High-precision calculation of the 4-loop contribution to the electron g-2 in QED[1]

Stefano Laporta

Dipartimento di Fisica e Astronomia, Università di Padova,
Istituto Nazionale Fisica Nucleare, Sezione di Padova,
Via Marzolo 8,
I-35131 Padova, Italy[2]

E-mail: `stefano.laporta@pd.infn.it`

Abstract. In this paper I show the result of the numerical evaluation of the mass-independent 4-loop contribution to the electron g-2 in QED with 1100 digits of precision. I show the semi-analytical fit to the numerical value, which contains harmonic polylogarithms of $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$ and $e^{\frac{i\pi}{2}}$, one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits. I show also some information about the method and the program used.

1. Introduction

The QED mass-independent contribution to the electron g-2 can be expanded perturbatively in power series of the fine structure constant α:

$$A_1 = A_1^{(2)}\left(\frac{\alpha}{\pi}\right) + A_1^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_1^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + A_1^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + A_1^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \dots . \qquad (1)$$

For further information about the theoretical prediction of electron g-2, see Ref. [1] in these proceedings. In Ref. [2] I have evaluated the 4-loop contribution $A_1^{(8)}$ up to 1100 digits of precision, finalizing a twenty-year effort [3–9] begun after the completion of the calculation of $A_1^{(6)}$ [10]. The first digits of the result are

$$A_1^{(8)} = -1.912245764926445574152647167439830054060873390658725345171329848\dots . \qquad (2)$$

The full-precision result is shown in table 1. The contribution to $A_1^{(8)}$ is given by 891 Feynman vertex diagrams, which can be obtained by inserting an external photon in 104 self-mass diagrams (depicted in Fig.1).

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-1.912245764926445574152647167439830054060873390658725345171329848006038443980651706142760892700003631583755841533147327005637851491285453911
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6160442613254490690004319151982300474881814943110384953782994062967586787538524978194698979313216219797575067670114290489796208505078559 2...
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Table 1. First 1100 digits of $A_1^{(8)}$.

2. Gauge-invariant sets

The contributions of single Feynman diagrams may be I.R. or U.V. divergent, and depend on the gauge chosen. Therefore, it is convenient to regroup the diagrams in gauge invariant sets, which are finite. The number of diagrams at 2, 3 and 4 loop are 7, 72 and 891, respectively; they can be arranged in 3, 9(see Ref. [11]) and 25 gauge invariant sets, respectively. Some elements of the 25 sets at 4 loops are shown in Fig.2. The numerical contributions of each 4-loop set truncated to 40 digits are listed in Ref. [2].

3. The numerical fit

The contributions of all 4-loop vertex diagrams can be expressed by means of 334 master integrals belonging to 220 topologies. Due to the expected analytical complexity of the result, a completely analytical calculation of all the master integrals (similar to that at three-loop) has seemed out of reach. So I have used a different approach:

- Compute very high precision values of the $\epsilon-$expansions of the master integrals in $D = 4-2\epsilon$ dimensions.
- Make the right analytical ansatz.
- Fit the rational coefficients of the ansatz by using the PSLQ algorithm [12–14]

As a result, one finds that the analytical expressions of the coefficients of the $\epsilon-$expansions of the master integrals contain values of harmonic polylogarithms [15] with argument 1, $\frac{1}{2}$, $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$ and $e^{\frac{i\pi}{2}}$, a family of one-dimensional integrals of products of elliptic integrals, and the finite terms of the $\epsilon-$expansions of six master integrals. Work is still in progress to fit analytically these six unknown elliptical constants. The result of the analytical fit for $A_1^{(8)}$ can be written as follows:

$$A_1^{(8)} = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \ , \tag{3}$$

$$\begin{aligned}
T =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 \\
&+ \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) - \frac{221581}{2160}\zeta(4)\ln 2 \\
&+ \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 \\
&- \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 \\
&- \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) - \frac{142793}{18}a_5\zeta(2) \\
&+ \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 \\
&+ \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 \\
&- \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \ ,
\end{aligned} \tag{4}$$

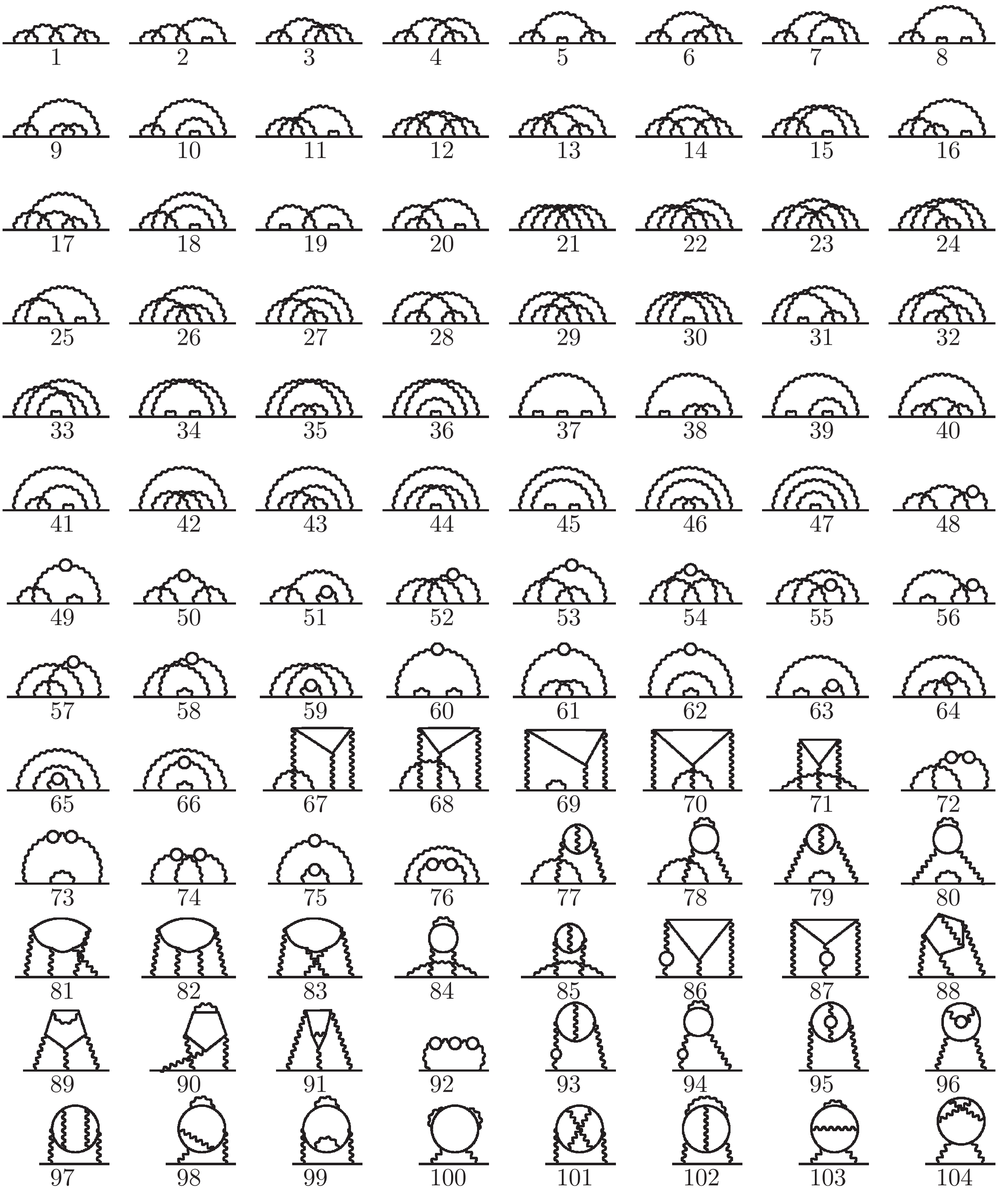

Figure 1. The 4-loop self-mass diagrams.

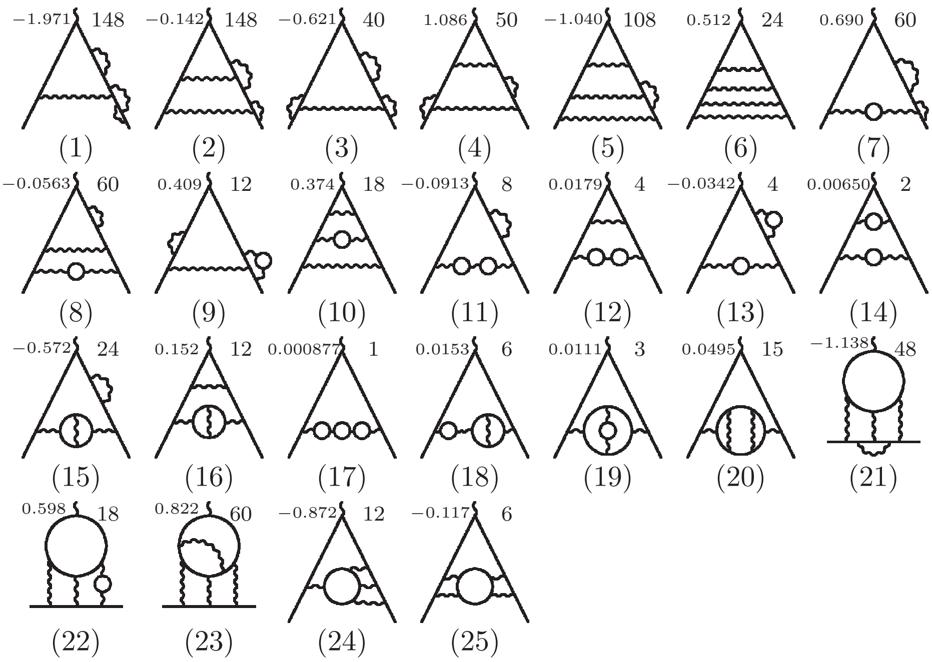

Figure 2. Typical representative diagrams of gauge-invariant sets. For each set only one diagrams is shown. Top left: contribution of the set to g-2; top right: number of diagrams of the set

$$\begin{aligned}V_a =& -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right)\\
&+\frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + 19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\\
&+\frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\\
&-\frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\\
&+\frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2\\
&+\frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\ ,\end{aligned} \tag{5}$$

$$\begin{aligned}V_b =& \frac{13487}{60}\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13487}{60}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\\
&+651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\\
&-\frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\\
&+\frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\\
&+\frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\\
&+\frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ ,\end{aligned} \tag{6}$$

$$W_b = \zeta(2)\Bigg(-\frac{28276}{25}\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\Bigg(4\mathrm{Re}H_{0,1,0,1,1}(i) + 4\mathrm{Im}H_{0,1,1}(i)\,\mathrm{Cl}_2\left(\frac{\pi}{2}\right) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\ln 2\Bigg)\Bigg)\ , \tag{7}$$

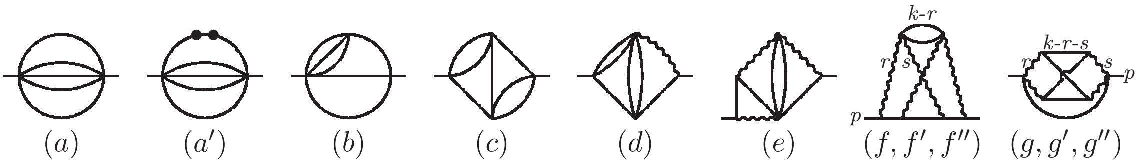

Figure 3. Minimal set of master integrals which contain all the elliptic constants. The double dot in (a') means that denominator is raised to the power three. (f, f', f'') and (g, g', g'') have numerators equal to $(1, p.k, (p.k)^2)$, respectively.

$$
\begin{aligned}
E_a =& \pi\left(-\frac{28458503}{691200}B_3+\frac{250077961}{18662400}C_3\right)+\frac{483913}{77760}\pi f_2(0,0,1)+\pi\left(\frac{4715}{1944}\ln 2\; f_2(0,0,1)+\frac{270433}{10935}f_2(0,2,0)\right.\\
&\left.-\frac{188147}{4860}f_2(0,1,1)+\frac{188147}{12960}f_2(0,0,2)\right)+\pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1)-\frac{5525}{432}\ln 2\; f_2(0,0,2)\right.\\
&+\frac{5525}{162}\ln 2\; f_2(0,1,1)-\frac{5525}{243}\ln 2\; f_2(0,2,0)+\frac{526015}{248832}f_2(0,0,3)-\frac{4675}{768}f_2(0,1,2)+\frac{1805965}{248832}f_2(0,2,1)\\
&\left.-\frac{3710675}{1119744}f_2(0,3,0)-\frac{75145}{124416}f_2(1,0,2)-\frac{213635}{124416}f_2(1,1,1)+\frac{168455}{62208}f_2(1,2,0)+\frac{69245}{124416}f_2(2,1,0)\right)\;,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
E_b =& \zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2)-\frac{556445}{6912}f_1(0,1,1)+\frac{54515}{972}f_1(0,2,0)-\frac{75145}{20736}f_1(1,0,1)\right)\\
&-\frac{4715}{1458}\zeta(2)f_1(0,0,1)\;,
\end{aligned} \tag{9}
$$

$$
U =-\frac{541}{300}C_{81a}-\frac{629}{60}C_{81b}+\frac{49}{3}C_{81c}-\frac{327}{160}C_{83a}+\frac{49}{36}C_{83b}+\frac{37}{6}C_{83c}\;. \tag{10}
$$

C_{8xy} are the ϵ^0 coefficients of the $\epsilon-$expansion of six master integrals (see f, f', f'', g, g', g'' of Fig.3); they appear only in the gauge-invariant sets 24 and 25. In the above expressions $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i}\, i^{-n}$, $b_6 = H_{0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $b_7 = H_{0,0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$, $\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(e^{i\theta})$. $H_{i_1,i_2,\ldots}(x)$ are the harmonic polylogarithms. The integrals f_j are defined as follows:

$$
f_m(i,j,k) \;=\; \int_1^9 ds\; D_1(s)\mathrm{Re}\left(\sqrt{3^{m-1}}D_m(s)\right)\left(s-\frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s)\;, \tag{11}
$$

$$
D_m(s) \;=\; \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(m-1-(2m-3)\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right)\;; \tag{12}
$$

$K(x)$ is the complete elliptic integral of the first kind. The constants B_3 and C_3, defined in Ref. [8], admit the hypergeometric representations:

$$
B_3 =\frac{4\pi^{\frac{3}{2}}}{3}\left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})}{}_4F_3\left(\begin{matrix}\frac{1}{6}\,\frac{1}{3}\,\frac{1}{3}\,\frac{1}{2}\\ \frac{5}{6}\,\frac{5}{6}\,\frac{2}{3}\end{matrix};1\right)+\frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})}{}_4F_3\left(\begin{matrix}\frac{1}{2}\,\frac{2}{3}\,\frac{2}{3}\,\frac{5}{6}\\ \frac{7}{6}\,\frac{7}{6}\,\frac{4}{3}\end{matrix};1\right)\right)\;, \tag{13}
$$

$$
C_3 =\frac{486\pi^2}{1925}\;{}_7F_6\left(\begin{matrix}\frac{7}{4}\,-\frac{1}{3}\,\frac{1}{3}\,\frac{2}{3}\,\frac{4}{3}\,\frac{3}{2}\,\frac{3}{2}\\ \frac{3}{4}\,1\,\frac{7}{6}\,\frac{11}{6}\,\frac{13}{6}\,\frac{17}{6}\end{matrix};1\right)\;, \tag{14}
$$

Fig.3 shows the fundamental elliptic master integrals which contains irreducible combinations of B_3, C_3 and $f_m(i,j,k)$.

The PSLQ basis needed to fit T contains $F_{10} - 1 = 54$ terms (F_i are the Fibonacci numbers, $F_1 = F_2 = 1$). Due to the presence of harmonic polylogarithms of complex argument, the general basis needed to perform the fit of V_a and V_b contains $F_{17} - 1 = 1596$ terms. The practical maximum size of basis used in PSLQ fits is ≈ 500, so some guessing is needed in order to identify which elements are really needed in the fits (for example, only 93 for V_a and V_b). The precision of the calculation of elements of the basis is up to 9600 digits. The multi-pair parallel version of the PSLQ algorithm [13] publicly available from Ref. [14] has been essential to work out these difficult analytical fits in reasonable times.

4. Method of calculation

4.1. Diagrams generation

All the 104 4-loop self-mass diagrams are generated with a *C* program written *from scratch*; for each self-mass diagram, the corresponding vertex diagrams are generated by inserting a photon in all possible ways. For the sake of subsequent checks, one keeps also the diagrams which give no contribution because of Furry's theorem.

4.2. Extraction of the contribution to g-2

The contribution to g-2 of the diagram is extracted from the unrenormalized amplitude of each vertex diagram using suitable projectors, with a FORM [20, 21] program. The results are expressions which typically contain 100-30000 different Feynman integrals.

4.3. Identification of master integrals

The total number of topologically distinct master integrals is 334. 82 topologies have more than one master integral. For these topologies, products of scalar products are chosen as numerators, the choice depending on the momentum flow of the diagram. For the sake of subsequent checks, one processes *separately* the contributions from each one of the 104 self-mass diagrams. The master integrals are calculated separately for each self-mass diagram; therefore the total number of the master integrals actually calculated is 4607. Because of differences in momentum routing and therefore in the numerators, topologies with more than one master integral will have a slightly different set of master integrals in different self-mass diagrams. This provides a *lot of* useful checks.

4.4. Reduction to master integrals

The reduction is exact in D, and is performed by generating large systems of integration-by-parts identities and solving them with the algorithm implemented in `SYS` [3]. These large systems contain $4 \times 10^6 - 50 \times 10^6$ identities, with physical disk space ranging from $4GB$ to $100GB$. The reduction to master integrals has been repeated with 32 and 64 bit versions of `SYS`. In addition, the principal self-mass diagrams were reprocessed using a different momentum flow, checking that reduction to master integrals remained the same (after converting different sets of master integrals).

4.5. Numerical calculation of master integrals

For the numerical calculation of master integrals I have used combinations of the the difference equation method [3, 4] and the differential equation method [17–19]. An approach consists in calculating with difference equations (inserting the exponent n of the first electron propagator) the integral obtained putting the photon mass λ equal to the electron mass m, and integrating a differential equation in λ from $\lambda = m$ to $\lambda = 0$. A second approach consists in calculating with difference equations the diagram obtained by putting the the external momentum $p = 0$, and integrating a differential equation in p^2 from $p^2 = 0$ to $p^2 = -m^2$. The systems of difference and differential equations for master integrals are obtained by building systems of integration-by-parts identities and solving them by using the algorithm [3, 4], using rational arithmetic in D.

The sizes of the systems of difference or differential equations to be numerically solved are in the range $1MB-3GB$. Difference equations are solved by using the Laplace transformation method (integral representation of solutions and differential equation of the integrand). Differential equations are solved numerically, by using series expansion with truncated expansions in $\epsilon=(4-D)/2$ as coefficients. The minimum number of terms of the expansion in ϵ is 9 ($\epsilon^{-4}\ldots\ \epsilon^{4}$). There are cancellations of ϵ terms in intermediate steps; no care is used to avoid cancellations, as the corresponding numerical zeroes provide extremely useful checks. In the worst case 37 terms of expansions are needed. The standard precision of calculations is 4 kbit (1232 digits). About 130 digits are lost due to cancellations, so that the result has 1100 digits exact. Note that the analytical fit of some selected master integrals required a precision much higher, up to 16 kbit (9864 digits).

4.6. Renormalization
The renormalization counterterms are generated with two procedures written *from scratch* in *C* and FORM. Their expressions are reduced to master integrals already known in analytical form. I have chosen the Feynman gauge. I checked explicitly the (internal) gauge invariance of the contributions for arbitrary gauge of the photon line going into 1- ,2- or 3-loop vacuum polarization diagrams.

4.7. The program
In order to perform this calculation, in 1995 I begun writing 24000-line a comprehensive *C* program *from scratch*, `SYS` [3], containing all the necessary parts, among which:

- a simplified fast algebraic symbolic manipulator,
- a numerical solver of systems of difference and differential equations,
- and a library of multiprecision routines, operating on integer and rational numbers, polynomials in one and two variables, floating point numbers, series expansions with floating point coefficients.

Another important feature is the thorough protection of large buffers and I/O with checksums. Throughout the calculation, I have found several subtle corruptions, due to different causes, like marginal coupling of non-ECC RAM modules (error-rate: 1 bit per week), failing RAID systems (error-rate: one corrupted block of 64KB every 100GB), etc....) This has resulted in an highly reliable result.

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