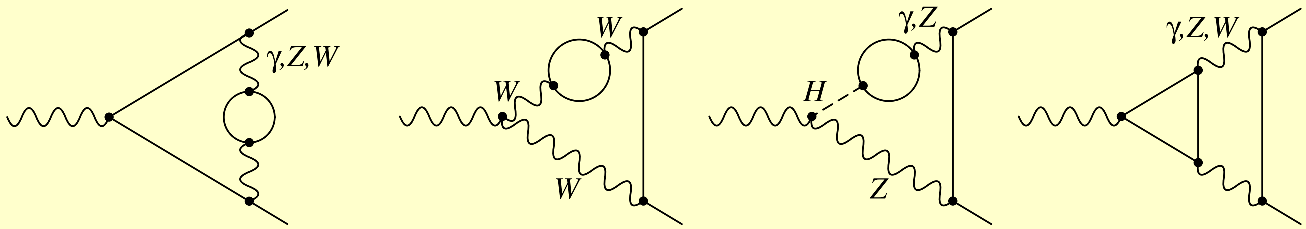
A. Freitas, arXiv:1609.09159, arXiv:1702.02996

1. $\mathcal{O}(\alpha^2)$ **bosonic corrections** $Zf\bar{f}$ **vertices**

2. Techniques for general 3-loop vacuum integrals

$\mathcal{O}(\alpha^2)$ bosonic corrections $Zf\bar{f}$ vertices

Known corrections to $Zf\bar{f}$ vertices:

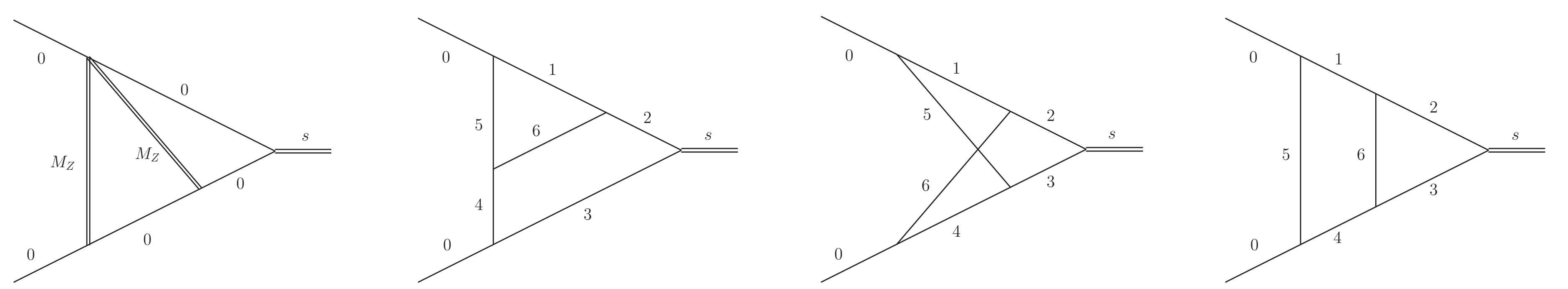


- One-loop — Sirlin, Marciano '80; Akhundov, Bardin, Riemann '86
- $\mathcal{O}(\alpha\alpha_s)$ QCD — Djouadi, Verzegnassi '87; Kniehl '90; Djouadi, Gambino '93; Fleischer, Tarasov, Jegerlehner, Raczka '92; Buchalla '93; Degrassi '93; Czarnecki, Kühn '96; Harlander, Seidensticker, Steinhauser '97
- "Fermionic" NNLO corrections (g_{Vf}, g_{Af}) — Czarnecki, Kühn '96; Harlander, Seidensticker, Steinhauser '98; Freitas '13,14
- Partial 3/4-loop corrections to ρ/T-parameter
  $\mathcal{O}(\alpha_t\alpha_s^2)$, $\mathcal{O}(\alpha_t^2\alpha_s)$, $\mathcal{O}(\alpha_t\alpha_s^3)$ — Chetyrkin, Kühn, Steinhauser '95; Faisst, Kühn, Seidensticker, Veretin '03; Boughezal, Tausk, v. d. Bij '05; Schröder, Steinhauser '05; Chetyrkin et al. '06; Boughezal, Czakon '06

$(\alpha_t \equiv \frac{y_t^2}{4\pi})$

Calculational approach for bosonic $\mathcal{O}(\alpha^2)$ corrections

- Two-loop diagrams without closed fermion loops
- On-shell renormalization
- Self-energies (incl. from renormlization) and vertices with sub-loop bubbles using dispersion relation technique
  S. Bauberger et al. '95
  Awramik, Czakon, Freitas '06
- Non-trivial vertex diagrams:
  - Sector decomposition (FIESTA 3 / SecDec 3) Smirnov '14; Borowka et al. '15
  - Mellin-Barnes representations (MB / AMBRE 3 / MBnumerics) Czakon '06
    Dubovyk, Gluza, Riemann '15; Usovitsch '17
  - No tensor reduction (besides trivial cancellations)
    $\longrightarrow$ About 700 different two-loop vertex integrals



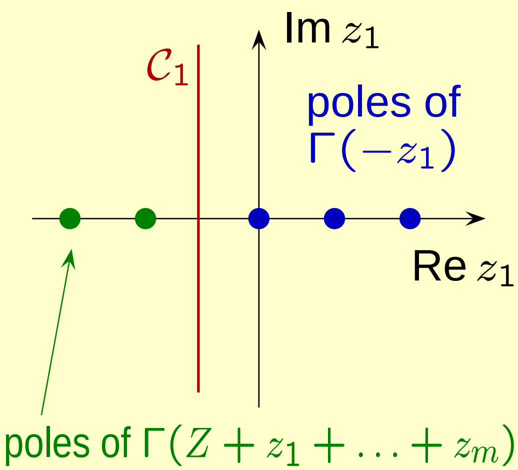
Mellin-Barnes representations

Transform Feynman integral with Mellin-Barnes representation

$$\frac{1}{(A_0+\ldots+A_m)^Z} = \frac{1}{(2\pi i)^m}\int_{\mathcal{C}_1} dz_1 \cdots \int_{\mathcal{C}_m} dz_m$$

$$\times\, A_1^{z_1}\cdots A_m^{z_m} A_0^{-Z-z_1-\ldots-z_m}$$

$$\times\, \frac{\Gamma(-z_1)\cdots\Gamma(-z_m)\Gamma(Z+z_1+\ldots+z_m)}{\Gamma(Z)},$$



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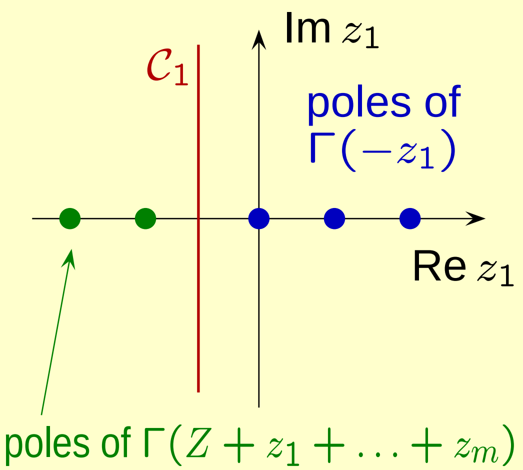
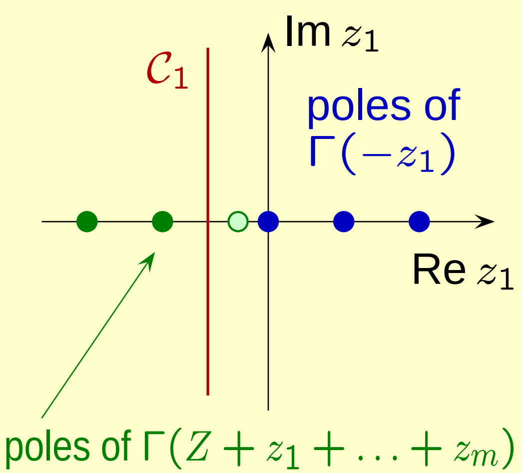


- Consistent choice of all $\mathcal{C}_i$ often requires $\varepsilon \neq 0$ $(Z = n+\epsilon)$
- For $\varepsilon \to 0$: residues from pole crossings $\to 1/\varepsilon^k$ terms

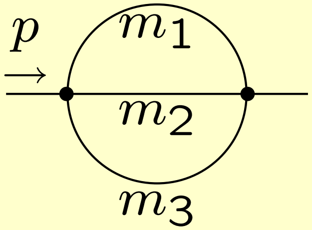
Czakon '06
Anastasiou, Daleo '06

- Do remaining $\mathcal{C}_i$ integrations numerically



$\varepsilon \longrightarrow 0$

Mellin-Barnes representations

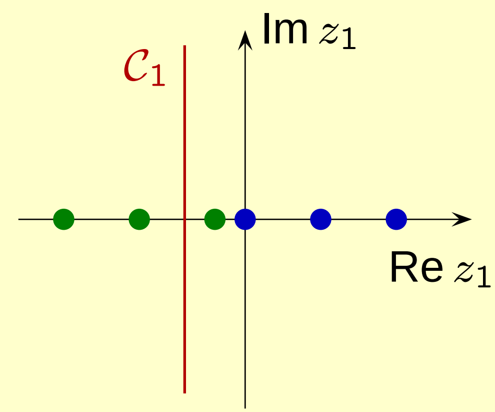


$$= \frac{-1}{(2\pi i)^3}\int dz_1 dz_2 dz_3\, (m_1^2)^{-\varepsilon-z_1-z_2}(m_2^2)^{z_2}(m_3^2)^{1-\varepsilon+z_1-z_3}(-p^2)^{z_3}$$
$$\times\, \Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_1+z_2)\Gamma(z_3-z_1)$$
$$\times\, \frac{\Gamma(1-\varepsilon-z_2)\Gamma(\varepsilon+z_1+z_2)\Gamma(\varepsilon-1-z_1+z_3)}{\Gamma(2-\varepsilon+z_3)}$$

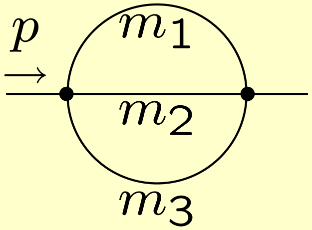
$z_3 = c_3 + iy_3, \qquad y_i \in (-\infty, \infty)$

$$(-p^2)^{z_3} = \underbrace{(p^2)^{c_3+iy_3} e^{-i\pi c_3}}_{\text{oscillating}}\, \underbrace{e^{\pi y_3}}$$

div. for $y_3 \to \infty$, eventually over-come by Γ funct.



Mellin-Barnes representations



$$= \frac{-1}{(2\pi i)^3} \int dz_1 dz_2 dz_3 \, (m_1^2)^{-\varepsilon - z_1 - z_2} (m_2^2)^{z_2} (m_3^2)^{1-\varepsilon+z_1-z_3} (-p^2)^{z_3}$$
$$\times \Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_1+z_2)\Gamma(z_3-z_1)$$
$$\times \frac{\Gamma(1-\varepsilon-z_2)\Gamma(\varepsilon+z_1+z_2)\Gamma(\varepsilon-1-z_1+z_3)}{\Gamma(2-\varepsilon+z_3)}$$

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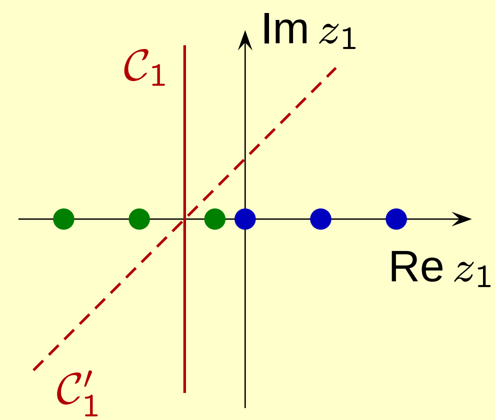
div. for $y_3 \to \infty$

$$y_i \to y_i - i\theta$$

$$(-p^2)^{z_3} = (p^2)^{c_3+iy_3} e^{-i\pi(c_3+\theta y_i)} e^{(\pi+\theta \log p^2) y_3}$$

Huang, Freitas '10



Mellin-Barnes representations

Counter rotations not always successful:

$$\frac{1}{(2\pi i)^2}\int dz_1 dz_2\, 2(m^2)^{-2}\Big(-\frac{p^2}{m^2}\Big)^{-z_1-z_2} \times \frac{\Gamma(-z_2)\Gamma^3(1+z_2)\Gamma(-z_1-z_2)\Gamma(1+z_1+z_2)\Gamma(-1-z_1-2z_2)}{\Gamma(1-z_1)}$$

For $p^2 = m^2$ contour rotation has no effect

Shift countour: $z_1 = c_1 + iy_1,\ z_2 = c_2 + n + iy_2$

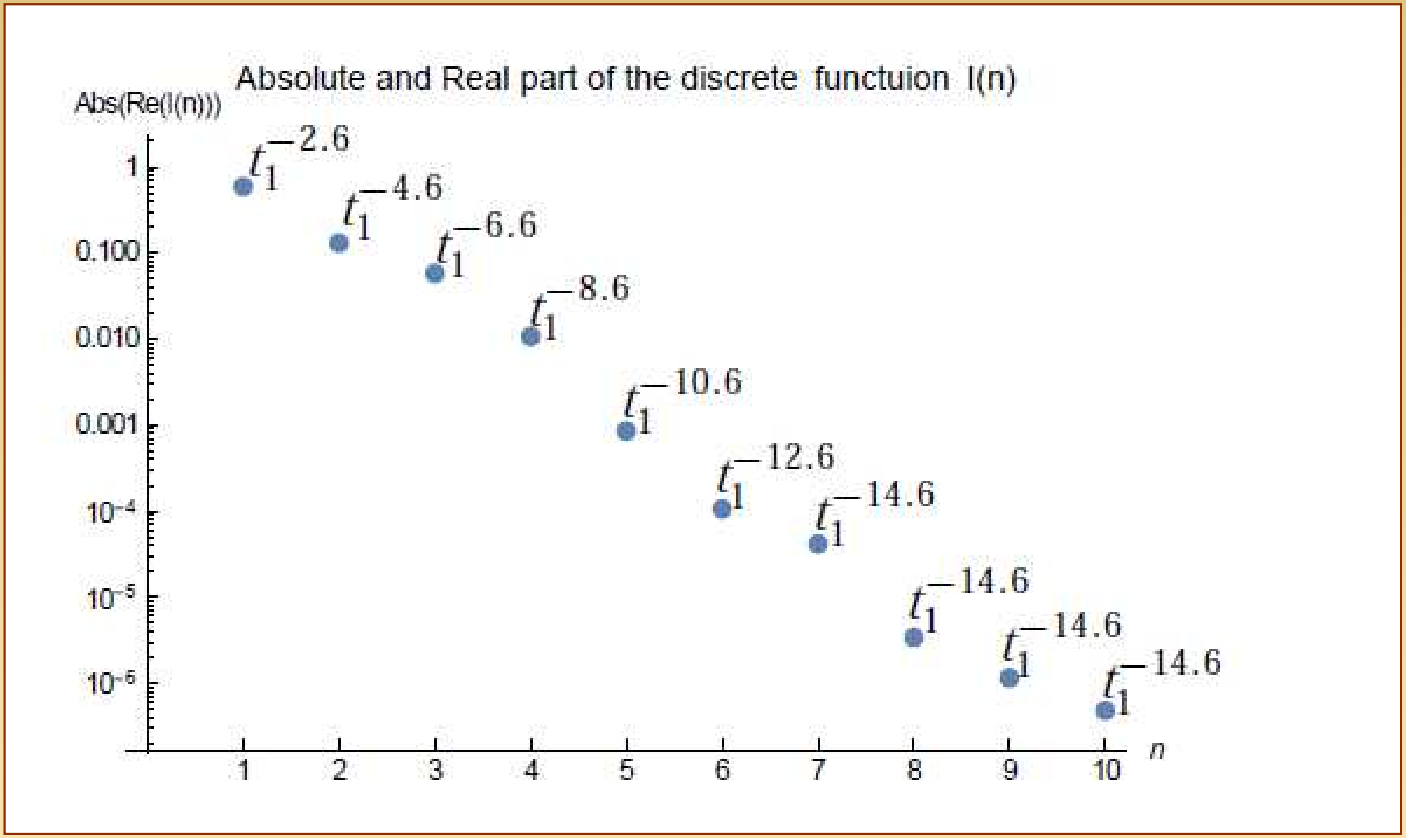
- Worst asymtotic behaviour of integrand for $y_1 \to -\infty$, $y_2 = 0$:

  $\sim y_1^{-2-2(c_2+n)}$ (for $n = 0$ and $c_2 = -0.7$: $\sim y_1^{-0.6}$)

- Pick up (finite number of) pole residues from contour shift

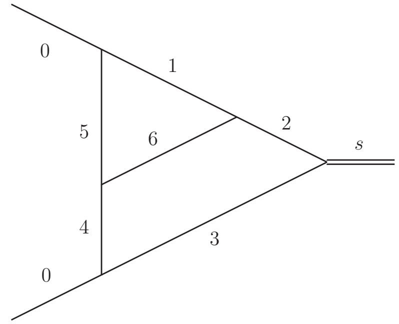
Mellin-Barnes representations

- Shifts improve asymptotic behaviour and size of numerical integral
- Automatic algorithms for finding suitable shifts in development (`MBnumerics`) Usovitsch '17



Examples

$m_1 = m_t,\ m_5 = m_6 = M_W,\ m_2 = m_3 = m_4 = 0$



SecDec: (24 hours)

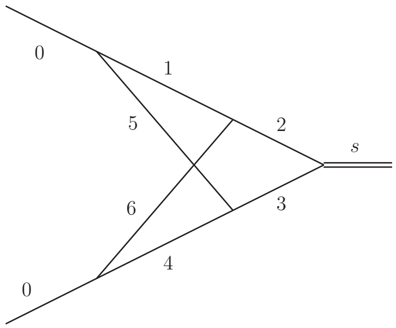
$$I_{SD} = 1.541 + 0.2487\, i + \frac{1}{\epsilon}(0.123615 - 1.06103\, i) + \frac{1}{\epsilon^2}(-0.3377373796 - 5 \times 10^{-10}\, i)$$

MBnumerics: (43 min.)

$$I_{MB} = 1.541402128186602 + 0.248804198197504\, i + \frac{1}{\epsilon}(0.1236145994284665 9 - 1.0610332704387688\, i) + \frac{1}{\epsilon^2}(-0.3377373795505797 0 + 3.6 \times 10^{-17}\, i)$$

Examples

$m_1 = M_Z$, rest zero



SecDec: error $\gg$ 1

MBnumerics: (finite part)

$$-0.7785996083 - 4.12351260\, i$$

Analytical: Fleischer, Kotikov, Veretin '98

$$-0.7785996090 - 4.12351259\, i$$

Comparison of techniques

Sector decomposition:

- Fully automated for (almost) any multi-loop diagram
  $\longrightarrow$ public tools available
- Numerical stability and precision difficult in some cases

Mellin-Barnes:

- Contour shift method applied successfully for 2-loop vertices
  $\longrightarrow$ Good numerical precision
- Extension to more loops/legs possible, but more work needed
- Partial automatization possible, but full automatization difficult
  (nested interdependent shifts for multi-dimensional integrals)
- Package `MBnumerics` under development — Usovitsch '17

General 3-loop vacuum integrals

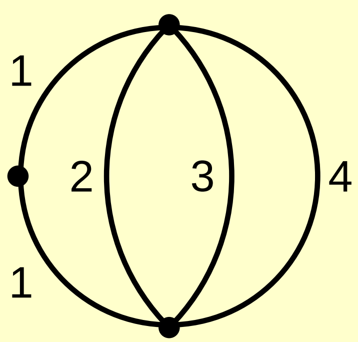
- Relevant for low-energy precision observables ($p^2 \ll M_{\mathrm{Z}}$)
- Coefficients of low-momentum expansions
- Building block for more general 3-loop calculations

Master integrals:

$$M(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6;\, m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2)$$

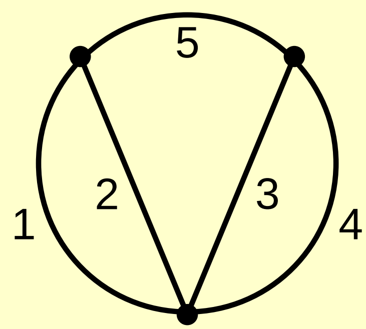
$$= i\frac{e^{3\gamma_{\mathrm{E}}\epsilon}}{\pi^{3D/2}}\int d^Dq_1\, d^Dq_2\, d^Dq_3\, [q_1^2-m_1^2]^{-\nu_1}[(q_1-q_2)^2-m_2^2]^{-\nu_2}$$

$$\times\, [(q_2-q_3)^2-m_3^2]^{-\nu_3}[q_3^2-m_4^2]^{-\nu_4}[q_2^2-m_5^2]^{-\nu_5}[(q_1-q_3)^2-m_6^2]^{-\nu_6}$$



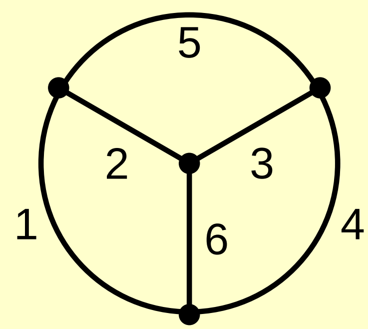
U_4

$= M(2,1,1,1,0,0)$



U_5

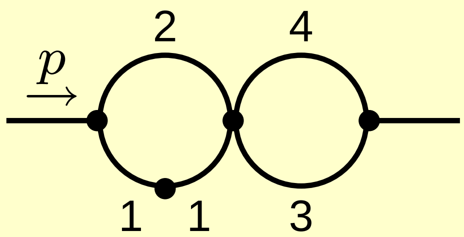
$= M(1,1,1,1,1,0)$



U_6

$= M(1,1,1,1,1,1)$

U_4



$$= B_{0,m_1}(p^2, m_1^2, m_2^2) B_0(p^2, m_3^2, m_4^2)$$

$$= \int_0^\infty ds \, \frac{\Delta I_{\mathrm{db}}(s)}{s - p^2 - i\varepsilon}$$

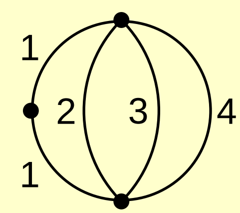
$$\Delta I_{\mathrm{db}}(s, m_1^2, m_2^2, m_3^2, m_4^2) = \Delta B_{0,m_1}(s, m_1^2, m_2^2) \, B_0(s, m_3^2, m_4^2) + B_{0,m_1}(s, m_1^2, m_2^2) \, \Delta B_0(s, m_3^2, m_4^2),$$

$$\Delta B_0(s, m_a^2, m_b^2) = \frac{1}{s} \lambda(s, m_a^2, m_b^2) \, \Theta\Big(s - (m_a + m_b)^2\Big)$$

$$\Delta B_{0,m_1}(s, m_a^2, m_b^2) = \frac{m_a^2 - m_b^2 - s}{s \, \lambda(s, m_a^2, m_b^2)} \Theta\Big(s - (m_a + m_b)^2\Big)$$

$$U_4(m_1^2, m_2^2, m_3^2, m_4^2) = -\frac{e^{\gamma_{\mathrm{E}} \epsilon}}{i\pi^{D/2}} \int d^D q_3 \int_0^\infty ds \, \frac{\Delta I_{\mathrm{db}}(s)}{q_3^2 - s + i\varepsilon}$$

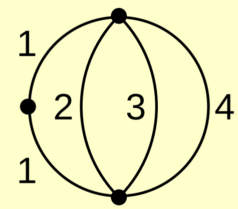
$$= -\int_0^\infty ds \, A_0(s) \, \Delta I_{\mathrm{db}}(s)$$



U_4

Problem: U_4 is divergent

Solution:



$$U_4(m_1^2,m_2^2,m_3^2,m_4^2) = U_4(m_1^2,m_2^2,0,0) + U_4(m_1^2,0,m_3^2,0) + U_4(m_1^2,0,0,m_4^2) - 2\,U_4(m_1^2,0,0,0) + U_{4,\text{sub}}(m_1^2,m_2^2,m_3^2,m_4^2)$$

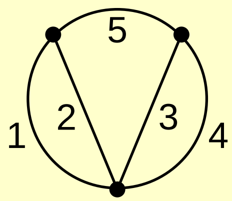
$\longrightarrow$ $U_4(m_X^2,m_Y^2,0,0)$ can be computed analytically

$\longrightarrow$ $U_{4,\text{sub}}$ is finite

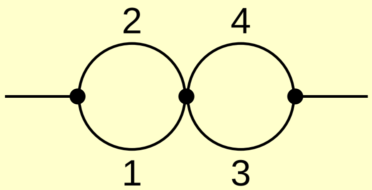
$$U_{4,\text{sub}}(m_1^2,m_2^2,m_3^2,m_4^2) = -\int_0^\infty ds\, A_{0,\text{fin}}(s)\,\Delta I_{\text{db,sub}}(s)$$

$$\begin{aligned} I_{\text{db,sub}}(s,m_1^2,m_2^2,m_3^2,m_4^2) = {} & \\ & \Delta B_{0,m_1}(s,m_1^2,m_2^2)\,\text{Re}\left\{B_0(s,m_3^2,m_4^2) - B_0(s,0,0)\right\} \\ & - \Delta B_{0,m_1}(s,m_1^2,0)\,\text{Re}\left\{B_0(s,0,m_3^2) + B_0(s,0,m_4^2) - 2B_0(s,0,0)\right\} \\ & + \text{Re}\left\{B_{0,m_1}(s,m_1^2,m_2^2)\right\}\left[\Delta B_0(s,m_3^2,m_4^2) - \Delta B_0(s,0,0)\right] \\ & - \text{Re}\left\{B_{0,m_1}(s,m_1^2,0)\right\}\left[\Delta B_0(s,0,m_3^2) + \Delta B_0(s,0,m_4^2) - 2\,\Delta B_0(s,0,0)\right] \end{aligned}$$

U_5



$$= -\frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int_0^\infty ds \int \frac{d^D q_3}{[q_3^2 - s][q_3^2 - m_5^2]} \times \text{Disc}\Big[\quad \Big]_s$$



$$= \int_0^\infty ds\, B_0(0, s, m_5^2)\, \text{Disc}[\ldots]_s$$

U_5 is divergent

Integration-by-parts relations:

$$U_5(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$$

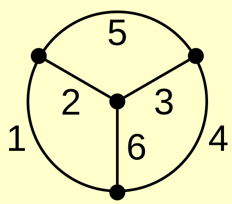
$$= F\Big[A_0(m_i),\, T_3(m_i, m_j, m_k),\, U_4(m_i, m_j, m_k, m_l)\Big]$$

$$+ \frac{\lambda_{125}^2 \lambda_{345}^2}{(3-D)^2 (m_2^2 - m_1^2 + m_5^2)(m_3^2 - m_4^2 + m_5^2)} M(2,1,1,2,1,0)$$

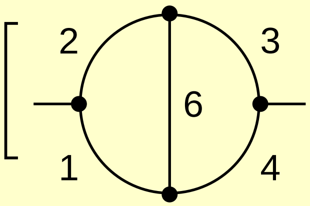
$F[\ldots]$ = some linear combination (lengthy)

$M(2,1,1,2,1,0)$ is finite

U_6



$$= -\frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int_0^\infty ds \int \frac{d^D q_3}{[q_3^2 - s][q_3^2 - m_5^2]} \times \text{Disc}\Big[$$



$$\Big]_s$$

$$= \int_0^\infty ds\, B_0(0, s, m_5^2)\, \text{Disc}[\ldots]_s$$

2-loop self-energy known in terms of 1-dimensional numerical integral

Bauberger, Böhm ’95

U_6 is divergent, but

$U_6(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) - U_6(m_6^2, m_6^2, m_6^2, m_6^2, m_6^2, m_6^2)$ is finite

and $U_6(m_6^2, m_6^2, m_6^2, m_6^2, m_6^2, m_6^2)$ is analytically known

Broadhurst ’98

Results

- U_4, U_5 given in terms of one-dimensional numerical integrals
- U_6 given in terms of twodimensional numerical integral
- Special cases (e.g. $m_1 = 0$) can also be handled

Public code: **TVID**

- Algebraic part (`Mathematica`) performs subtraction of UV-divergencies
- Numerical part (`C++`) performs numerical integrals

  Timing (single core Xeon 3.7 GHz):
  $\lesssim$ 0.1 s for U_4, U_5
  $\lesssim$ 30 s for U_6

- At least ten digit agreement with literature (for one/two-scale cases)

  Broadhurst '98; Chetyrkin, Steinhauser '99
  Grigo, Hoff, Marquard, Steinhauser '12

- Available at `www.pitt.edu/~afreitas/`

Conclusions

- **Numerical techniques** are promising for multi-scale multi-loop integrals, but no one-size-fits-all method
- **Sector decomposition:** very general, but numerical convergence sometimes slow and not guaranteed
- **Mellin-Barnes integrals with contour shifts:** good numerical accuracy, but requires extra work for new classes of integrals
- **Dispersion relations for sub-loop bubbles:** very efficient method for 3-loop vacuum integrals,
  can be extended to certain 3-loop integrals with external legs

Backup slides

Numerical integration over Feynman parameters

- After removal of singularities through sector decomposition:

$$I^{(1)}_{\text{reg}} = \int_0^1 dx_1 \ldots dx_{n-1} \, (A - i\epsilon)^{-k}$$

- Physical thresholds: A changes sign in ingration region
  $\longrightarrow$ Problematic for numerical integrators

  $\longrightarrow$ Deform integration into complex plane: Nagy, Soper '06

$$x_i = z_i - i\lambda z_i(1-z_i)\frac{\partial A}{\partial z_i}, \qquad 0 \leq z_i \leq 1.$$

$$A(\vec{x}) = A(\vec{z}) - i\lambda \sum_i z_i(1-z_i)\left(\frac{\partial A}{\partial z_i}\right)^2 + \mathcal{O}(\lambda^2).$$

  Typical choice: $\lambda \sim 0.5 - 1$

- Potential issues:
  $\longrightarrow$ $\partial A/\partial z_i$ may vanish in certain sub-spaces
  $\longrightarrow$ Thresholds may be at edge of integration region

Variables mapping

Map MB integrals onto interval [0,1]:

$$z_i = x_i + i\,\frac{1}{\tan(-\pi t_i)}, \qquad t_i \in (0,1)$$

Jacobian: $\dfrac{\pi}{\sin^2(\pi t_i)}$

In addition, $\Gamma \to e^{\ln \Gamma}$ improves numerical stability

U_4 for $m_1 = 0$

U_4 with $m_1 = 0$ has IR singularity!

$$
\begin{aligned}
U_4(0, m_2^2, m_3^2, m_4^2) = {} & B_0(0,0,0)\, T_3(m_2^2, m_3^2, m_4^2) \\
& - B_0(0, \delta^2, \delta^2)\, T_3(m_2^2, m_3^2, m_4^2) \\
& + U_4(\delta^2, m_2^2, m_3^2, m_4^2) + \mathcal{O}(\delta^2)
\end{aligned}
$$

($\delta \ll m_i$)

$\log \delta$ dependence of $U_4(\delta^2, m_2^2, m_3^2, m_4^2)$ can be extracted explicitly to avoid numerical instabilities

Checks

(finite part shown)

$x = 0.8^2$	This work	Grigo, Hoff, Marquard, Steinhauser '12
$U_4(1,1,1,x)$ $U_4(1,x,x,x)$ $M(1,1,1,1,0,0;\ 1,1,1,x)$ $M(1,1,1,1,0,0;\ 1,x,x,x)$	3.641562533670 4.209536621473 37.77079673659 33.73316262161	3.641562533537 4.209536621428 37.77079673639 33.73316262154
	This work	Chetyrkin, Steinhauser '99
$U_5(1,1,0,0,1)$	55.659622461206329	55.659622461206330