New results on g-2 calculation[1]

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Abstract. In this paper I summarize the current situation of the theoretical calculation of the electron g-2, including the result from the recent calculation of the QED mass-independent 4-loop contribution with a precision of 1100 digits.

A particle of mass m and spin s possesses a magnetic moment μ

$$\mu = g\frac{eh}{4\pi mc}s \ , \tag{1}$$

where g is the gyromagnetic ratio. According to the Dirac's theory [1], an electron has $g = 2$. This agreed with the experimental measurements until Kusch and Foley [2] measured a value of the *anomaly* a_e slightly different from zero:

$$a_e = \frac{g-2}{2} = 0.001\ 15(4) \ . \tag{2}$$

The deviation is due to the interaction of the electron with photons; using Q.E.D. Schwinger [3,4] was able to calculate at the first order that

$$a_e = \frac{\alpha}{2\pi} = 0.001\ 161\ldots \ , \tag{3}$$

where α is the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \ . \tag{4}$$

The current measurements of a_e are based on the Penning trap method, developed by the group at the University of Washington. This trap uses an axial magnetic field and a quadrupole electric field; the anomaly is expressed as the ratio of two frequencies, which can measured to a very high precision. For the development of this technique, the Nobel prize in Physics 1989 was awarded to H. Dehmelt. Their final results were [5]:

$$a_{e^-}^{exp} = 1\ 159\ 652\ 188.4(4.3) \times 10^{-12} \quad (4.3\ \text{ppb}) \ , \tag{5}$$

$$a_{e^+}^{exp} = 1\ 159\ 652\ 187.9(4.3) \times 10^{-12} \quad (4.3\ \text{ppb}) \ . \tag{6}$$

[1] Invited talk to ACAT 2017, University of Washington, Seattle, 21-25 August 2017

[2] since 1 September 2017

$A_1^{(4)}$	$A_1^{(6)}$	$A_1^{(8)}$	$A_1^{(10)}$
−2.973. . . [12]	1.49(20) [31]	−1.434(138) [24]	7.795(336) [29]
−0.328 478 965 579 . . . [8, 9]	1.195(26) [32]	−1.5098(384) [35]	6.599(223) [29]
	1.17611(42) [33]	−1.7283(35) [25]	
	1.181259(40) [34]	−1.9144(35) [26]	
	1.181241456587200006. . . [10]	−1.9106(20) [28]	
		−1.91298(84) [29]	
		−1.912245764926445. . . [11]	

Table 1. Numerical results of the evaluations of $A_1^{(4)}$, $A_1^{(6)}$, $A_1^{(8)}$ and $A_1^{(10)}$.

The most recent value obtained with the same technique by the Harvard group is [6, 7]:

$$a_{e^-}^{exp} = 1\ 159\ 652\ 180.73(.28) \times 10^{-12} \quad (0.24 \text{ ppb}) \ . \tag{7}$$

In the standard model

$$a_e^{SM} = a_e^{QED} + a_e^{\text{weak}} + a_e^{\text{hadr}} \ . \tag{8}$$

The QED contribution can be split up in mass-independent and mass-dependent parts:

$$a_e^{QED} = A_1 + A_2\left(\frac{m_e}{m_\mu}\right) + A_2\left(\frac{m_e}{m_\tau}\right) + A_3\left(\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau}\right) \ . \tag{9}$$

The functions $A^{(i)}$ can be expanded in power series

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + A_i^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + A_i^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \dots \ . \tag{10}$$

The mass-independent coefficients at 1, 2 and 3 loop are known in analytical form [3, 4, 8–10]:

$$A_1^{(2)} = \frac{1}{2} \ , \tag{11}$$

$$A_1^{(4)} = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328\ 478\ 965\ 579\ \dots \ , \tag{12}$$

$$\begin{aligned} A_1^{(6)} &= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right] \\ &- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} = 1.181\ 241\ 456\ \dots \ , \end{aligned}$$

where $\zeta(n) = \sum_{i=0}^{\infty} i^{-n}$, $a_n = \sum_{i=0}^{\infty} 2^{-i} i^{-n}$. In table 1 we list some older theoretical evaluations of the two, three and four loop coefficients. In Ref. [11] I have evaluated up to 1100 digits of precision the 4-loop contribution $A_1^{(8)}$, finalizing a twenty-year effort [13–19] begun after the completion of the calculation of $A_1^{(6)}$ [10]. The first digits of the result are

$$A_1^{(8)} = -1.912245764926445574152647167439830054060873390658725345171329848\dots \ . \tag{13}$$

The full-precision result is shown in table 3. The result (13) is in excellent agreement (0.9σ) with the numerical value

$$A_1^{(8)}(\text{Ref. [29]}) = -1.91298(84) \ , \tag{14}$$

latest result of a really impressive pluridecennial effort [20–29], and with the independent value

$$A_1^{(8)}(\text{Ref. [30]}) = -1.87(12) \ . \tag{15}$$

contribution	value in units of 10^{-12}
$A_1^{(2)}(\alpha/\pi)$	1 161 409 733.631(720)
$A_1^{(4)}(\alpha/\pi)^2$	− 1 772 305.065(3)
$A_1^{(6)}(\alpha/\pi)^3$	14 804.203
$A_1^{(8)}(\alpha/\pi)^4$	− 55.667
$A_1^{(10)}(\alpha/\pi)^5$	0.446(15)
$A_2^{(4)}(m_e/m_\mu)(\alpha/\pi)^2$	2.804
$A_2^{(6)}(m_e/m_\mu)(\alpha/\pi)^3$	− 0.092
$A_2^{(8)}(m_e/m_\mu)(\alpha/\pi)^4$	0.026
$A_2^{(10)}(m_e/m_\mu)(\alpha/\pi)^5$	− .0002
$A_2^{(4)}(m_e/m_\tau)(\alpha/\pi)^2$	0.010
$A_2^{(6)}(m_e/m_\tau)(\alpha/\pi)^3$	− 0.0008
a_e(hadronic v.p.)	1.866(11)
a_e(hadronic v.p.,NLO)	− 0.223(1)
a_e(hadronic v.p.,NNLO)	0.028(1)
a_e(hadronic l-l)	0.035(10)
a_e(weak)	0.0297(5)

Table 2. Contributions to a_e.

At 5-loop level there is only the numerical evaluation by the Kinoshita's group

$$A_1^{(10)}(\text{Ref. [29]}) = 6.599(223) \ . \tag{16}$$

Concerning the mass-dependent part $A_2(r)$, $A_2^{(4)}(r)$ is known in analytical form [36], as well as $A_2^{(6)}(r)$ [37–41]; the first terms of the expansion for small r of the 4-loop coefficient $A_2^{(8)}(r)$ are known analytically [42, 43] . $A_2^{(10)}(m_e/m_\mu)$ and $A_2^{(10)}(m_e/m_\tau)$ have been calculated numerically [29]; the first terms of the expansion for small mass rations of $A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)$ and $A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$ are known analytically [43]. The hadronic and weak contribution are

$$a_e(\text{hadronic v.p.}) = \ \ 1.866(11) \times 10^{-12} \quad (\text{see Ref. [44]}) \ , \tag{17}$$
$$a_e(\text{hadronic v.p.,NLO}) = -0.223(1) \times 10^{-12} \quad (\text{see Ref. [45]}) \ , \tag{18}$$
$$a_e(\text{hadronic v.p.,NNLO}) = \ \ 0.028(1) \times 10^{-12} \quad (\text{see Ref. [45]}) \ , \tag{19}$$
$$a_e(\text{hadronic l-l }) = \ \ 0.035(10) \times 10^{-12} \quad (\text{see Ref. [46]}) \ , \tag{20}$$
$$a_e(\text{weak}) = \ \ 0.0297(5) \times 10^{-12} \quad (\text{see Ref. [47]}) \ . \tag{21}$$

Inserting Eq.s(11-16,17-21), the known $A_2^{(j)}$ and the measurement of the fine structure constant [48, 49]

$$\alpha^{-1} = 137.035\ 998\ 996(85)\ (0.62\ \text{ppb}) \ ,$$

into Eq.s(8-10) one finds

$$a_e^{\text{th}} = 1\ 159\ 652\ 182.031(15)(15)(720) \times 10^{-12} \ , \tag{22}$$

where the first error comes from $A_1^{(10)}$, the second one from the hadronic and electroweak corrections, the last one from α, respectively. The values of the single contributions to a_e are listed in table 2. Conversely, assuming the validity of the theory and using the experimental measurement (7) one finds

$$\alpha^{-1}(a_e) = 137.035\ 999\ 1500(18)(18)(330)(0.25\ \text{ppb}) \ ,$$

where the errors come from $A_1^{(10)}$, hadronic and electroweak corrections, and a_e, respectively.

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-1.912245764926445574152647167439830054060873390658725345171329848006038443980651706142760892700003631583755841533147327005637851491285453 91
90280432705027382230434557895704556272930994129669976027778221157847203390641519081665270979708674381150121551479722743221642734319279759586
07405005783738496070187432831402483802519224946074229855893046350614049225266343109442400023563568812806206454940132249775943004292888367617
48899236915180878086989705263578533753776964117024536196013497574494361268486175162606832387186747303831505962741878015305514879400536977798
36946427868432691843117588958115974356695043304834907361342658649953116387811743475385423488364085584441882237217456706871041823307430517443
05573945961171550858961148995261266061246994073118403927472340023464969531735482584817998224097373710773657404645135211230912425281111372153
02154453721014811121159848970884223279879720484201445122828451516585236561786594592600991733031721302865467212345340500349104700728924487200
61604426132544906900043191519823004748818149431103849537829940629675867875385249781946989793132162197975750676701142904897962085050785592...
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Table 3. First 1100 digits of $A_1^{(8)}$.

Methods of calculation of $A_1^{(8)}$

I sketch here the methods of calculation used in the literature, only for comparison. For further information on the technical aspects of my calculation of the 4-loop coefficient $A_1^{(8)}$, see Ref. [51] in these proceedings.

In QED the contributions to g-2 at n loops can be expressed as combinations of n-loop 4-dimensional Feynman integrals, belonging to a variety of Feynman diagrams.

- In Ref. [29], the n-loop 4-dimensional integrals are transformed in $(3n-2)$-dimensional integrals of (huge) rational functions of Feynman parameters. The integrals are computed using the MonteCarlo adaptative routine VEGAS [50]; an enormous amount of computing time is needed to sample adequately the integrands; for more information, see Ref. [20–29].
- My method, used in [11], consists in:
  - (i) reduction of contributions from each Feynman diagram to a small number (334 for $A_1^{(8)}$) of n-loop D-dimensional master integrals by using a suitable algorithm [10, 13];
  - (ii) determination of systems of difference or differential equations satisfied by the master integrals [13];
  - (iii) high precision calculation of these integrals by solving these systems of equations by means of rapidly convergent series expansions [13, 14];

  This method allowed to obtain 1100 digits of $A_1^{(8)}$ (and up to 9800 digits for some selected important integrals). See Ref. [51] for further details.
- In Ref. [30] the contributions of the various diagrams are reduced to combinations of a small number of master integrals. Most of these master integrals are computed with MonteCarlo methods.
- I note the alternative approach recently introduced by S.Volkov in Ref. [52, 53]. It is also based on MonteCarlo integration. It seems promising at 5-loop level.

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