New results on the g-2 calculation

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Anomalous magnetic moment of a electron

Intrinsic magnetic moment μ of the electron

$$\mu = g \frac{e\,h}{4\pi mc} \qquad g \text{ giromagnetic ratio}$$

Dirac's equation (1928) $\rightarrow$ $g = 2$ (good agreement with experiments for 20 years)

Kush and Foley (1947) $\rightarrow a_e = \frac{g-2}{2} = 0.001\,15(4) \neq 0!$ (Zeeman effect, Ga atoms)

Schwinger (1948) $\rightarrow a_e = \frac{\alpha}{2\pi} = 0.001\,161\ldots$ Quantum ElectroDynamics effect!

$$\alpha = \frac{e^2}{hc} \approx 1/137 \text{ is the fine structure constant}$$

A bit of history: 2nd order

QED contribution to a_e can be expanded in power series of α

$$a_e^{QED} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

Karplus and Kroll (1950) $\rightarrow$ analytical calculation $\rightarrow C_2 = -2.973\ldots$

Franken and Liebes (1956) $\rightarrow a_e = 0.001\,165(11)$ $\rightarrow$ disagreement with C_2

Petermann, Sommerfeld (1957) $\rightarrow$ analytical recalculation $\rightarrow C_2 = -0.328\,478\,965\,579\,19$

QED contribution to a_e can be expanded in power series of α

$$a_e^{QED} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - 0.328\,478\,965\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

Michigan group(1971) $\rightarrow a_e = 1\,159\,657\,7(35) \times 10^{-10} \rightarrow C_3$ needed!

Levine, Wright(1971) $\rightarrow C_3 = 1.49(20)$ good agreement

Cvitanovic, Kinoshita(1974) $\rightarrow C_3 = 1.195(26)$

Kinoshita(1990) $\rightarrow C_3 = 1.17611(42)$ 0.005164 shift due to an error

Kinoshita et al.(1995) $\rightarrow C_3 = 1.181259(40)$

S.L., Remiddi(1996) $\rightarrow C_3 = 1.181\,241\,456\,587\,200\,006\ldots$ analytical

Penning trap electron g-2 measurement

Storage of a single electron in a Penning trap (electrical quadrupole + axial B-field)

$$\omega_c = 2\frac{eB}{mc} \qquad \text{cyclotron frequency}$$

$$\omega_s = g\frac{eB}{mc} \qquad \text{spin precession frequency}$$

$$\omega_a = \omega_s - \omega_c \qquad \text{spin flip frequency}$$

$$a = \frac{g-2}{2} = \frac{\omega_a}{\omega_c} \qquad \text{frequencies ratio}$$

Dramatic improvement of experimental precision

$a^{exp}_{e^-} = 1\,159\,652\,188.4(4.3) \times 10^{-12}$ 4.3 ppb UW,Dehmelt et al 1987 (Nobel Prize 1989)

$a^{exp}_{e^+} = 1\,159\,652\,187.9(4.3) \times 10^{-12}$ 4.3 ppb UW,Dehmelt et al 1987 (Nobel Prize 1989)

$a^{exp}_{e^-} = 1\,159\,652\,180.85(.76) \times 10^{-12}$ 0.66 ppb Harvard, Gabrielse 2006

$a^{exp}_{e^-} = 1\,159\,652\,180.73(.28) \times 10^{-12}$ 0.24 ppb Harvard, Gabrielse 2008

part per billion precision requires knowledge of α^4 and α^5 coefficients, as well as QED muon-loop and non-QED small contributions (i.e. $\leq 3 \times 10^{-12}$)

dominant   small terms (i.e. $\leq 3 \times 10^{-12}$)

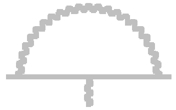
$$a_e^{SM} = a_e^{QED} \overbrace{+a_e^{QED}(\mu) + a_e^{QED}(\tau) + a_e^{QED}(\mu,\tau) + a_e(\text{hadr}) + a_e(\text{weak})}$$

- a_e^{QED} mass-independent:
  - 1-loop analytical, (Schwinger 1948)
  - 2-loop analytical (Petermann,Sommerfeld 1956)
  - 3-loop analytical (S.L.,Remiddi 1996)
  - 4-loop semi-analytical (S.L. 2017)
  - 5-loop numerical (3% precision) (Kinoshita 2017)
- $a_e^{QED}(X)$ mass-dependent:
  - 2-loop analytical (Elend 1966)
  - 3-loop analytical (S.L.,Remiddi 1992; S.L. 1994)
  - 4-loop analytical expansion in small mass ratio m_e/m_X, (Kurz, Liu, Marquard, Steinhauser 2013)
  - 5-loop numerical (10% precision) (Kinoshita 2014)
- $a_e^{QED}(X)$ small because scales as $(m_e/X)^2$

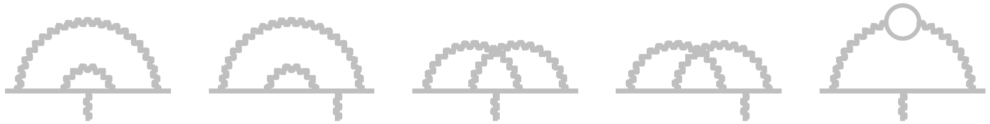
Mass-independent QED contribution, 1-3 loop

$$a_e(QED) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$$C_1 = \frac{1}{2}$$ (Schwinger 1948) *1 diagram*



$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

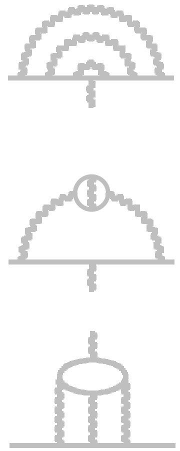


$$= -0.328\,478\,965\,579\,\ldots,$$ (Petermann, Sommerfield 1957) *7 diagrams*

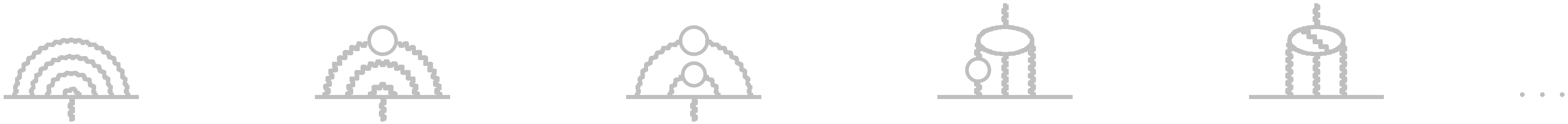
$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right]$$

$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

$$= 1.181\,241\,456\,\ldots,$$ (S.L., E.Remiddi 1996) *72 diagrams*



$\zeta(p) = \sum_{n=0}^{\infty} \frac{1}{n^p}$, $a_4 = \sum_{n=0}^{\infty} \frac{1}{2^n n^4}$, …



$C_4 \rightarrow$ **891 diagrams** obtained by inserting a photon in 104 self-mass diagrams

Previous numerical values obtained by using MonteCarlo integration

$C_4 = -1.434(138)$ (Kinoshita, 1990)
$C_4 = -1.5098(384)$ (Kinoshita, 1999)
$C_4 = -1.7283(35)$ (Kinoshita, 2003) -0.24 shift due to the discovery of one error
$C_4 = -1.9144(35)$ (Kinoshita, 2007) -0.22 shift due to the discovery of another error
$C_4 = -1.91298(84)$ (Kinoshita, 2014)

My new high-precision result is (S.L., 2017)

$$C_4 = -1.912245764926445574152647167439830054060873390658725345171329\ldots$$

good agreement with Kinoshita 2014 (0.9σ) $\longrightarrow$ important independent check!

5 loop coefficient → **12672 diagrams**

$C_5 = 7.795(336)$ (Kinoshita et al., 2014) -1.2 shift due to the discovery of an error

$C_5 = 6.599(223)$ (Kinoshita et al., 2017) 3% precision

The electron anomaly is dominated by the QED terms. The other interactions contributes to the 10^{-12} level.

$$\alpha^{-1}(Rubidium : 2016) = 137.035\,998\,996(85) \qquad (0.62\ ppb)$$

$$\begin{aligned}
C_1(\alpha/\pi) &= 1\,161\,409\,733.631(720) \times 10^{-12} \\
C_2(\alpha/\pi)^2 &= -\,1\,772\,305.065(3) \times 10^{-12} \\
C_3(\alpha/\pi)^3 &= 14\,804.203 \times 10^{-12} \\
C_4(\alpha/\pi)^4 &= -\,55.667 \times 10^{-12} \\
C_5(\alpha/\pi)^5 &= 0.446(15) \times 10^{-12} \\
a_e^{QED}(\mu) &= 2.738 \times 10^{-12} \\
a_e^{QED}(\tau) &= 0.009 \times 10^{-12} \\
a_e(\text{hadronic v.p.,2-loop}) &= 1.866(11) \times 10^{-12} \\
a_e(\text{hadronic v.p.,3-loop}) &= -\,0.223(1) \times 10^{-12} \\
a_e(\text{hadronic l-l }) &= 0.035(10) \times 10^{-12} \\
a_e(\text{weak}) &= 0.028(1) \times 10^{-12}
\end{aligned}$$

Comparison of a_e the determination of fine structure constant

$$\alpha^{-1}(Rubidium : 2016) = 137.035\,998\,996(85) \qquad (0.62\ ppb)$$

$$a_e^{SM}(\alpha) = 1\,159\,652\,182.031\overset{C_5}{(15)}\overset{had}{(15)}\overset{\alpha}{(720)} \times 10^{-12}$$

$$a_e^{exp} = 1\,159\,652\,180.730(280) \times 10^{-12} \qquad 0.25\ ppb$$

$$a_e^{SM}(\alpha) - a_e^{exp} = 1.30(77) \times 10^{-12} \quad 1.6\sigma \text{ agreement}$$

same order of magnitude of the hadronic contribution and muon-loop vacuum polarization

Assuming QED is correct

$$\alpha^{-1}(a_e) = 137.035\,999\,1500\overset{C_5}{(18)}\overset{had}{(18)}\overset{a_e}{(330)}$$

$$= 137.035\,999\,1500(332) \qquad (0.25\ ppb)$$

$\alpha(a_e)$ more precise than $\alpha^{exp}(Rubidium) \rightarrow \alpha(a_e)$ used in CODATA least-square adjustment of fundamental constants.

I have calculated C_4 with 1100 digits; let us see the digits...

1100 digits of 4-loop coefficient

$C_4 =$

```
-1.9122457649264455741526471674398300540608733906587253451713298480060
3844398065170614276089270000363158375584153314732700563785149128545391
9028043270502738223043455789570455627293099412966997602777822115784720
3390641519081665270979708674381150121551479722743221642734319279759586
0740500578373849607018743283140248380251922494607422985589304635061404
9225266343109442400023563568812806206454940132249775943004292888367617
4889923691518087808698970526357853375377696411702453619601349757449436
1268486175162606832387186747303831505962741878015305514879400536977798
3694642786843269184311758895811597435669504330483490736134265864995311
6387811743475385423488364085584441882237217456706871041823307430517443
0557394596117155085896114899526126606124699407311840392747234002346496
9531735482584817998224097373710773657404645135211230912425281111372153
0215445372101481112115984897088422327987972048420144512282845151658523
6561786594592600991733031721302865467212345340500349104700728924487200
6160442613254490690004319151982300474881814943110384953782994062967586
787538524978194698979313216219797575067670114290489796208505...
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1100 digits? Why such an high precision?

Because it allows to fit an analytical expression to the numerical value by using the "PSLQ algorithm".

- PSLQ algorithm (Ferguson and Bailey 1992)
- find integer relation between real numbers or bounds on size of coefficients.
- requires high precision; at least number of digits of coefficients * number of real numbers
- a parallel version of the algorithm exists (Bailey and Broadhurst 1999)

Differently from C_1, C_2, C_3, the analytical expression of C_4 is very complicated!
It can be divided in 5 parts:

1. usual constants (harmonic) polylogarithms of 1 and 1/2.
2. harmonic polylogarithms of arguments $e^{i\pi/3}$ and $e^{2i\pi/3}$ new
3. harmonic polylogarithms of arguments $e^{i\pi/2}$ new
4. elliptic constants with semi-analytic representation new
5. unknown elliptic constants

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \qquad \text{121 terms}$$

$$\begin{aligned}
T =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) \\
&+ \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
&- \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) \\
&- \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 \\
&- \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) \\
&+ \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 \\
&+ \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 \\
&+ \frac{407771}{432}\zeta^2(3)\ln 2 - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 \\
&- \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2
\end{aligned}$$

$a_n = \mathrm{Li}_n(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

terms of weight 5 cancel out

$$\begin{aligned}
V_a =& -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) \\
&+\frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&-\frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi \\
&-\frac{223}{243}\zeta(4)\pi\ln 2 + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&-\frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

$$\mathrm{Cl}_n(\theta) = \mathrm{ImLi_n}(\mathrm{e}^{\mathrm{i}\theta})$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{13487}{60}\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13487}{60}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) \\
&+ \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&- \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi \\
&+ \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&+ \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&+ \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)
\end{aligned}$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned} W_b =& -\frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 \\ &+104\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\ &-2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg) \end{aligned}$$

$\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ is the Catalan's constant $\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$

analytical fit part 4

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
E_a =& \pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) \\
&+\pi\left(\frac{4715}{1944}\ln 2\; f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
&+\pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\; f_2(0,0,2) + \frac{5525}{162}\ln 2\; f_2(0,1,1)\right. \\
&-\frac{5525}{243}\ln 2\; f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
&-\frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) \\
&\left.+\frac{69245}{124416}f_2(2,1,0)\right)
\end{aligned}$$

$$\begin{aligned}
E_b =& -\frac{4715}{1458}\zeta(2)f_1(0,0,1) \\
&+\zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2) - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\right) \; .
\end{aligned}$$

$$A_3 = \int_0^1 dx \frac{K_c(x)K_c(1\text{-}x)}{\sqrt{1-x}} = \frac{2\pi^{\frac{3}{2}}}{3}\left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{6}\;\frac{1}{3}\;\frac{1}{3}\;\frac{1}{2}\\ \frac{5}{6}\;\frac{5}{6}\;\frac{2}{3}\end{matrix};1\right) - \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{2}\;\frac{2}{3}\;\frac{2}{3}\;\frac{5}{6}\\ \frac{7}{6}\;\frac{7}{6}\;\frac{4}{3}\end{matrix};1\right)\right)$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{4\pi^{\frac{3}{2}}}{3}\left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{6}\;\frac{1}{3}\;\frac{1}{3}\;\frac{1}{2}\\ \frac{5}{6}\;\frac{5}{6}\;\frac{2}{3}\end{matrix};1\right) + \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{2}\;\frac{2}{3}\;\frac{2}{3}\;\frac{5}{6}\\ \frac{7}{6}\;\frac{7}{6}\;\frac{4}{3}\end{matrix};1\right)\right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{486\pi^2}{1925}\,{}_7F_6\left(\begin{matrix}\frac{7}{4}\;-\frac{1}{3}\;\frac{1}{3}\;\frac{2}{3}\;\frac{4}{3}\;\frac{3}{2}\;\frac{3}{2}\\ \frac{3}{4}\;1\;\frac{7}{6}\;\frac{11}{6}\;\frac{13}{6}\;\frac{17}{6}\end{matrix};1\right)\,,$$

$$K_c(x) = \frac{2\pi}{\sqrt{27}}\,{}_2F_1\left(\begin{matrix}\frac{1}{3}\;\frac{2}{3}\\ 1\end{matrix};x\right)\,,\qquad E_c(x) = \frac{2\pi}{\sqrt{27}}\,{}_2F_1\left(\begin{matrix}\frac{1}{3}\;-\frac{1}{3}\\ 1\end{matrix};x\right)\,.$$

A_3 cancels out in the diagram contributions

f_j are defined as follows:

$$
\begin{aligned}
f_1(i,j,k) &= \int_1^9 ds\, D_1^2(s) \left(s - \frac{9}{5}\right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) \ , \\
f_2(i,j,k) &= \int_1^9 ds\, D_1(s) \mathrm{Re}\left(\sqrt{3} D_2(s)\right) \left(s - \frac{9}{5}\right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) \ , \\
D_1(s) &= \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \ , \\
D_2(s) &= \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \ ;
\end{aligned}
$$

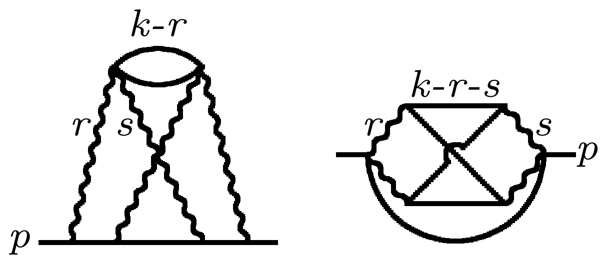
$K(x)$ is the complete elliptic integral of the first kind.

$D_1(s) \sim$ discontinuity of the 2-loop sunrise diagram with equal masses in $D = 2$ dimensions.

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

The term containing the ϵ^0 coefficients of the ϵ−expansion of six master integrals (see f, f', f'', g, g', g''):

$$U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c} \ .$$

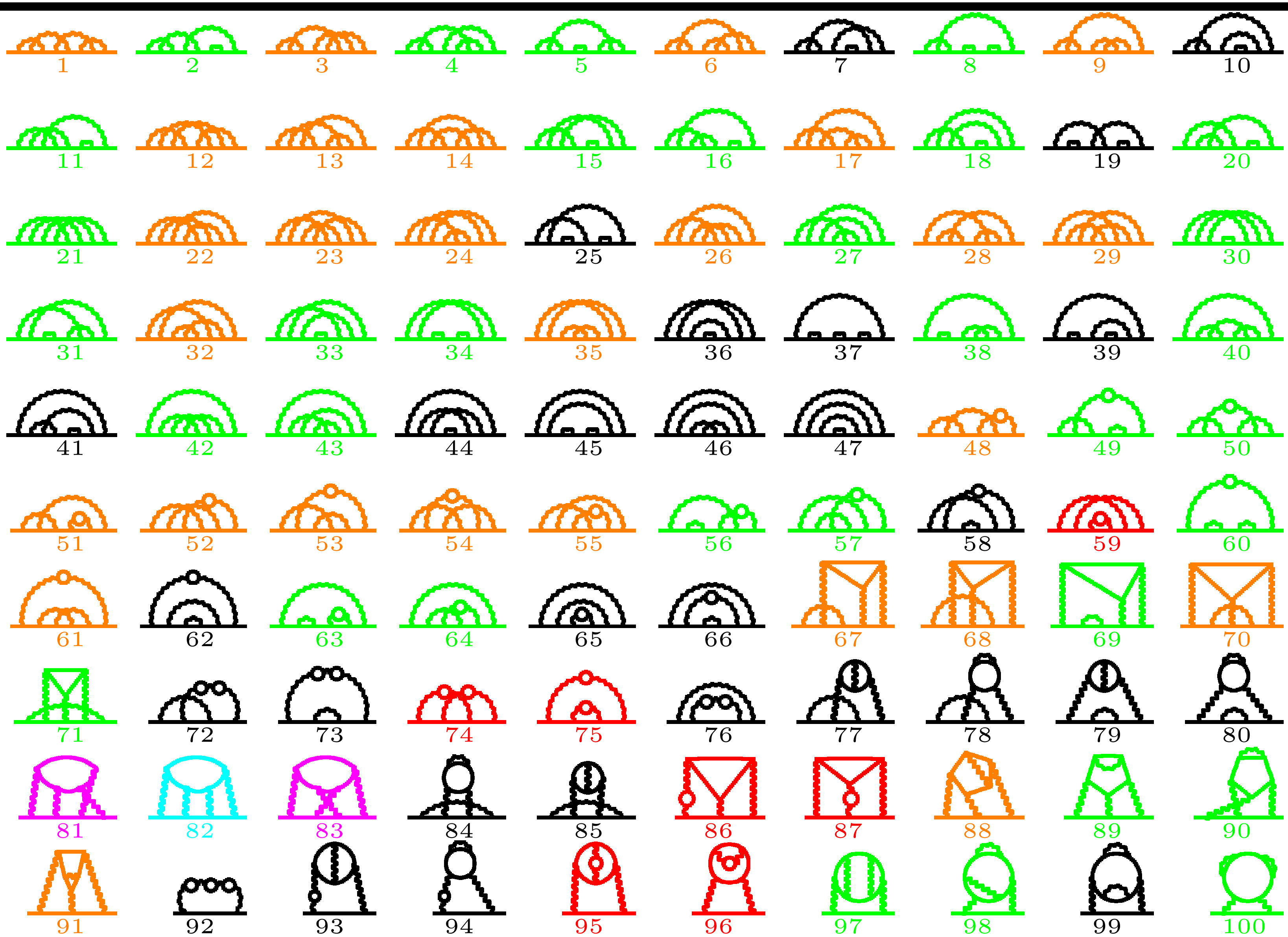


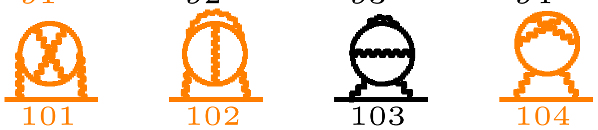
(f, f', f'') $\quad$ (g, g', g'')

(f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$

These master integrals appear in topologies 81 and 83 (gauge-invariant sets 24 and 25, vacuum polarization diagrams containing a light-light scattering).

the 104 4-loop electron self-masses





the 891 vertex diagrams can be obtained by inserting one photon in all possible way in each of the above 104 self-mass diagrams

Observations

- a_e is dominated by the QED contribution
- hadronic and muon-loop vacuum polarizations contributes to 10^{-12} level
- 1,2,3-loop QED coefficient are known exactly
- 4-loop QED coefficient is now known "near-exactly" (more than 1100 digits)
- the ultimate limit is the error in the hadronic contribution $\approx 10^{-14}$
- that corresponds to $0.15\left(\frac{\alpha}{\pi}\right)^5$ or $64\left(\frac{\alpha}{\pi}\right)^6$
- historically checks with the experiment or independent theoretical results have often highlighted inconsistences in QED contributions
- for this reason an independent calculation of 5-loop coefficient would be important

- QED expresses g-2 contributions as combinations of n-loop 4-dimensional integrals
- Kinoshita's group calculations are based on the transformation of n-loop 4-dim integrals in $(3n-2)$-dimensional integrals of (huge) rational functions of Feynman parameters. Integrals are computed using MonteCarlo routine (VEGAS) and needs an enormous amount of computing time to sample adequately the integrands.
- My method, consists in two phases
  1. reduction of contributions from each Feynman diagram to a small number (334 for C_4) of irreducible n-loop D-dimensional integrals by using a suitable algorithm (S.L. 1996, 2001).
  2. determination of sistems of difference or differential equations satisfied by the irreducible integrals (S.L. 2001)
  3. high precision calculation of these integrals by solving these systems of equations by means of rapidly convergent series expansions

  This method allowed to obtain 1100 digits of C_4 (and 9800 digits for some selected important integrals).

- An alternative approach has been recently introduced by S.Volkov (2017). It is based on MonteCarlo integration. It seems promising at 5-loop level.

In order to perform this calculation, in 1995 I begun writing a C program, `SYS`, containing all the necessary ingredients:

- a symplified fast algebraic (invoking repeatedly FORM, that I had successfully used for C_3, has a not negligible time cost)
- a numerical solver of systems of difference and differential equations
- a library of arbitrary precision mathematical routines, integer and floating point (in mid-1990 the GMP library was still in its infancy).

The program SYS

- C program, about 23000 lines.
- The program automatically determines the master integrals of a diagram, it builds and solves the systems of difference or differential equations.
- Input: description of the diagram, number of terms of the expansion in $D-4$.
- The program contains a simplified algebraic manipulator, used to solve systems of identities among integrals with this kind of coefficients: arbitrary precision integers, rationals, ratios of polynomials in one and two variables (for example D and x) with integer coefficients.
- Efficient management of systems of identities of size up to the limit of disk space (tested up to 500 million of identities).
- Numerical solution of systems of difference and differential equations up to 900 equations, using arbitrary precision floating point complex numbers and truncated series in ϵ.
- All the coefficients of the expansions in ϵ are worked out in numerical form,

even those of divergent terms.

- Floating number precision: up to 9800 digits (essentially one sums expansions in *one* variable).
- Arithmetic libraries which deal with operations on arbitrary precision integers, polynomials, rationals, arbitrary precision floating point numbers and truncated series in ϵ were written on purpose by the author. *Independent* of all other available libraries.
- Several Multicore/multinode parallel versions of the program were written on purpose.
- Sistematic protection of large buffers, I/O with crc/checksums. Found several subtle corruptions in the years, like marginal coupling of non-ECC RAM modules (1 bit changed per week), failing RAID systems (corrupted blocks of 64KBytes), etc. . . .)

  This has permitted to obtain an high reliability result.

Possible new results

There are some quantities which can be calculated using the results of the QED C_4 calculation (table of irreducible integrals):

- the high-precision calculation of the slope $F_2'(0)$ (in progress)
- the high-precision calculation of renormalization constants in QED at 4 loop (in progress)
- the high-precision calculation of the on-shell renormalization constants in QCD at 4 loop; due to the presence of the gluon-gluon vertex in QCD, this will require time-consuming high-precision calculations of some (~ 50) additional irreducible (master) integrals. Adequate computing resources are necessary.
- High-precision C_5? Surely a many-years task.

Acknowledgements

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The End