

A Java library to perform S-expansions of Lie algebras

ACAT 2017

Nelson Merino

***Laboratoire APC,
Université Denis Diderot - PARIS VII
Paris - France***

Collaboration with : C. Inostroza, I. Kondrashuk and F. Nadal
Based on : arXiv:1703.04036

Contents

- 1) Introduction.**
- 2) The S-expansion method.**
- 3) Need of automatizing the procedure.**
- 4) Conclusions.**

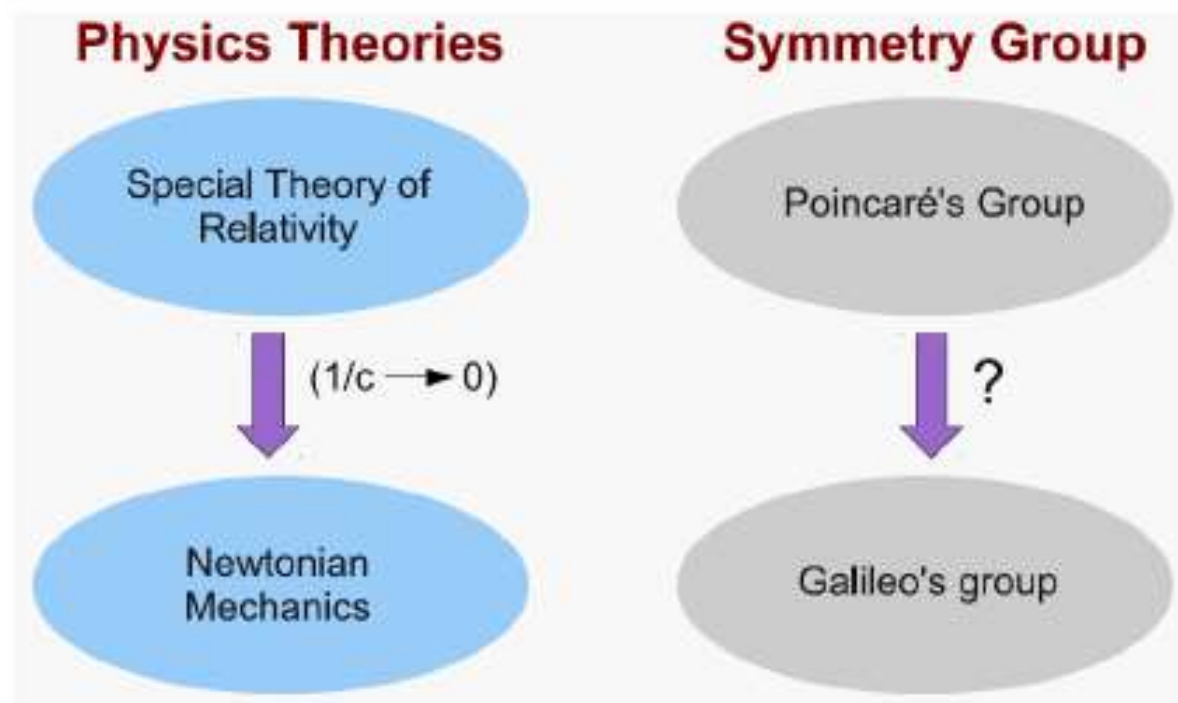
Contents

- 1) Introduction.**
- 2) The S-expansion method.
- 3) Need of automatizing the procedure.
- 4) Conclusions.

Expansion methods are examples of procedures giving **non-trivial relations** between Lie algebras and groups.

They allow to understand **interrelations** between physical theories.

The first example, proposed by I.E. Segal in 1951,

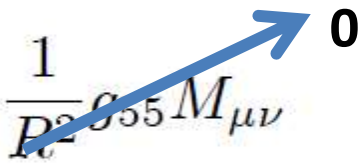


1953: Inonu-Wigner contractions

Ex: $ISO(3,1)$ from $SO(3,2)$

$$[M_{ab}, M_{cd}] = g_{ac}M_{bd} - g_{bc}M_{ad} - g_{ad}M_{bc} + g_{bd}M_{ac}$$
$$a, c = 1, \dots, 5$$

Defining $M_{5\mu} = RP_\mu$ with $\mu, \nu = 1, \dots, 4$ it leads to

$$[P_\mu, P_\nu] = \frac{1}{R^2} [M_{5\mu}, M_{5\nu}] = \frac{1}{R^2} g_{55} M_{\mu\nu}$$

$$[M_{\mu\nu}, P_\rho] = \frac{1}{R} [M_{\mu\nu}, M_{5\rho}] = \frac{1}{R} (-g_{\mu\rho}M_{\nu 5} + g_{\nu\rho}M_{\mu 5})$$
$$= \frac{1}{R} (-g_{\mu\rho}RP_\nu + g_{\nu\rho}RP_\mu) = -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu$$

Brief history of these methods

1953: Inonu-Wigner contractions

2000 -2003: Generalized Inonu-Wigner contractions, Weimar Woods

2003: Expansion method, Hatsuda, Sakaguchi
[arXiv:hep-th/0106114]
& de Azcarraga *et al* [arXiv: hep-th/0212347]

2006: S-expansion method, Izaurieta *et al*
[arXiv:heo-th/0606215]

Expanded algebras with
higher dimensión.

Contents

- 1) Introduction
- 2) The S-expansion method**
- 3) Need of automatizing the procedure
- 4) Conclusions

S-expansion method [1]

Ingredients:

- A Lie algebra \mathcal{G} , i.e., a vector space with basis $\{\mathbf{T}_A\}_{A=1}^{\dim \mathcal{G}}$ provided with a Lie product,

$$[\mathbf{T}_A, \mathbf{T}_B] = C_{AB}^C \mathbf{T}_C$$

- A finite abelian semigroup S , i.e., a set $\{\lambda_\alpha\}_{\alpha=1}^n$ with composition law

$$\lambda_\alpha \lambda_\beta = \lambda_{\gamma(\alpha, \beta)} = K_{\alpha\beta}^\rho \lambda_\rho$$
$$K_{\alpha\beta}^\rho = \begin{cases} 1, & \text{when } \rho = \gamma(\alpha, \beta) \\ 0, & \text{in other case} \end{cases}$$

is closed and asociative.

C_{AB}^C : structure constants

$K_{\alpha\beta}^\rho$: selectors

[1] F. Izaurieta, E. Rodríguez, P. Salgado, J.Math.Phys.47:123512,2006
arXiv: 0606215

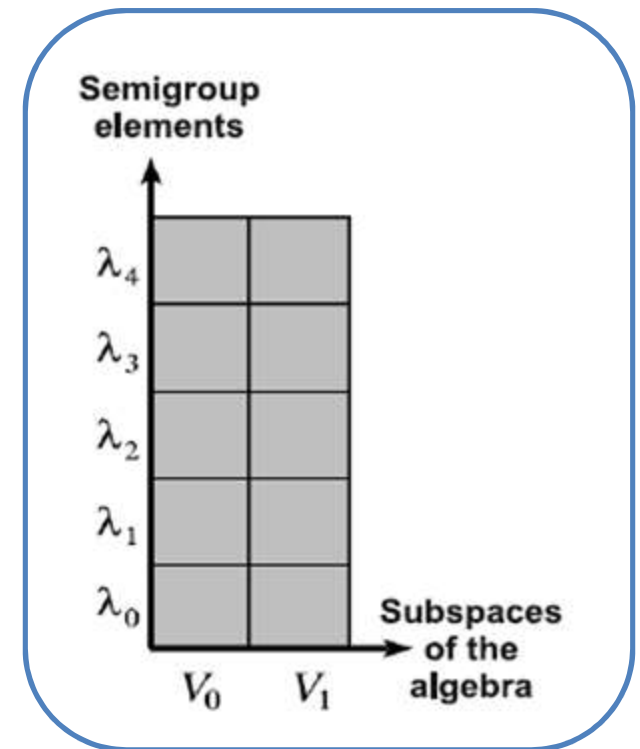
Steps:

I) Construction of $\mathcal{G}_S = S \otimes \mathcal{G}$ with basis $\{\mathbf{T}_{(A,\alpha)} = \lambda_\alpha \otimes \mathbf{T}_A\}$ and define the induced Lie product:

$$\begin{aligned} [\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}] &= \lambda_\alpha \lambda_\beta \otimes [\mathbf{T}_A, \mathbf{T}_B] \\ &= C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} \mathbf{T}_{(C,\gamma)} \end{aligned}$$

$$C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}^\rho C_{AB}^C$$

First result: $\mathcal{G}_S = S \otimes \mathcal{G}$ is a Lie algebra, called the **expanded algebra**

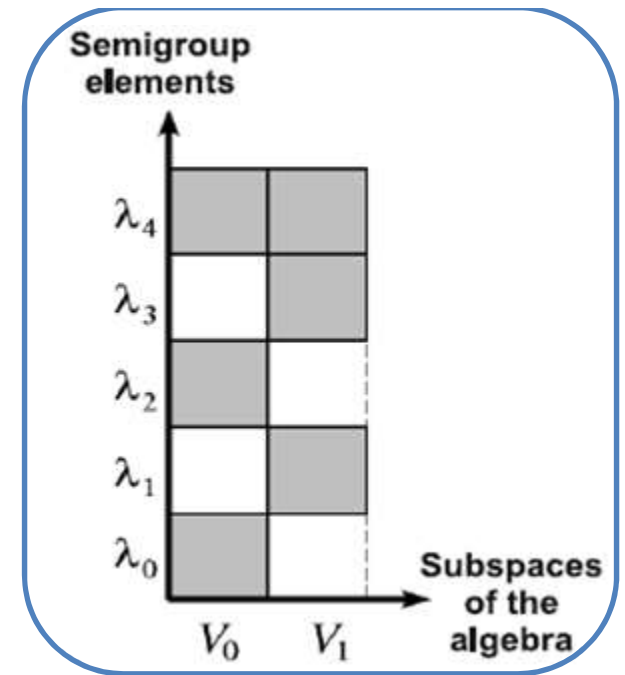


II) Suppose that $\mathcal{G} = \bigoplus_{p \in I} V_p$, $S = \bigcup_{p \in I} S_p$
satisfying respectively:

$$[V_p, V_q] \subset \bigoplus_{r \in i(p,q)} V_r \quad \text{and} \quad S_p \times S_q \subset \bigcup_{r \in i(p,q)} V_r$$

Perform the following construction $\mathcal{G}_{S,R} = \bigoplus_{p \in I} S_p \otimes V_p$

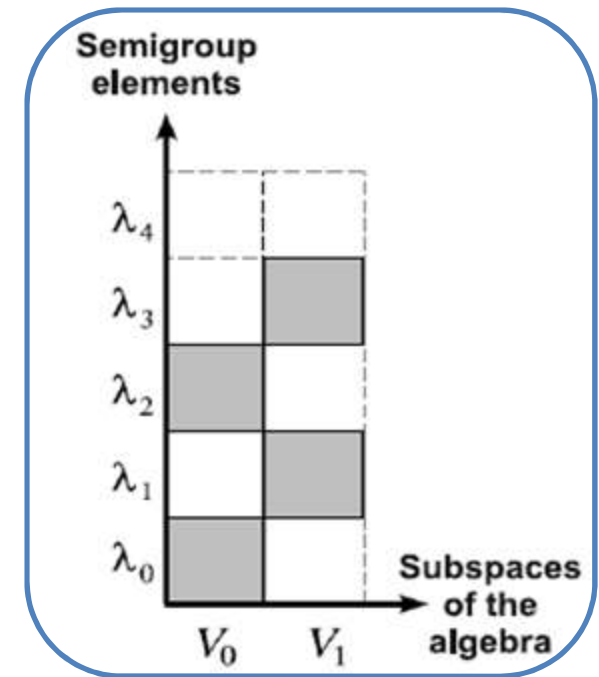
Second result: $\mathcal{G}_{S,R}$ is a subalgebra called the **resonant subalgebra**



III) Suppose that there is an element 0_S which satisfies $0_S \lambda_\alpha = 0_S \quad \forall \lambda_\alpha \in S$

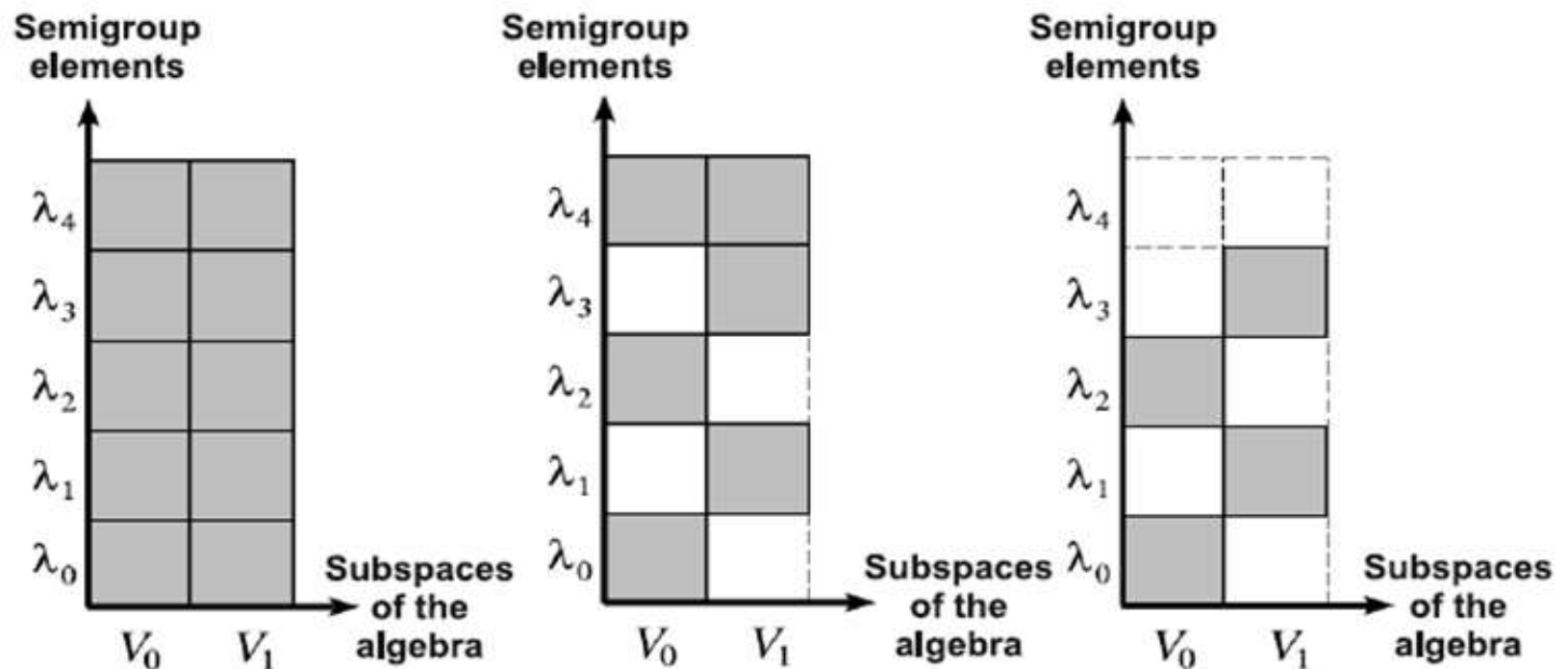
Third result: the sector $0_S \otimes \mathcal{G}$ can be removed from the expanded algebra to obtain the so called **reduced algebra**.

Observation: Steps II and III are independent, but can also be applied simultaneously.



In general the S-expansion method consist in a serie of steps:

- Obtain the S-expanded algebra
- Find the (resonant) subalgebras
- Perform a reduction.



- ➡ Inönü-Wigner contraction,
- ➡ its generalizations (Weimar-Woods contractions),
- ➡ and the dAIPV-expansion method

can be obtained as an S -expansion (as a 0_S -reduction of the resonant subalgebra) with one of the semigroups $S_E^{(N)}$ defined by:

$$S_E^{(N)} = \{\lambda_0, \lambda_1, \dots, \lambda_N, \lambda_{N+1}\} \text{ with}$$

$$\begin{aligned} \lambda_\alpha \lambda_\beta &= \lambda_{\alpha+\beta}, \text{ if } \alpha + \beta < N + 1 \\ \lambda_\alpha \lambda_\beta &= \lambda_{N+1}, \text{ if } \alpha + \beta \geq N + 1 \end{aligned}$$

$$S = S_0 \cup S_1 \text{ with } \left\{ \begin{array}{l} S_0 = \{\lambda_{2m}, \text{ with } m = 0, \dots, [\frac{N}{2}]\} \cup \{\lambda_{N+1}\} \\ S_1 = \{\lambda_{2m+1}, \text{ with } m = 0, \dots, [\frac{N-1}{2}]\} \cup \{\lambda_{N+1}\} \\ S_0 \times S_0 \subset S_0 \\ S_0 \times S_1 \subset S_1 \\ S_1 \times S_1 \subset S_0 \end{array} \right.$$

Example: Inönü-Wigner contraction as an $S_E^{(1)}$ -expansion

Starting from $\mathcal{G} = SO(3, 2) = V_0 \oplus V_1$ where

$$V_0 = \{\bar{J}_{ab}\} ; \quad V_1 = \{\bar{P}_a\} \quad \text{and} \quad \begin{cases} [\bar{J}_{ab}, \bar{J}_{cd}] \sim \bar{J}_{ab} \\ [\bar{J}_{ab}, \bar{P}_c] \sim \bar{P}_a \\ [\bar{P}_a, \bar{P}_b] \sim \bar{J}_{ab} \end{cases} \quad \text{i.e.,} \quad \begin{cases} [V_0, V_0] \sim V_0 \\ [V_0, V_1] \sim V_1 \\ [V_1, V_1] \sim V_0 \end{cases}$$

Consider the semigroup $S_E^{(1)} = \{\lambda_0, \lambda_1, \lambda_2\}$ with

$$\begin{aligned} \lambda_\alpha \lambda_\beta &= \lambda_{\alpha+\beta}, \text{ if } \alpha + \beta < 2 \\ \lambda_\alpha \lambda_\beta &= \lambda_2, \quad \text{if } \alpha + \beta \geq 2 \end{aligned}$$

$$S_E^{(1)} = S_0 \cup S_1$$

$$S_0 = \{\lambda_0, \lambda_2\}$$

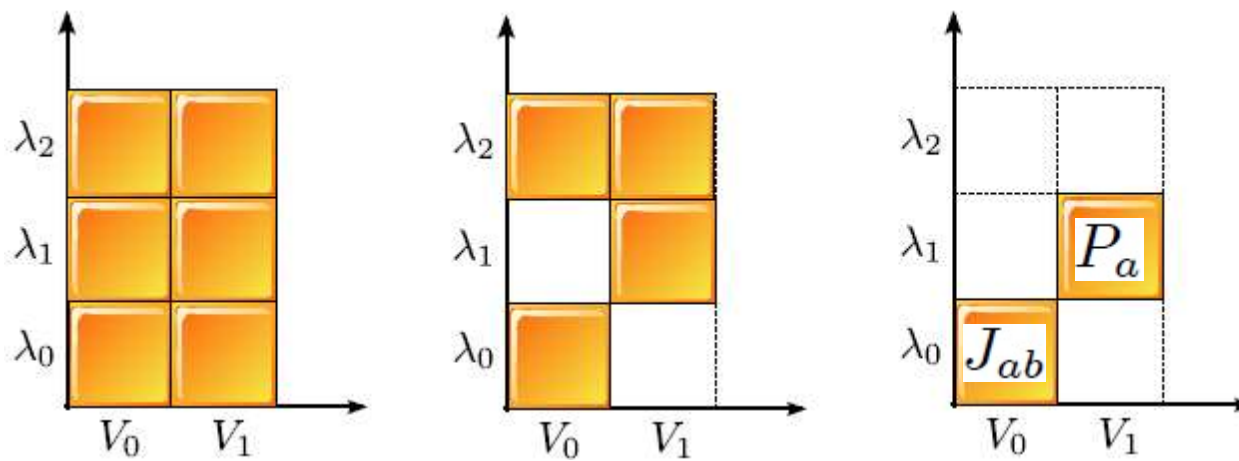
$$S_1 = \{\lambda_1, \lambda_2\}$$

$$S_0 \times S_0 \subset S_0$$

$$S_0 \times S_1 \subset S_1$$

$$S_1 \times S_1 \subset S_0$$

Example: Inönü-Wigner contraction as an $S_E^{(1)}$ -expansion



$$\begin{aligned} J_{ab} &= \lambda_0 \otimes \bar{J}_{ab} \\ P_a &= \lambda_1 \otimes \bar{P}_a \end{aligned}$$

$$\mathcal{G}_{R,\text{red}}^{S_E^{(1)}} = [(S_0 \otimes V_0) \oplus (S_1 \otimes V_1)]_{\text{red}} = \langle \{ \lambda_0 \bar{J}_{ab}, \lambda_1 \bar{P}_a \} \rangle = \langle \{ J_{ab}, P_a \} \rangle$$

$$[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}] = \lambda_\alpha \lambda_\beta \otimes [\mathbf{T}_A, \mathbf{T}_B]$$

$$[J_{ab}, J_{cd}] = \lambda_0 \lambda_0 \otimes [\bar{J}_{ab}, \bar{J}_{cd}] \sim \lambda_0 \bar{J}_{ab} \sim J_{ab}$$

$$[J_{ab}, P_c] = \lambda_0 \lambda_1 \otimes [\bar{J}_{ab}, \bar{P}_c] \sim \lambda_1 \bar{P}_c \sim P_c$$

$$[P_a, P_b] = \lambda_1 \lambda_1 \otimes [\bar{P}_a, \bar{P}_b] \sim \lambda_2 \bar{J}_{ab} \sim 0$$

IW contraction.

Contents

- 1) Introduction
- 2) The S-expansion method
- 3) Need of automatazing the procedure**
- 4) Conclusions

Need of automatizing the procedure

All generalized contractions, mentioned before, can be reproduced in the frame of the S-expansion method by using one of the semigroups

$$S_E^{(N)} = \{\lambda_0, \dots, \lambda_{N+1}\} \text{ defined by:}$$

$$\begin{aligned} \lambda_\alpha \lambda_\beta &= \lambda_{\alpha+\beta}, \text{ if } \alpha + \beta < N + 1 \\ \lambda_\alpha \lambda_\beta &= \lambda_{N+1}, \text{ if } \alpha + \beta \geq N + 1 \end{aligned}$$

Expansions with other semigroups can generate algebras that cannot be reached, nor by any contraction neither by an expansion of de Azcarraga et al.

As many physical applications have appeared by using this method we wanted to answer the following question:

Given two Lie Algebras, can they be related by an S-Expansion?

To answer this one should consider the complete family of abelian semigroups, i.e., to take into account the history of their classification:

order	# Of Semigroups	
1	1	
2	4	
3	18	
4	126	[Forsythe '54]
5	1.160	[Motzkin, Selfridge '55]
6	15.973	[Plemmons '66]
7	836.021	[Jurgensen, Wick '76]
8	1.843.120.128	[Sato, Yama, Tokizawa '94]
9	52.989.400.714.478	[Distler, Kelsey, Mitchell '09]
10	12.418.001.077.381.302.684	[Distler, Jefferson, Kelsey, Kotthoff '16]

and find a way to implement the S-expansion procedure.

[2] John A. Hildebrand, *Handbook of finite semigroup programs*.

Table 2. List of semigroups of the third rank.

$S_{(3)}^1$	λ_1	λ_2	λ_3	$S_{(3)}^2$	λ_1	λ_2	λ_3	$S_{(3)}^3$	λ_1	λ_2	λ_3	$S_{(3)}^6$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_2	λ_1	λ_1	λ_1	λ_2	λ_1	λ_1	λ_1	λ_2	λ_1	λ_1	λ_2
λ_3	λ_1	λ_1	λ_1	λ_3	λ_1	λ_1	λ_2	λ_3	λ_1	λ_1	λ_3	λ_3	λ_1	λ_2	λ_3
$S_{(3)}^7$	λ_1	λ_2	λ_3	$S_{(3)}^9$	λ_1	λ_2	λ_3	$S_{(3)}^{10}$	λ_1	λ_2	λ_3	$S_{(3)}^{12}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_2	λ_2	λ_1	λ_2	λ_2	λ_2	λ_1	λ_2	λ_3
λ_3	λ_1	λ_1	λ_3	λ_3	λ_1	λ_2	λ_2	λ_3	λ_1	λ_2	λ_3	λ_3	λ_1	λ_3	λ_2
$S_{(3)}^{15}$	λ_1	λ_2	λ_3	$S_{(3)}^{16}$	λ_1	λ_2	λ_3	$S_{(3)}^{17}$	λ_1	λ_2	λ_3	$S_{(3)}^{18}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_3	λ_1	λ_1	λ_1	λ_3	λ_1	λ_1	λ_2	λ_2	λ_1	λ_1	λ_2	λ_3
λ_2	λ_1	λ_1	λ_3	λ_2	λ_1	λ_2	λ_3	λ_2	λ_2	λ_1	λ_1	λ_2	λ_2	λ_3	λ_1
λ_3	λ_3	λ_3	λ_1	λ_3	λ_3	λ_3	λ_1	λ_3	λ_2	λ_1	λ_1	λ_3	λ_3	λ_1	λ_2

Contents

- 1) Introduction
- 2) The S-expansion method
- 3) Need of automatize the procedure
- 4) Conclusions**

Conclusions (of the first part)

We have implemented computer programs, in the form of a **Java Library**, to perform S-expansions of Lie algebras with any semigroup (up to order 6).

The methods of this library allow us:

1. to find isomorphisms between any given pair of semigroups
2. identify all the semigroups with a zero element
3. to study all resonant decomposition of any semigroup,
4. to represent any semigroup and Lie algebra by means of their respective *selectors* and *structure constants*
5. to perform all possible S-expansions (*GxS*, *resonant subalgebra*, *reduced algebra*, *reduction of the resonant subalgebra*) of a given Lie algebra.
6. to study further conditions for some specific problem

A description of this library, as well as some examples related with applications, will be given in the next presentation, by C. Inostroza.

THANKS!