Four-point function in general kinematics through geometrical splitting and reduction

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Abstract. It is shown how the geometrical splitting of N-point Feynman diagrams can be used to simplify the parametric integrals and reduce the number of variables in the occurring functions. As an example, a calculation of the dimensionally-regulated one-loop four-point function in general kinematics is presented.

1. Introduction

In the general off-shell case, one-loop N-point diagrams (shown in figure 1) depend on $\frac{1}{2}N(N-1)$ momentum invariants $k_{jl}^2 = (p_j - p_l)^2$ and N masses of the internal particles m_i. Here and below, for the corresponding scalar integrals we follow the notation $J^{(N)}\left(n; \{\nu_i\}|\{k_{jl}^2\}, \{m_i\}\right)$ used in [1], where ν_i are the powers of the internal scalar propagators, and the space-time dimension is denoted as n, so that we can also deal with the dimensionally-regulated integrals with $n = 4 - 2\varepsilon$ [2]. Below we will mainly consider the cases when all $\nu_i = 1$.

A geometrical interpretation of kinematic invariants and other quantities related to N-point Feynman diagrams helps us to understand the analytical structure of the results for these diagrams. As an example, singularities of the general three-point function can be described pictorially through a tetrahedron constructed out of the external momenta and internal masses. Such a geometrical visualization can be used to derive Landau equations defining the positions of possible singularities [3] (see also in [4]).

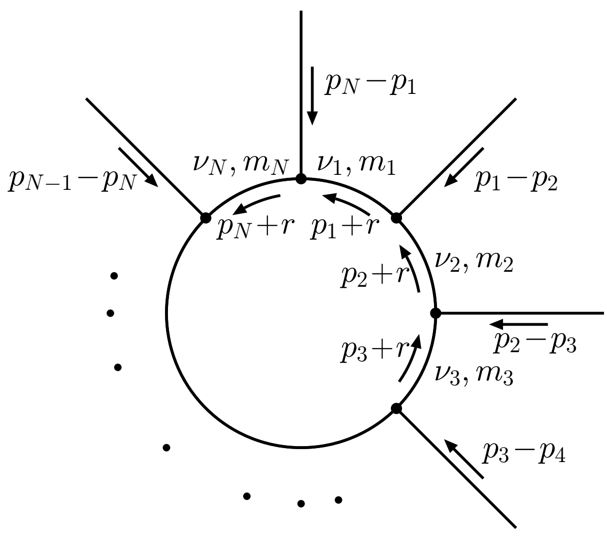

Figure 1. N-point one-loop diagram

In [5, 6, 7] it was demonstrated how such geometrical ideas could be used for an analytical calculation of one-loop N-point diagrams. For the geometrical interpretation, a "basic simplex" in N-dimensional Euclidean space is employed (a triangle for $N = 2$, a tetrahedron for $N = 3$, etc.), and the obtained results are expressed in terms of an integral over an $(N-1)$-dimensional spherical (or hyperbolic) simplex, which corresponds to the intersection

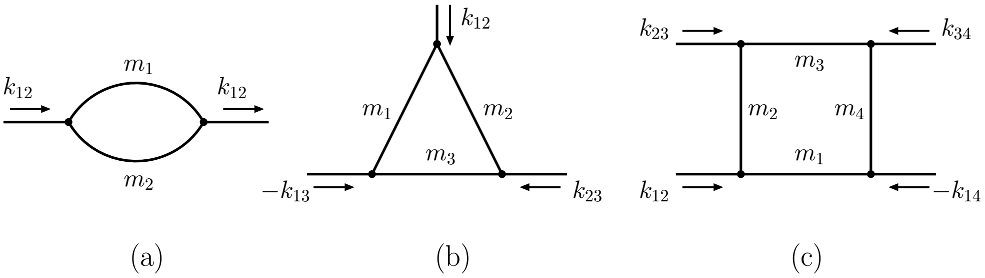

Figure 2. Momenta and masses in (a) two-point (b) three-point and (c) four-point diagrams.

of the basic simplex and the unit hypersphere (or the corresponding hyperbolic hypersurface), with a weight function depending on the angular distance θ between the integration point and the point 0, corresponding to the height of the basic simplex (see in [5]). For $n = N$ this weight function is equal to 1, and the results simplify: for the case $n = N = 3$ see in [8], and for the case $n = N = 4$ see in [9, 10]. Other interesting examples of using the geometrical approach can be found, e.g., in [11].

In this paper we will demonstrate that the natural way of splitting the basic simplex, as prescribed within the geometrical approach discussed above, leads to a reduction of the effective number of independent variables in separate contributions obtained as a result of such splitting [12]. Moreover, by considering examples with $N \leq 4$ we will show that this reduction leads to simplifications in the corresponding Feynman parametric integrals, which can be explicitly calculated in terms of the (generalized) hypergeometric functions.

2. Two-point function

For the two-point function (see figure 2a), there is only one external momentum invariant k_{12}^2, and the sides of the corresponding basic triangle are m_1, m_2 and $K_{12} \equiv \sqrt{k_{12}^2}$ (see figure 3a). The angle τ_{12} between the sides m_1 and m_2 is defined through $\cos\tau_{12} \equiv c_{12} = (m_1^2 + m_2^2 - k_{12}^2)/(2m_1m_2)$, and (in the spherical case) the integration goes over the arc τ_{12} of the unit circle, as shown in figure 3b.

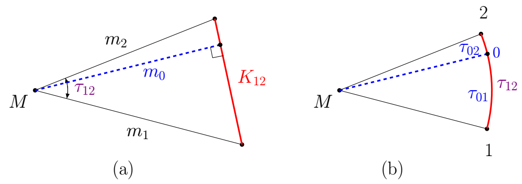

Figure 3. Two-point case: (a) the basic triangle and (b) the arc τ_{12}.

For splitting we use the height of the basic triangle, $m_0 = m_1 m_2 \sin\tau_{12}/\sqrt{k_{12}^2}$, and obtain two triangles with the sides $(m_1, m_0, K_{01} \equiv \sqrt{k_{01}^2})$ and $(m_2, m_0, K_{02} \equiv \sqrt{k_{02}^2})$, respectively. By construction, $K_{01} + K_{02} = K_{12}$. Here $k_{01}^2 = (k_{12}^2 + m_1^2 - m_2^2)^2/(4k_{12}^2)$ and $k_{02}^2 = (k_{12}^2 - m_1^2 + m_2^2)^2/(4k_{12}^2)$ (note that $k_{01}^2 = m_1^2 - m_0^2$ and $k_{02}^2 = m_2^2 - m_0^2$). Each of the resulting integrals can be associated with a two-point function, and we arrive at the following decomposition [12]:

$$\begin{aligned} J^{(2)}\left(n; 1, 1 | k_{12}^2; m_1, m_2\right) &= \frac{1}{2k_{12}^2}\left\{(k_{12}^2 + m_1^2 - m_2^2) J^{(2)}\left(n; 1, 1 | k_{01}^2; m_1, m_0\right)\right. \\ &\qquad \left. + (k_{12}^2 - m_1^2 + m_2^2) J^{(2)}\left(n; 1, 1 | k_{02}^2; m_2, m_0\right)\right\} . \qquad (1) \end{aligned}$$

This is an example of a functional relation between integrals with different momenta and masses, similar to those described in [13]. Moreover, as shown in [12], we can represent the right-hand side in terms of the equal-mass integrals $J^{(2)}\left(n; 1, 1 | 4k_{01}^2; m_1, m_1\right)$ and $J^{(2)}\left(n; 1, 1 | 4k_{02}^2; m_2, m_2\right)$.

Let us look at the number of variables. In the original integral $J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right)$ we have three independent variables: two masses and one momentum invariant (out of them we can construct two dimensionless variables). In the integral $J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right)$ we have one extra condition on the variables, $k_{01}^2=m_1^2-m_0^2$, so that we get two independent variables (i.e., one dimensionless variable).

This can also be seen in the integrands of Feynman parametric integrals: for the original two-point integral, the quadratic form is

$$J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) \Rightarrow [\alpha_1\alpha_2 k_{12}^2 - \alpha_1 m_1^2 - \alpha_2 m_2^2]\,, \tag{2}$$

whereas for one of the resulting integrals after splitting, remembering that $\alpha_1+\alpha_2=1$, we get

$$J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right) \Rightarrow [\alpha_1\alpha_2 k_{01}^2 - \alpha_1 m_1^2 - \alpha_2 m_0^2] = -[\alpha_1^2 k_{01}^2 + m_0^2]\,. \tag{3}$$

In this way, we obtain the following result in arbitrary dimension:

$$\begin{aligned} J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right) &= \mathrm{i}\pi^{n/2}\Gamma(2-n/2)\int\limits_0^1\!\!\int\limits_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\;\delta(\alpha_1+\alpha_2-1)}{[\alpha_1^2 k_{01}^2+m_0^2]^{2-n/2}} \\ &= \mathrm{i}\pi^{n/2}\frac{\Gamma(2-n/2)}{(m_0^2)^{2-n/2}}\;{}_2F_1\left(\left.\begin{matrix}1/2,\;2-n/2\\ 3/2\end{matrix}\right|-\frac{k_{01}^2}{m_0^2}\right)\,, \end{aligned} \tag{4}$$

where ${}_2F_1$ is the Gauss hypergeometric function. Similar expression for the second integral, $J^{(2)}\left(n;1,1|k_{02}^2;m_0,m_2\right)$, can be obtained by permutation $1\leftrightarrow 2$. Therefore, the result for the two-point function in arbitrary dimension can be expressed in terms of a combination of two ${}_2F_1$ functions of a single dimensionless variable (see, e.g., in [5, 14]) whose ε-expansion is known to any order [15, 16].

3. Three-point function

For the three-point function (see figure 2b), there are three external momentum invariants, k_{12}^2, k_{13}^2 and k_{23}^2, and the sides of the corresponding basic tetrahedron are m_1, m_2, m_3, $K_{12}\equiv\sqrt{k_{12}^2}$, $K_{13}\equiv\sqrt{k_{13}^2}$ and $K_{23}\equiv\sqrt{k_{23}^2}$ (see figure 4a). The angles τ_{12}, τ_{13} and τ_{23} between the sides m_1, m_2 and m_3 are defined through $\cos\tau_{jl}\equiv c_{jl}=(m_j^2+m_l^2-k_{jl}^2)/(2m_jm_l)$, and (in the spherical case) the integration extends over the spherical triangle 123 of the unit sphere, see in figure 4b.

For the splitting we use the height of the basic tetrahedron, m_0, and obtain three tetrahedra, as shown in figure 5a. One of them has the sides m_1, m_2, m_0, $K_{12}\equiv\sqrt{k_{12}^2}$, $K_{01}\equiv\sqrt{k_{01}^2}$ and $K_{02}\equiv\sqrt{k_{02}^2}$, and the sides for the others can be obtained by permutation of the indices.

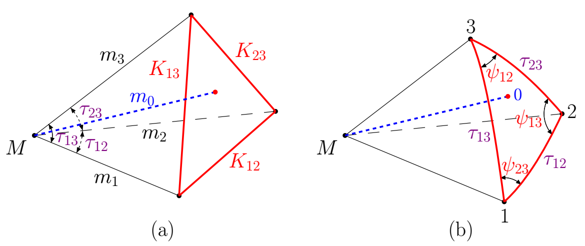

Figure 4. Three-point case: (a) the basic tetrahedron and (b) the solid angle.

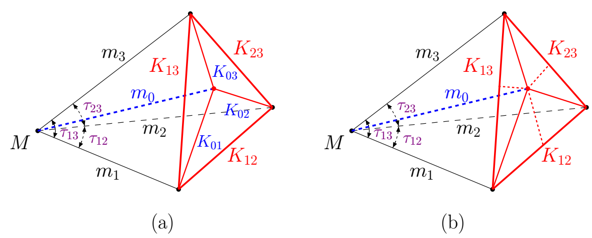

Figure 5. (a) Splitting the basic tetrahedron into three tetrahedra and (b) further splitting into six tetrahedra.

Here $k_{01}^2 = m_1^2 - m_0^2$, $k_{02}^2 = m_2^2 - m_0^2$, $k_{03}^2 = m_3^2 - m_0^2$, and $m_0 = m_1 m_2 m_3 \sqrt{D^{(3)}/\Lambda^{(3)}}$, where $\Lambda^{(3)} = \frac{1}{4}\left[2k_{12}^2 k_{13}^2 + 2k_{13}^2 k_{23}^2 + 2k_{23}^2 k_{12}^2 - (k_{12}^2)^2 - (k_{13}^2)^2 - (k_{23}^2)^2\right]$, and $D^{(3)} = \det \|c_{jl}\|$ is the Gram determinant, see in [5, 6] for more details. Each of the resulting integrals can be associated with a specific three-point function, and we arrive at the following decomposition:

$$\begin{aligned} J^{(3)}\left(n;1,1,1|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right) &= \frac{m_1^2 m_2^2 m_3^2}{\Lambda^{(3)}} \left\{ \frac{F_1^{(3)}}{m_1^2} J^{(3)}\left(n;1,1,1|k_{23}^2,k_{03}^2,k_{02}^2;m_0,m_2,m_3\right) \right. \\ &\quad + \frac{F_2^{(3)}}{m_2^2} J^{(3)}\left(n;1,1,1|k_{03}^2,k_{13}^2,k_{01}^2;m_1,m_0,m_3\right) \\ &\quad \left. + \frac{F_3^{(3)}}{m_3^2} J^{(3)}\left(n;1,1,1|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right) \right\}, \end{aligned} \tag{5}$$

with

$$F_3^{(3)} = \frac{1}{4m_1^2 m_2^2}\left[k_{12}^2\left(k_{13}^2+k_{23}^2-k_{12}^2+m_1^2+m_2^2-2m_3^2\right) - (m_1^2 - m_2^2)\left(k_{13}^2 - k_{23}^2\right)\right], \tag{6}$$

etc., so that $\sum_{i=1}^{3}(F_i^{(3)}/m_i^2) = \Lambda^{(3)}/(m_1^2 m_2^2 m_3^2)$.

By dropping perpendiculars onto the sides $K_{12} \equiv \sqrt{k_{12}^2}$, etc., each of the resulting tetrahedra can be split into two, so that in total we get six "birectangular" tetrahedra, as shown in figure 5b. In this way, we get the following relations for the integrals on the right-hand side of equation (5):

$$\begin{aligned} & J^{(3)}\left(n;1,1,1|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right) \\ &\quad = \frac{1}{2k_{12}^2}\left\{(k_{12}^2 + m_1^2 - m_2^2) J^{(3)}\left(n;1,1,1|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)\right. \\ &\qquad \left. + (k_{12}^2 - m_1^2 + m_2^2) J^{(3)}\left(n;1,1,1|k_{02}^2,k_{00'}^2,k_{20'}^2;m_{0'},m_2,m_0\right)\right\}, \end{aligned} \tag{7}$$

etc. The notation $0'$ is explained in figure 6; in particular, $m_{0'}$ is the distance between the points M and $0'$, whereas $K_{10'} = \sqrt{k_{10'}^2}$ and $K_{20'} = \sqrt{k_{20'}^2}$ are the distances between the points $(1,0')$ and $(2,0')$, respectively, so that $K_{10'} + K_{20'} = K_{12}$. Note that $k_{10'}^2 = (k_{12}^2 + m_1^2 - m_2^2)^2/(4k_{12}^2)$ and $k_{20'}^2 = (k_{12}^2 - m_1^2 + m_2^2)^2/(4k_{12}^2)$, similarly to the reduction of the two-point function.

Let us analyze the number of variables in the occurring three-point integrals. In $J^{(3)}\left(n;1,1,1|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$ we have six independent variables: three masses and three momentum invariants (out of them we can construct five dimensionless variables). In $J^{(3)}\left(n;1,1,1|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$ we have two extra conditions on the variables, $k_{01}^2 = m_1^2 - m_0^2$ and $k_{02}^2 = m_2^2 - m_0^2$, so that we get four independent variables (i.e., three dimensionless variables). For the integral $J^{(3)}\left(n;1,1,1|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$ we have three relations, $k_{01}^2 = m_1^2 - m_0^2$, $k_{00'}^2 = k_{01}^2 - k_{10'}^2$ and $k_{00'}^2 = m_{0'}^2 - m_0^2$. Therefore, the result for the three-point function in

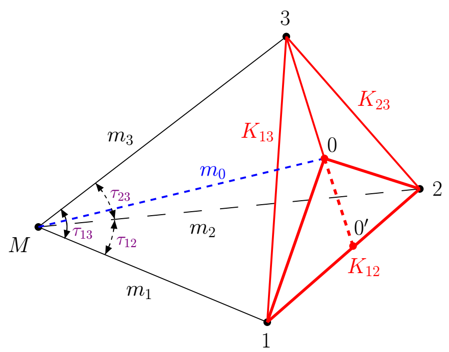

Figure 6. Last step of splitting the basic three-dimensional tetrahedron.

arbitrary dimension should be expressible in terms of a combination of functions of two dimensionless variables: indeed, we know that it can be presented in terms of the Appell hypergeometric function F_1 (see, e.g., in [6, 17, 18]).

This can also be seen in the integrands of the corresponding Feynman parametric integrals: for the original three-point integral, the quadratic form is

$$J^{(3)}\left(n;1,1,1|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right) \Rightarrow [\alpha_1\alpha_2 k_{12}^2+\alpha_1\alpha_3 k_{13}^2+\alpha_2\alpha_3 k_{23}^2-\alpha_1 m_1^2-\alpha_2 m_2^2-\alpha_3 m_3^2], \quad (8)$$

and for one of the resulting integrals after splitting, remembering that $\alpha_1+\alpha_2+\alpha_3=1$, we get

$$\begin{aligned} J^{(3)}&\left(n;1,1,1|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)\\ &\Rightarrow [\alpha_1\alpha_2 k_{10'}^2+\alpha_1\alpha_3 k_{01}^2+\alpha_2\alpha_3 k_{00'}^2-\alpha_1 m_1^2-\alpha_2 m_{0'}^2-\alpha_3 m_0^2]\\ &\Rightarrow -[\alpha_1^2 k_{10'}^2+(\alpha_1+\alpha_2)^2 k_{00'}^2+m_0^2]. \end{aligned} \quad (9)$$

In this way, we obtain the following result in arbitrary dimension:

$$\begin{aligned} &J^{(3)}\left(n;1,1,1|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)\\ &= -\mathrm{i}\pi^{n/2}\Gamma(3-n/2)\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{[\alpha_1^2 k_{10'}^2+(\alpha_1+\alpha_2)^2 k_{00'}^2+m_0^2]^{3-n/2}}\\ &= -\frac{\mathrm{i}\pi^{n/2}\Gamma(2-n/2)}{2(m_0^2)^{2-n/2}k_{00'}^2}\left\{\sqrt{\frac{k_{00'}^2}{k_{10'}^2}}\arctan\sqrt{\frac{k_{10'}^2}{k_{00'}^2}}\right.\\ &\qquad\left.-\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2}F_1\left(1/2,1,2-n/2;3/2\Big|-\frac{k_{10'}^2}{k_{00'}^2},-\frac{k_{10'}^2}{m_{0'}^2}\right)\right\}, \end{aligned} \quad (10)$$

where F_1 is Appell hypergeometric function of two variables,

$$F_1\left(a,b_1,b_2;c|x,y\right)=\sum_{j_1,j_2}\frac{(a)_{j_1+j_2}\,(b_1)_{j_1}\,(b_2)_{j_2}}{(c)_{j_1+j_2}}\,\frac{x^{j_1}\,y^{j_2}}{j_1!\,j_2!}\,.$$

Similar results for other five contributions can be obtained by permutation. Using known transformation formulae for F_1 we can see that the obtained expression (10) is equivalent to the result presented in [6].

4. Four-point function

For the four-point function (see figure 2c), there are six external momentum invariants. Out of them, k_{12}^2, k_{23}^2, k_{34}^2 and k_{14}^2 are the squared momenta of the external legs, whilst k_{13}^2 and k_{24}^2 correspond to the Mandelstam variables s and t. The sides of the corresponding basic four-dimensional simplex are m_1, m_2, m_3, m_4, and six additional sides $K_{jl}\equiv\sqrt{k_{jl}^2}$, as shown in figure 7a. The six angles τ_{jl} between the corresponding sides m_j and m_l are defined through $\cos\tau_{jl}\equiv c_{jl}=(m_j^2+m_l^2-k_{jl}^2)/(2m_jm_l)$, and (in the spherical case) the integration extends over the spherical tetrahedron 1234 of the unit hypersphere, as shown in figure 7b (for the hyperbolic case one can use analytic continuation).

For splitting we use the height of the basic simplex, m_0, and obtain four simplices, as shown in figures 8 and 8a. One of them has the sides m_1, m_2, m_3, m_0, $K_{12}\equiv\sqrt{k_{12}^2}$, $K_{13}\equiv\sqrt{k_{13}^2}$,

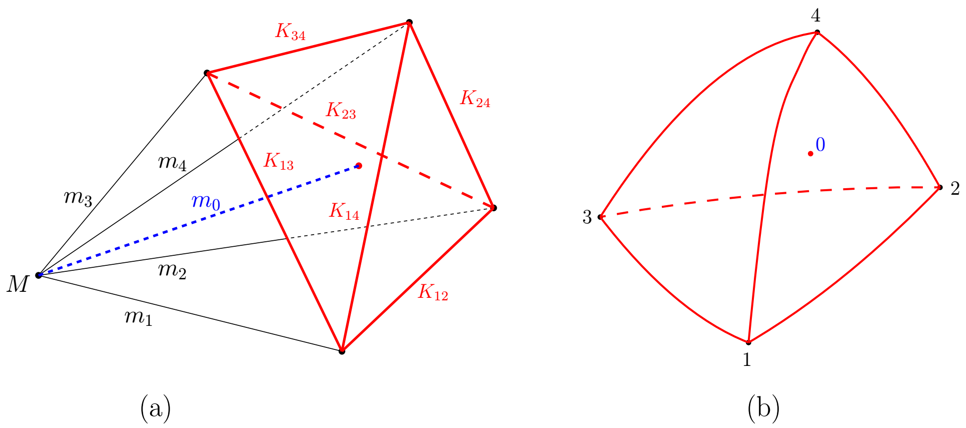

Figure 7. Four-point case: (a) the basic simplex and (b) the spherical tetrahedron.

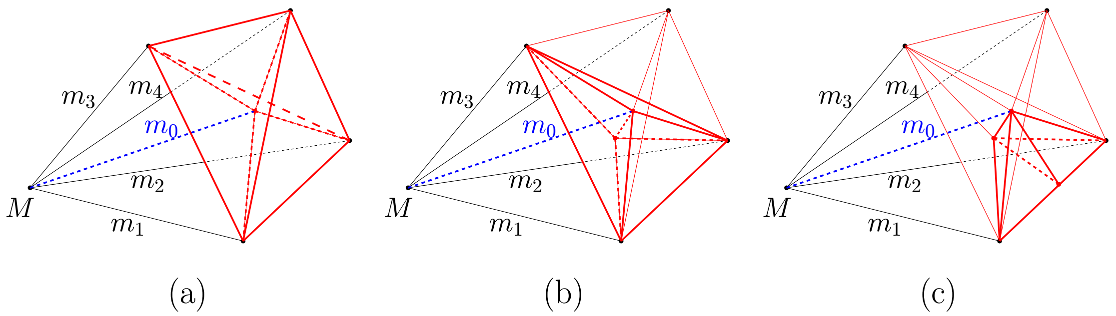

Figure 8. Four-point case: steps of splitting the basic four-dimensional simplex.

$K_{23} \equiv \sqrt{k_{23}^2}$, $K_{01} \equiv \sqrt{k_{01}^2}$, $K_{02} \equiv \sqrt{k_{02}^2}$ and $K_{03} \equiv \sqrt{k_{03}^2}$, and the sides of the others can be obtained by permutation of the indices. As before, $k_{0i}^2 = m_i^2 - m_0^2$ $(i = 1, 2, 3, 4)$, whereas $m_0 = m_1 m_2 m_3 m_4 \sqrt{D^{(4)}/\Lambda^{(4)}}$, where $D^{(4)} = \det \|c_{jl}\|$ and $\Lambda^{(4)} = \det \|(k_{j4} \cdot k_{l4})\|$, see in [5] for more details. Each of the four resulting integrals can be associated with a certain four-point function. At the next step, in each of the four tetrahedra (drawn in red) we drop the perpendiculars onto the triangle sides, as shown in figure 8b, splitting each of them into three, and then dividing each of the resulting tetrahedra into two, by dropping perpendiculars onto the $\sqrt{k_{jl}^2}$ sides, as shown in figure 8c. As a result of this splitting, we get $4 \cdot 3 \cdot 2 = 24$ simplices.

Let us look at the number of variables. In the integral $J^{(4)}\left(n; 1, 1, 1, 1 | \{k_{jl}^2\}; \{m_i\}\right)$ we have ten independent variables: four masses and six momentum invariants (out of them we can construct nine dimensionless variables). After the first step (figure 8a) we have three conditions on the variables, $k_{01}^2 = m_1^2 - m_0^2$, $k_{02}^2 = m_2^2 - m_0^2$ and $k_{03}^2 = m_3^2 - m_0^2$, so that we get seven independent variables (i.e., six dimensionless variables). After the second step (figure 8b), we get two extra conditions due to the right triangles, and after the third step (figure 8c) we get one more condition. As a result, for each of the 24 resulting four-point

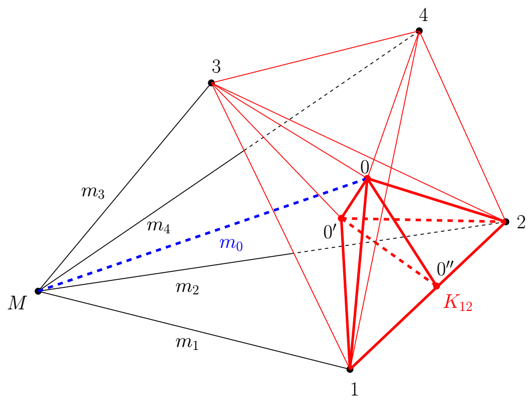

Figure 9. Last step of splitting the basic four-dimensional simplex.

functions we have six relations, so that we end up with four independent variables (i.e., three dimensionless variables). Therefore, the result for the four-point function in arbitrary dimension should be expressible in terms of a combination of functions of three dimensionless variables, such as, e.g., Lauricella functions and their generalizations (see, e.g., in [18, 19]).

This can also be seen in the integrands of the corresponding Feynman parametric integrals: for the original four-point integral, the quadratic form is

$$\begin{aligned} &J^{(4)}\left(n;1,1,1,1|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right) \\ &\qquad\Rightarrow [\alpha_1\alpha_2 k_{12}^2+\alpha_1\alpha_3 k_{13}^2+\alpha_1\alpha_4 k_{14}^2+\alpha_2\alpha_3 k_{23}^2+\alpha_2\alpha_4 k_{24}^2+\alpha_3\alpha_4 k_{34}^2 \\ &\qquad\quad -\alpha_1 m_1^2-\alpha_2 m_2^2-\alpha_3 m_3^2-\alpha_4 m_4^2]\,, \end{aligned} \tag{11}$$

whereas after the last step of splitting, remembering that $\alpha_1+\alpha_2+\alpha_3+\alpha_4=1$, we get

$$\begin{aligned} &J^{(4)}\left(n;1,1,1,1|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right) \\ &\qquad\Rightarrow [\alpha_1\alpha_2 k_{10''}^2+\alpha_1\alpha_3 k_{10'}^2+\alpha_1\alpha_4 k_{01}^2+\alpha_2\alpha_3 k_{0'0''}^2+\alpha_2\alpha_4 k_{00''}^2+\alpha_3\alpha_4 k_{00'}^2 \\ &\qquad\quad -\alpha_1 m_1^2-\alpha_2 m_{0''}^2-\alpha_3 m_{0'}^2-\alpha_4 m_0^2] \\ &\qquad\Rightarrow -[\alpha_1^2 k_{10''}^2+(\alpha_1+\alpha_2)^2 k_{0'0''}^2+(\alpha_1+\alpha_2+\alpha_3)^2 k_{00'}^2+m_0^2]\,. \end{aligned} \tag{12}$$

The notations $0'$ and $0''$ are explained in figure 9; in particular, $m_{0'}$ is the distance between M and $0'$, $m_{0''}$ is the distance between M and $0''$, whereas $K_{00'}=\sqrt{k_{00'}^2}$, $K_{0'0''}=\sqrt{k_{0'0''}^2}$ and $K_{10''}=\sqrt{k_{10''}^2}$ are the distances between the corresponding points $(0,0')$, $(0',0'')$ and $(1,0'')$, respectively. In this way, we obtain the following result in arbitrary dimension:

$$\begin{aligned} &J^{(4)}\left(n;1,1,1,1|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right) \\ &= \mathrm{i}\pi^{n/2}\Gamma(4-n/2)\int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\;\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{[\alpha_1^2 k_{10''}^2+(\alpha_1+\alpha_2)^2 k_{0'0''}^2+(\alpha_1+\alpha_2+\alpha_3)^2 k_{00'}^2+m_0^2]^{4-n/2}} \\ &= \frac{\mathrm{i}\pi^{n/2}\Gamma(3-n/2)}{2k_{0'0''}^2(m_0^2)^{3-n/2}}\left\{\sqrt{\frac{k_{0'0''}^2}{k_{10''}^2}}\arctan\sqrt{\frac{k_{10''}^2}{k_{0'0''}^2}}\;{}_2F_1\left(\begin{matrix}1/2,3-n/2\\3/2\end{matrix}\Big|-\frac{k_{00'}^2}{m_0^2}\right)\right. \end{aligned} \tag{13}$$

$$\left.-\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2} F_N\left(1,1,3-n/2,1/2,(n-3)/2,1/2;3/2,3/2,3/2\Big|-\frac{k_{10''}^2}{k_{0'0''}^2},-\frac{k_{00'}^2}{m_0^2},-\frac{k_{10'}^2}{m_{0'}^2}\right)\right\}, \tag{14}$$

where F_N is one of the Lauricella-Saran functions [20],

$$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2|x,y,z)=\sum_{j_1,j_2,j_3}\frac{(a_1)_{j_1}(a_2)_{j_2}(a_3)_{j_3}(b_1)_{j_1+j_3}(b_2)_{j_2}}{(c_1)_{j_1}(c_2)_{j_2+j_3}}\,\frac{x^{j_1}y^{j_2}z^{j_3}}{j_1!j_2!j_3!}\,.$$

5. General remarks and conclusions

The geometrical approach allows us to relate the one-loop N-point Feynman diagrams to certain (hyper)volume integrals in non-Euclidean geometry. Geometrical splitting provides a straightforward way of reducing general integrals to those with lesser number of independent variables. Furthermore, in this way we can predict the set and the number of these variables in the resulting integrals. As shown in [12], for an N-point diagram (depending, in the general off-shell case, on $\frac{1}{2}(N-1)(N+2)$ dimensionless variables), after splitting in $N!$ pieces we will get a combination of $N!$ integrals, each of them depending only on $N-1$ variables. For example, in the four-point case we will get functions of three variables, rather than nine.

Geometrically, we can calculate the resulting integrals in the framework of non-Euclidean geometry or, alternatively, represent them again in terms of Feynman parameters. Since some of the variables are connected, the quadratic forms in the integrands of the resulting parametric integrals can be simplified. In this way, for $N = 2, 3, 4$ we get the following quadratic forms:

$$J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right) \Rightarrow -[\alpha_1^2 k_{01}^2 + m_0^2]\,,$$
$$J^{(3)}\left(n;1,1,1|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right) \Rightarrow -[\alpha_1^2 k_{10'}^2 + (\alpha_1+\alpha_2)^2 k_{00'}^2 + m_0^2]\,,$$
$$J^{(4)}\left(n;1,1,1,1|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$$
$$\Rightarrow -[\alpha_1^2 k_{10''}^2 + (\alpha_1+\alpha_2)^2 k_{0'0''}^2 + (\alpha_1+\alpha_2+\alpha_3)^2 k_{00'}^2 + m_0^2]\,.$$

Evaluating these integrals for an arbitrary dimension n we can get explicit expressions in terms of (generalized) hypergeometric functions: ${}_2F_1$ for $N = 2$, F_1 for $N = 3$, and F_N for $N = 4$. For $N > 4$, in the quadratic forms we should also expect sums of squares of partial sums of α's (the coefficient of m_0^2 can be understood as the square of the sum of all α's, equal to one).

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