Full NLO corrections for DIS structure functions in the dipole factorization formalism

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Outline

- Introduction: dipole factorization for DIS at low x_{Bj}
- One-loop correction to the $\gamma^*_{T,L} \to q\bar{q}$ light-front wave-functions:

  Direct calculation

  G.B., PRD94 (2016)
- DIS at NLO in the dipole factorization

  Example: F_L case

  Cancellation of the UV divergences between the $q\bar{q}$ and $q\bar{q}g$ terms

  G.B., *in preparation*

Introduction

At low x_{Bj}, many DIS observables can be expressed within dipole factorization, including gluon saturation $\rightarrow$ rich phenomenology.

In particular: Dipole amplitude obtained from fits of HERA data for DIS structure functions in the dipole factorization at LO+LL with rcBK
Albacete *et al.*, PRD80 (2009); EPJC71 (2011)
Kuokkanen *et al.*, NPA875 (2012);
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In the last 10 years, many theoretical (including numerical) progresses towards NLO/NLL accuracy for gluon saturation/CGC.

Obviously, DIS structure functions at NLO in the dipole factorization are required to push the fits beyond LO+LL accuracy.

DIS at NLO: previous results

2 independent calculations had been performed earlier for NLO corrections to photon impact factor and/or DIS cross-section:

1. Balitsky, Chirilli, PRD83 (2011); PRD87 (2013)
   Using covariant perturbation theory. Results provided as
   - Current correlator in position space
   - Impact factor for $k_\perp$ factorization $\rightarrow$ Good for BFKL phenomenology

2. G.B., PRD85 (2012)
   Using light-front perturbation theory. Results provided as
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However, in both papers only the $q\bar{q}g$ contribution was calculated explicitly, whereas NLO corrections to the $q\bar{q}$ contribution were guessed.
Methods used for that:
In Balitsky, Chirilli, PRD83 (2011):
Matching with earlier vacuum results ?

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Methods used for that:
In G.B., PRD85 (2012):
Unitary argument $\rightarrow$ wrong: missed photon finite WF renormalization
$\Rightarrow$ NLO $q\bar{q}$ terms needs to be calculated separately

Dipole factorization for eikonal DIS

Total cross section for (virtual) photon scattering on a gluon shockwave background, in light-front perturbation theory:

$$\begin{aligned}
\sigma_\lambda^\gamma &= 2N_c \widetilde{\sum_{q_0\bar{q}_1 \text{ F. states}}} \frac{2\pi\delta(k_0^+ + k_1^+ - q^+)}{2q^+} \left|\widetilde{\psi}_{\gamma_\lambda\to q_0\bar{q}_1}\right|^2 \mathrm{Re}\left[1 - \mathcal{S}_{01}\right] \\
&+ 2N_c C_F \widetilde{\sum_{q_0\bar{q}_1 g_2 \text{ F. states}}} \frac{2\pi\delta(k_0^+ + k_1^+ + k_2^+ - q^+)}{2q^+} \\
&\qquad \times \left|\widetilde{\psi}_{\gamma_\lambda\to q_0\bar{q}_1 g_2}\right|^2 \mathrm{Re}\left[1 - \mathcal{S}_{012}^{(3)}\right] + \cdots
\end{aligned}$$

$\widetilde{\psi}_{\gamma_\lambda\to f}$: color-stripped light-front wavefunctions of the incoming photon for the Fock-state decomposition in mixed-space $(k^+, \mathbf{x})$

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&\qquad \times \left|\widetilde{\psi}_{\gamma_\lambda \to q_0\bar{q}_1 g_2}\right|^2 \mathrm{Re}\left[1 - \mathcal{S}^{(3)}_{012}\right] + \cdots
\end{aligned}$$

Dipole operator: $$\mathcal{S}_{01} \equiv \frac{1}{N_c} \mathrm{Tr}\left(U_F(\mathbf{x}_0)\, U_F^\dagger(\mathbf{x}_1)\right)$$

"Tripole" operator: $$\mathcal{S}^{(3)}_{012} \equiv \frac{1}{N_c\, C_F} \mathrm{Tr}\left(t^b U_F(\mathbf{x}_0)\, t^a U_F^\dagger(\mathbf{x}_1)\right) U_A(\mathbf{x}_2)_{ba}$$

Dipole factorization for eikonal DIS

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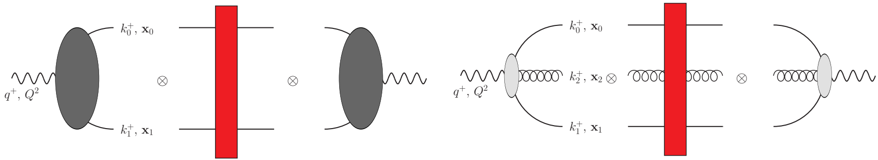
$$+ 2N_c C_F \widetilde{\sum_{q_0\bar{q}_1 g_2 \text{ F. states}}} \frac{2\pi\delta(k_0^+ + k_1^+ + k_2^+ - q^+)}{2q^+}$$
$$\times \left|\widetilde{\psi}_{\gamma_\lambda \to q_0\bar{q}_1 g_2}\right|^2 \mathrm{Re}\left[1 - \mathcal{S}^{(3)}_{012}\right] + \cdots$$



Calculation of the $\gamma_{T,L} \to q\bar{q}$ LF wave-functions at NLO

- Calculation done in Light-front perturbation theory for QCD+QED

- Cut-off k^+_{min} introduced to regulate the small k^+ divergences

  $\Rightarrow$ associated with low-x leading logs to be resummed with BK/JIMWLK evolution at the end

- UV divergences from various tensor transverse integrals, but no UV renormalization at this order.

  $\Rightarrow$ UV divergences (and finite regularization artifacts) have to cancel at cross-section level

  $\Rightarrow$ Use (Conventional) Dimensional Regularization, and pay attention to rational terms in $(D-4)/(D-4)$

Diagrams for the $\gamma_T \to q\bar{q}$ LF wave-function only

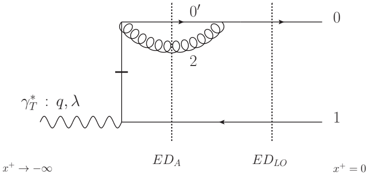


Diagram A’

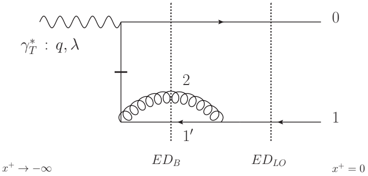


Diagram B’

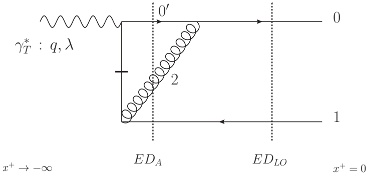


Diagram 1’

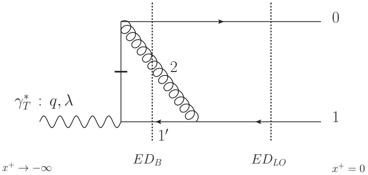
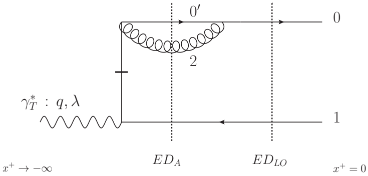


Diagram 2’

Diagrams for the $\gamma_T \to q\bar{q}$ LF wave-function only



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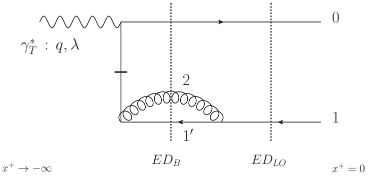


Diagram B'

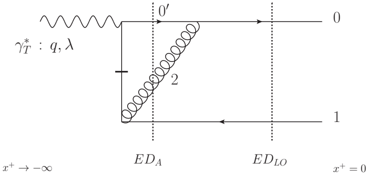


Diagram 1'

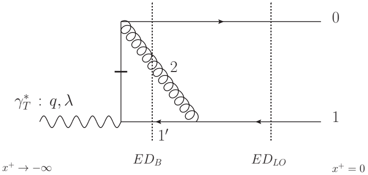


Diagram 2'

All four vanish due to Lorentz symmetry!

Diagrams for γ_T and γ_L LFWFs: 3 steps graphs

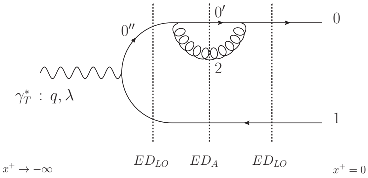
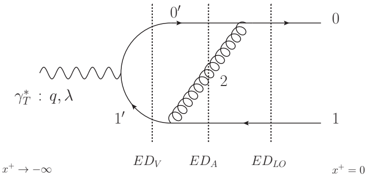


Diagram A



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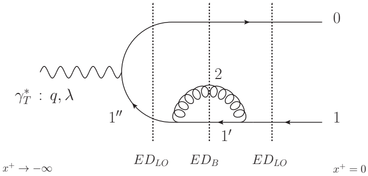


Diagram B

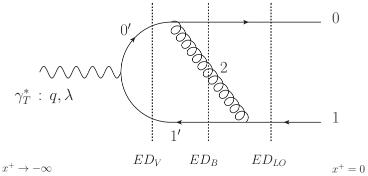


Diagram 2

Diagrams for γ_T and γ_L LFWFs: 2 steps graph

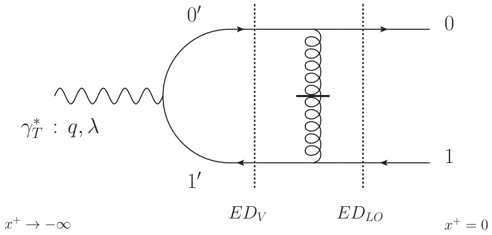


Diagram 3

- In the γ_T case: vanishes due to Lorentz symmetry
- In the γ_L case: non-zero, and cancels the unphysical power-like small k^+ divergence of the other vertex correction graphs.

Results for NLO $\gamma_{T,L} \to q\bar{q}$ LFWFs in momentum space

$$\psi_{\gamma^*_{T,L}\to q_0\bar{q}_1} = \left[1 + \left(\frac{\alpha_s\, C_F}{2\pi}\right)\, \mathcal{V}^{T,L}\right] \psi^{\text{tree}}_{\gamma^*_{T,L}\to q_0\bar{q}_1} + \mathcal{O}(e\,\alpha_s^2)$$

$$\begin{aligned}\mathcal{V}^L = \;& 2\left[\log\left(\frac{k^+_{\text{min}}}{\sqrt{k_0^+ k_1^+}}\right) + \frac{3}{4}\right]\left[\Gamma\left(2-\frac{D}{2}\right)\left(\frac{\overline{Q}^2}{4\pi\,\mu^2}\right)^{\frac{D}{2}-2} - 2\log\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\overline{Q}^2}\right)\right] \\ & + \frac{1}{2}\left[\log\left(\frac{k_0^+}{k_1^+}\right)\right]^2 - \frac{\pi^2}{6} + 3 + O(D-4)\end{aligned}$$

$$\mathcal{V}^T = \mathcal{V}^L + 2\left[\log\left(\frac{k^+_{\text{min}}}{\sqrt{k_0^+ k_1^+}}\right) + \frac{3}{4}\right]\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\mathbf{P}^2}\right)\log\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\overline{Q}^2}\right) + O(D-4)$$

Notations: $\overline{Q}^2 \equiv \frac{k_0^+ k_1^+}{(q^+)^2}\, Q^2$,

and relative transverse momentum: $\mathbf{P} \equiv \mathbf{k}_0 - \frac{k_0^+}{q^+}\,\mathbf{q} = -\mathbf{k}_1 + \frac{k_1^+}{q^+}\,\mathbf{q}$

Remark: results consistent with the ones of Boussarie, Grabovsky, Szymanowski and Wallon, JHEP11(2016)149

Results for NLO $\gamma_{T,L} \to q\bar{q}$ LFWFs in mixed space

$$\widetilde{\psi}_{\gamma^*_{T,L}\to q_0\bar{q}_1} = \left[1 + \left(\frac{\alpha_s\, C_F}{2\pi}\right)\, \widetilde{\mathcal{V}}^{T,L}\right] \widetilde{\psi}^{\text{tree}}_{\gamma^*_{T,L}\to q_0\bar{q}_1} + \mathcal{O}(e\,\alpha_s^2)$$

$$\begin{aligned}
\widetilde{\mathcal{V}}^{T} &= \widetilde{\mathcal{V}}^{L} + O\,(D-4) \\
&= 2\left[\log\left(\frac{k^+_{\text{min}}}{\sqrt{k_0^+ k_1^+}}\right) + \frac{3}{4}\right]\left[\frac{1}{(2-\frac{D}{2})} - \Psi(1) + \log\left(\pi\,\mu^2\, \mathbf{x}_{01}{}^2\right)\right] \\
&\quad + \frac{1}{2}\left[\log\left(\frac{k_0^+}{k_1^+}\right)\right]^2 - \frac{\pi^2}{6} + 3 + O\,(D-4)
\end{aligned}$$

- In mixed space: NLO corrections $\Rightarrow$ rescaling of the LO $\gamma_{T,L} \to q\bar{q}$ LFWFs by a factor independent of the photon polarization and virtuality !
- Leftover logarithmic *UV* and low k^+ divergences to be dealt with at cross-section level.

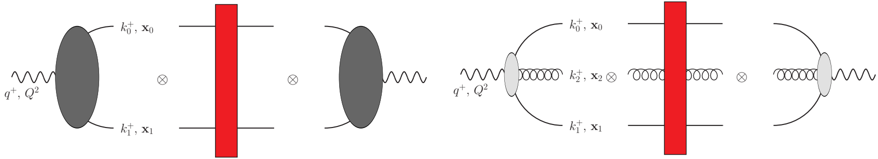
Results for NLO $\gamma_{T,L} \to q\bar{q}$ LFWFs in mixed space

$$\widetilde{\psi}_{\gamma^*_{T,L}\to q_0\bar{q}_1} = \left[1 + \left(\frac{\alpha_s\, C_F}{2\pi}\right)\, \widetilde{\mathcal{V}}^{T,L}\right] \widetilde{\psi}^{\text{tree}}_{\gamma^*_{T,L}\to q_0\bar{q}_1} + \mathcal{O}(e\,\alpha_s^2)$$

$$\begin{aligned}\widetilde{\mathcal{V}}^{T} &= \widetilde{\mathcal{V}}^{L} + O\left(D-4\right)\\ &= 2\left[\log\left(\frac{k^+_{\text{min}}}{\sqrt{k_0^+ k_1^+}}\right) + \frac{3}{4}\right]\left[\frac{1}{\left(2-\frac{D}{2}\right)} - \Psi(1) + \log\left(\pi\,\mu^2\,\mathbf{x}_{01}{}^2\right)\right]\\ &\quad + \frac{1}{2}\left[\log\left(\frac{k_0^+}{k_1^+}\right)\right]^2 - \frac{\pi^2}{6} + \frac{5}{2} + \frac{1}{2} + O\left(D-4\right)\end{aligned}$$

- In mixed space: NLO corrections $\Rightarrow$ rescaling of the LO $\gamma_{T,L} \to q\bar{q}$ LFWFs by a factor independent of the photon polarization and virtuality !
- $(D-4)/(D-4)$ rational term $1/2$: from γ^μ algebra in D dimensions $\Rightarrow$ UV regularization scheme dependent!

From LFWFs to DIS cross-section



$\widetilde{\psi}^{\gamma^*_{T,L}}_{\gamma^*_{T,L}\to \bar{q}}$ now known at NLO accuracy in Dim Reg.

$\Rightarrow$ Need to be combined with the $q\bar{q}g$ contribution in the dipole factorization formula at NLO

$\Rightarrow$ $\widetilde{\psi}_{\gamma^*_{T,L}q\bar{q}g}$ is required also in Dim Reg, in order to cancel UV divergences as well as scheme dependent artifacts.

Only the case of σ^{γ}_{L} will be discussed in the following for simplicity. The case of σ^{γ}_{T} can be dealt with in the same way, but gives much longer expressions.

$q\bar{q}$ contribution to σ_L^γ at NLO in dim. reg.

$$\widetilde{\psi}^{\text{tree}}_{\gamma_L^*\to q_0\bar{q}_1} = -e\,e_f\,\mu^{2-\frac{D}{2}}\,(2\pi)^{1-\frac{D}{2}}\,2Q\,\frac{k_0^+k_1^+}{(q^+)^2}\left(\frac{\overline{Q}}{|\mathbf{x}_{01}|}\right)^{\frac{D}{2}-2}\mathrm{K}_{\frac{D}{2}-2}\Big(|\mathbf{x}_{01}|\,\overline{Q}\Big)\,\overline{u_G}(0)\,\gamma^+v_G(1)$$

$$\sigma_L^\gamma\Big|_{q\bar{q}} = 2N_c \widetilde{\sum_{q_0\bar{q}_1\ \text{F. states}}} \frac{2\pi\delta(k_0^++k_1^+-q^+)}{2q^+}\left|\widetilde{\psi}^{\text{tree}}_{\gamma_L\to q_0\bar{q}_1}\right|^2 \mathrm{Re}\left[1-\mathcal{S}_{01}\right]$$
$$\times\left[1+\left(\frac{\alpha_s\,C_F}{2\pi}\right)\widetilde{\mathcal{V}}^L\right]^2 + O(\alpha_{em}\,\alpha_s^2)$$

$$\sigma_L^\gamma\Big|_{q\bar{q}} = 4N_c\,\alpha_{em}\sum_f e_f^2\int\frac{\mathrm{d}^{D-2}\mathbf{x}_0}{2\pi}\int\frac{\mathrm{d}^{D-2}\mathbf{x}_1}{2\pi}\int_0^{+\infty}dk_0^+\int_0^{+\infty}dk_1^+\,\delta(k_0^++k_1^+-q^+)$$
$$\times\frac{4Q^2}{(q^+)^5}(k_0^+k_1^+)^2\left[\frac{\overline{Q}^2}{(2\pi)^2\mu^2x_{01}^2}\right]^{\frac{D}{2}-2}\left[\mathrm{K}_{\frac{D}{2}-2}\Big(|\mathbf{x}_{01}|\,\overline{Q}\Big)\right]^2$$
$$\times\left[1+\left(\frac{\alpha_s\,C_F}{\pi}\right)\widetilde{\mathcal{V}}^L\right]\mathrm{Re}\left[1-\mathcal{S}_{01}\right]+O(\alpha_{em}\,\alpha_s^2)$$

Tree-level diagrams for $\gamma_L \to q\bar{q}g$ LFWFs

2 diagrams contribute to $\gamma_L \to q\bar{q}g$ (and 4 to $\gamma_T \to q\bar{q}g$):

Diagram (a) Diagram (b)

$\to$ Standard calculation in momentum space using LFPT rules, but to be done in dimensional regularization

Then: Fourier transform to mixed space

$\gamma_L \to q\bar{q}g$ LFWF in mixed space

Result:

$$\widetilde{\psi}^{\text{Tree}}_{\gamma_L^* \to q_0 \bar{q}_1 g_2} = e\, e_f\, g\, \varepsilon^{j*}_{\lambda_2} \frac{2Q}{(q^+)^2}$$

$$\times \Bigg\{ k_1^+ \;\; \overline{u_G}(0)\gamma^+ \Big[(2k_0^+ + k_2^+)\delta^{jm} + \frac{k_2^+}{2} [\gamma^j, \gamma^m] \Big] v_G(1) \;\; \mathcal{I}^m\Big(\mathbf{x}_{0+2;1}, \mathbf{x}_{20}; \overline{Q}^2_{(a)}, \mathcal{C}_{(a)}\Big)$$

$$- k_0^+ \;\; \overline{u_G}(0)\gamma^+ \Big[(2k_1^+ + k_2^+)\delta^{jm} - \frac{k_2^+}{2} [\gamma^j, \gamma^m] \Big] v_G(1) \;\; \mathcal{I}^m\Big(\mathbf{x}_{0;1+2}, \mathbf{x}_{21}; \overline{Q}^2_{(b)}, \mathcal{C}_{(b)}\Big) \Bigg\}$$

with the notations:

$$\overline{Q}^2_{(a)} = \frac{k_1^+(q^+ - k_1^+)}{(q^+)^2}\, Q^2 \quad \text{and} \quad \overline{Q}^2_{(b)} = \frac{k_0^+(q^+ - k_0^+)}{(q^+)^2}\, Q^2$$

$$\mathcal{C}_{(a)} = \frac{q^+\, k_0^+\, k_2^+}{k_1^+ (k_0^+ + k_2^+)^2} \quad \text{and} \quad \mathcal{C}_{(b)} = \frac{q^+\, k_1^+\, k_2^+}{k_0^+ (k_1^+ + k_2^+)^2}$$

And parent dipole vectors defined as:

$$\mathbf{x}_{n+m;p} \;=\; -\mathbf{x}_{p;n+m} \equiv \left(\frac{k_n^+ \mathbf{x}_n + k_m^+ \mathbf{x}_m}{k_n^+ + k_m^+} \right) - \mathbf{x}_p$$

$q\bar{q}g$ contribution to σ_L^γ at NLO in dim. reg.

$$
\begin{aligned}
\sigma_L^\gamma\big|_{q\bar{q}g} =&\ 2N_c C_F \widetilde{\sum_{q_0\bar{q}_1 g_2 \text{ F. states}}} \frac{2\pi\delta(k_0^+ + k_1^+ + k_2^+ - q^+)}{2q^+} \left|\widetilde{\psi}_{\gamma_L\to q_0\bar{q}_1 g_2}\right|^2 \mathrm{Re}\left[1-\mathcal{S}_{012}^{(3)}\right] \\
=&\ 4N_c\,\alpha_{em} \sum_f e_f^2 \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+ \int_{k_{\min}^+}^{+\infty} \frac{dk_2^+}{k_2^+}\, \delta(k_0^+ + k_1^+ + k_2^+ - q^+) \\
&\times 2\alpha_s C_F \int \mathrm{d}^{D-2}\mathbf{x}_0 \int \mathrm{d}^{D-2}\mathbf{x}_1 \int \mathrm{d}^{D-2}\mathbf{x}_2\ \mathrm{Re}\left[1-\mathcal{S}_{012}^{(3)}\right] \frac{4Q^2}{(q^+)^5} \\
&\times\Bigg\{ (k_1^+)^2 \left[2k_0^+(k_0^+ + k_2^+) + \frac{(D-2)}{2}\left(k_2^+\right)^2\right] \left|\mathcal{I}^m\left((a)\right)\right|^2 \\
&\quad + (k_0^+)^2 \left[2k_1^+(k_1^+ + k_2^+) + \frac{(D-2)}{2}\left(k_2^+\right)^2\right] \left|\mathcal{I}^m\left((b)\right)\right|^2 \\
&\quad - k_0^+ k_1^+ \left[2(k_0^+ + k_2^+)k_1^+ + 2k_0^+(k_1^+ + k_2^+) - (D-4)(k_2^+)^2\right] \\
&\qquad \times \mathrm{Re}\left(\mathcal{I}^m\left((a)\right)^* \mathcal{I}^m\left((b)\right)\right)\Bigg\} + O(\alpha_{em}\,\alpha_s^2)
\end{aligned}
$$

UV divergences of the $q\bar{q}g$ contribution to σ_L^{γ}

UV divergences :

- At $\mathbf{x}_2 \to \mathbf{x}_0$ for $|(a)|^2$ contribution
- At $\mathbf{x}_2 \to \mathbf{x}_1$ for $|(b)|^2$ contribution

UV divergences of the $q\bar{q}g$ contribution to σ_L^γ

UV divergences :

- At $\mathbf{x}_2 \to \mathbf{x}_0$ for $|(a)|^2$ contribution
- At $\mathbf{x}_2 \to \mathbf{x}_1$ for $|(b)|^2$ contribution

Traditional method to deal with these UV divergences:

1. Make the subtraction $\left[1 - \mathcal{S}_{012}^{(3)}\right] \to \left[1 - \mathcal{S}_{012}^{(3)}\right] - \left[1 - \mathcal{S}_{01}\right]$ in $\sigma_L^\gamma|_{q\bar{q}g}$
2. Add the corresponding term to $\sigma_L^\gamma|_{q\bar{q}}$

It works for the divergences, but it is far from optimal in the present case!
$\Rightarrow$ Let us present an improvement of that method.

Properties of the Fourier integral

$$\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) \equiv (\mu^2)^{2-\frac{D}{2}} \int \frac{d^{D-2}\mathbf{P}}{(2\pi)^{D-2}} \int \frac{d^{D-2}\mathbf{K}}{(2\pi)^{D-2}} \frac{\mathbf{K}^m\, e^{i\mathbf{K}\cdot\mathbf{r}'}\, e^{i\mathbf{P}\cdot\mathbf{r}}}{\left[\mathbf{P}^2+\overline{Q}^2\right]\left\{\mathbf{K}^2+\mathcal{C}\left[\mathbf{P}^2+\overline{Q}^2\right]\right\}}$$

Introducing Schwinger variables:

$$\begin{aligned}\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) \quad &= \quad \mathbf{r}'^m \left(\mathbf{r}'^2\right)^{1-\frac{D}{2}} \frac{i}{2} (2\pi)^{2-D} \left(\mu^2\right)^{2-\frac{D}{2}} \\ &\times \int_0^{+\infty} d\sigma\, \sigma^{1-\frac{D}{2}}\, e^{-\sigma \overline{Q}^2}\, e^{-\frac{\mathbf{r}^2}{4\sigma}}\, \Gamma\left(\frac{D}{2}-1, \frac{\mathbf{r}'^2\mathcal{C}}{4\sigma}\right)\end{aligned}$$

Properties of the Fourier integral

$$\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) \equiv (\mu^2)^{2-\frac{D}{2}} \int \frac{d^{D-2}\mathbf{P}}{(2\pi)^{D-2}} \int \frac{d^{D-2}\mathbf{K}}{(2\pi)^{D-2}} \frac{\mathbf{K}^m\, e^{i\mathbf{K}\cdot\mathbf{r}'}\, e^{i\mathbf{P}\cdot\mathbf{r}}}{\left[\mathbf{P}^2+\overline{Q}^2\right]\left\{\mathbf{K}^2+\mathcal{C}\left[\mathbf{P}^2+\overline{Q}^2\right]\right\}}$$

Introducing Schwinger variables:

$$\begin{aligned}\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) &= \mathbf{r}'^m \left(\mathbf{r}'^2\right)^{1-\frac{D}{2}} \frac{i}{2}\,(2\pi)^{2-D} \left(\mu^2\right)^{2-\frac{D}{2}} \\ &\times \int_0^{+\infty} d\sigma\; \sigma^{1-\frac{D}{2}}\; e^{-\sigma\overline{Q}^2}\; e^{-\frac{\mathbf{r}^2}{4\sigma}}\; \Gamma\left(\frac{D}{2}-1, \frac{\mathbf{r}'^2\mathcal{C}}{4\sigma}\right)\end{aligned}$$

For $D=4$:

$$\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) = \frac{i}{(2\pi)^2} \left(\frac{\mathbf{r}'^m}{\mathbf{r}'^2}\right) \mathrm{K}_0\left(\overline{Q}\sqrt{\mathbf{r}^2+\mathcal{C}\,\mathbf{r}'^2}\right)$$

Properties of the Fourier integral

$$\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) \equiv (\mu^2)^{2-\frac{D}{2}}\int\frac{d^{D-2}\mathbf{P}}{(2\pi)^{D-2}}\int\frac{d^{D-2}\mathbf{K}}{(2\pi)^{D-2}}\frac{\mathbf{K}^m\, e^{i\mathbf{K}\cdot\mathbf{r}'}\, e^{i\mathbf{P}\cdot\mathbf{r}}}{\left[\mathbf{P}^2+\overline{Q}^2\right]\left\{\mathbf{K}^2+\mathcal{C}\left[\mathbf{P}^2+\overline{Q}^2\right]\right\}}$$

Introducing Schwinger variables:

$$\begin{aligned}\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) &= \mathbf{r}'^m\,\left(\mathbf{r}'^2\right)^{1-\frac{D}{2}}\,\frac{i}{2}\,(2\pi)^{2-D}\,\left(\mu^2\right)^{2-\frac{D}{2}}\\ &\times\int_0^{+\infty} d\sigma\,\sigma^{1-\frac{D}{2}}\,e^{-\sigma\overline{Q}^2}\,e^{-\frac{\mathbf{r}^2}{4\sigma}}\,\Gamma\left(\frac{D}{2}-1,\frac{\mathbf{r}'^2\mathcal{C}}{4\sigma}\right)\end{aligned}$$

For $D=4$:

$$\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right) = \frac{i}{(2\pi)^2}\left(\frac{\mathbf{r}'^m}{\mathbf{r}'^2}\right)\mathrm{K}_0\left(\overline{Q}\sqrt{\mathbf{r}^2+\mathcal{C}\,\mathbf{r}'^2}\right)$$

UV behavior: For $|\mathbf{r}'|\to 0$: $\mathcal{I}^m\left(\mathbf{r},\mathbf{r}';\overline{Q}^2,\mathcal{C}\right)\sim\mathcal{I}^m_{UV}\left(\mathbf{r},\mathbf{r}';\overline{Q}^2\right)$

$$\mathcal{I}^m_{UV}\left(\mathbf{r},\mathbf{r}';\overline{Q}^2\right) \equiv \mathbf{r}'^m\left(\mathbf{r}'^2\right)^{1-\frac{D}{2}}\frac{i}{(2\pi)^2}\Gamma\left(\frac{D}{2}-1\right)\left(\frac{2\overline{Q}}{(2\pi)^2\mu^2|\mathbf{r}|}\right)^{\frac{D}{2}-2}\mathrm{K}_{\frac{D}{2}-2}\left(\overline{Q}\,|\mathbf{r}|\right)$$

Building the UV subtraction terms

Next attempt to deal with the UV divergences : make the subtraction

$$\left\{ \left| \mathcal{I}^m \left((a)\right) \right|^2 \mathrm{Re}\left[1 - \mathcal{S}_{012}^{(3)}\right] - \left| \mathcal{I}_{UV}^m \left(\mathbf{x}_{01}, \mathbf{x}_{20}; \overline{Q}_{(a)}^2\right) \right|^2 \mathrm{Re}\left[1 - \mathcal{S}_{01}\right] \right\}$$

Cancels indeed the UV divergence at $\mathbf{x}_2 \to \mathbf{x}_0$, but produces an IR divergence at $|\mathbf{x}_{20}| \to +\infty$, absent in the original term!

Building the UV subtraction terms

Final idea: subtract the IR divergence from the UV subtraction term, as

$$\Bigg\{ \Big|\mathcal{I}^{m}\left((a)\right)\Big|^{2} \mathrm{Re}\Big[1-\mathcal{S}^{(3)}_{012}\Big] - \Bigg[\Big|\mathcal{I}^{m}_{UV}\Big(\mathbf{x}_{01},\mathbf{x}_{20};\overline{Q}^{2}_{(a)}\Big)\Big|^{2}$$

$$-\mathrm{Re}\left(\mathcal{I}^{m*}_{UV}\Big(\mathbf{x}_{01},\mathbf{x}_{20};\overline{Q}^{2}_{(a)}\Big)\,\mathcal{I}^{m}_{UV}\Big(\mathbf{x}_{01},\mathbf{x}_{21};\overline{Q}^{2}_{(a)}\Big)\right)\Bigg] \mathrm{Re}\Big[1-\mathcal{S}_{01}\Big]\Bigg\}$$

This difference leads to a UV and IR finite integral in $\mathbf{x}_2$.

Building the UV subtraction terms

Final idea: subtract the IR divergence from the UV subtraction term, as

$$\Big\{ \Big|\mathcal{I}^m\left((a)\right)\Big|^2 \mathrm{Re}\Big[1-\mathcal{S}^{(3)}_{012}\Big] - \Big[\Big|\mathcal{I}^m_{UV}\Big(\mathbf{x}_{01},\mathbf{x}_{20};\overline{Q}^2_{(a)}\Big)\Big|^2$$
$$-\mathrm{Re}\Big(\mathcal{I}^{m*}_{UV}\Big(\mathbf{x}_{01},\mathbf{x}_{20};\overline{Q}^2_{(a)}\Big)\,\mathcal{I}^m_{UV}\Big(\mathbf{x}_{01},\mathbf{x}_{21};\overline{Q}^2_{(a)}\Big)\Big)\Big]\,\mathrm{Re}\Big[1-\mathcal{S}_{01}\Big]\Big\}$$

This difference leads to a UV and IR finite integral in $\mathbf{x}_2$.
$\Rightarrow$ The $D \to 4$ limit is now safe to take:

$$\to \Big\{ \frac{1}{(2\pi)^4}\,\frac{1}{\mathbf{x}_{20}^2}\,\Big[\mathrm{K}_0\Big(Q\,X_{012}\Big)\Big]^2\,\mathrm{Re}\Big[1-\mathcal{S}^{(3)}_{012}\Big]$$
$$-\frac{1}{(2\pi)^4}\Big[\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2}\cdot\Big(\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2}-\frac{\mathbf{x}_{21}}{\mathbf{x}_{21}^2}\Big)\Big]\,\Big[\mathrm{K}_0\Big(\overline{Q}^2_{(a)}\,|\mathbf{x}_{01}|\Big)\Big]^2\,\mathrm{Re}\Big[1-\mathcal{S}_{01}\Big]\Big\}$$

$$Q^2X_{012}^2 \equiv \frac{Q^2}{(q^+)^2}\Big[k_0^+k_1^+x_{01}^2+k_0^+k_2^+x_{02}^2+k_1^+k_2^+x_{12}^2\Big] = \frac{q\bar{q}g\ \text{form. time}}{\gamma^*\ \text{life time}}$$

UV-subtracted $q\bar{q}g$ contribution to σ_L^γ

Subtracting both UV divergences this way:

$$\sigma_L^\gamma|_{q\bar{q}g} - \sigma_L^\gamma|_{UV,|(a)|^2} - \sigma_L^\gamma|_{UV,|(b)|^2}$$
$$= 4N_c\,\alpha_{em}\sum_f e_f^2 \int \frac{\mathrm{d}^2\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^2\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+\, \frac{4Q^2}{(q^+)^5}\, \frac{\alpha_s C_F}{\pi}$$
$$\times \int_{k^+_{\min}}^{+\infty} \frac{dk_2^+}{k_2^+}\, \delta(k_0^+ + k_1^+ + k_2^+ - q^+) \int \frac{\mathrm{d}^2\mathbf{x}_2}{2\pi} \Big[q \text{ term} + \bar{q} \text{ term} + \text{leftover} \Big]$$

UV-subtracted $q\bar{q}g$ contribution to σ_L^γ

Subtracting both UV divergences this way:

$$\sigma_L^\gamma\big|_{q\bar{q}g} - \sigma_L^\gamma\big|_{UV,|(a)|^2} - \sigma_L^\gamma\big|_{UV,|(b)|^2}$$
$$= 4N_c\,\alpha_{em} \sum_f e_f^2 \int \frac{\mathrm{d}^2\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^2\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+ \, \frac{4Q^2}{(q^+)^5}\,\frac{\alpha_s C_F}{\pi}$$
$$\times \int_{k^+_{\text{min}}}^{+\infty} \frac{dk_2^+}{k_2^+}\,\delta(k_0^+ + k_1^+ + k_2^+ - q^+) \int \frac{\mathrm{d}^2\mathbf{x}_2}{2\pi} \Big[{\color{red} q \text{ term}} + \bar{q} \text{ term} + \text{leftover} \Big]$$

With:

$$q \text{ term} = (k_1^+)^2 \left[2k_0^+(k_0^+ + k_2^+) + (k_2^+)^2\right] \left[\frac{\mathbf{x}_{20}}{x_{20}^2}\cdot\left(\frac{\mathbf{x}_{20}}{x_{20}^2} - \frac{\mathbf{x}_{21}}{x_{21}^2}\right)\right]$$
$$\times \left\{ \left[\mathrm{K}_0(QX_{012})\right]^2 \mathrm{Re}\left[1 - \mathcal{S}_{012}\right] - \Big(\mathbf{x}_2 \to \mathbf{x}_0\Big) \right\}$$

UV-subtracted $q\bar{q}g$ contribution to σ_L^γ

Subtracting both UV divergences this way:

$$\sigma_L^\gamma|_{q\bar{q}g} - \sigma_L^\gamma|_{UV,|(a)|^2} - \sigma_L^\gamma|_{UV,|(b)|^2}$$
$$= 4N_c\,\alpha_{em}\sum_f e_f^2 \int \frac{\mathrm{d}^2\mathbf{x}_0}{2\pi}\int \frac{\mathrm{d}^2\mathbf{x}_1}{2\pi}\int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+ \,\frac{4Q^2}{(q^+)^5}\,\frac{\alpha_s C_F}{\pi}$$
$$\times \int_{k_{\min}^+}^{+\infty} \frac{dk_2^+}{k_2^+}\,\delta(k_0^+ + k_1^+ + k_2^+ - q^+)\int \frac{\mathrm{d}^2\mathbf{x}_2}{2\pi}\Big[q \text{ term} + \bar{q} \text{ term} + \text{leftover}\Big]$$

With:

$$\bar{q}\text{ term} = (k_0^+)^2\,\Big[2k_1^+(k_1^+ + k_2^+) + (k_2^+)^2\Big]\,\left[\frac{\mathbf{x}_{21}}{x_{21}^2}\cdot\left(\frac{\mathbf{x}_{21}}{x_{21}^2} - \frac{\mathbf{x}_{20}}{x_{20}^2}\right)\right]$$
$$\times\left\{\Big[\mathrm{K}_0(QX_{012})\Big]^2\,\mathrm{Re}\Big[1 - \mathcal{S}_{012}\Big] - \Big(\mathbf{x}_2 \to \mathbf{x}_1\Big)\right\}$$

UV-subtracted $q\bar{q}g$ contribution to σ_L^γ

Subtracting both UV divergences this way:

$$\sigma_L^\gamma\big|_{q\bar{q}g} - \sigma_L^\gamma\big|_{UV,|(a)|^2} - \sigma_L^\gamma\big|_{UV,|(b)|^2}$$
$$= 4N_c\, \alpha_{em} \sum_f e_f^2 \int \frac{\mathrm{d}^2\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^2\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+ \, \frac{4Q^2}{(q^+)^5} \, \frac{\alpha_s C_F}{\pi}$$
$$\times \int_{k_{\min}^+}^{+\infty} \frac{dk_2^+}{k_2^+} \, \delta(k_0^+ + k_1^+ + k_2^+ - q^+) \int \frac{\mathrm{d}^2\mathbf{x}_2}{2\pi} \Big[q \text{ term} + \bar{q} \text{ term} + \text{leftover} \Big]$$

With:

$$\text{leftover} \;=\; (k_2^+)^2 \Big[(k_0^+)^2 + (k_1^+)^2\Big] \left(\frac{\mathbf{x}_{20}}{x_{20}^2} \cdot \frac{\mathbf{x}_{21}}{x_{21}^2} \right) \Big[\mathrm{K}_0(QX_{012})\Big]^2 \,\mathrm{Re}\Big[1 - \mathcal{S}_{012}\Big]$$

Combining the UV terms with the $q\bar{q}$ contribution to σ_L^γ

In dim. reg., the UV subtraction terms can be written as

$$\begin{aligned}
&\sigma_L^\gamma\big|_{UV,|(a)|^2} + \sigma_L^\gamma\big|_{UV,|(b)|^2} \\
&= 4N_c\,\alpha_{em} \sum_f e_f^2 \int \frac{\mathrm{d}^{D-2}\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^{D-2}\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+\, \delta(k_0^+ + k_1^+ - q^+) \\
&\times \frac{4Q^2}{(q^+)^5}\,(k_0^+ k_1^+)^2 \left[\frac{\overline{Q}^2}{(2\pi)^2\mu^2 x_{01}^2}\right]^{\frac{D}{2}-2} \left[\mathrm{K}_{\frac{D}{2}-2}\Big(|\mathbf{x}_{01}|\,\overline{Q}\Big)\right]^2 \\
&\times \left(\frac{\alpha_s\, C_F}{\pi}\right) \left[\widetilde{\mathcal{V}}^L_{UV,|(a)|^2} + \widetilde{\mathcal{V}}^L_{UV,|(b)|^2}\right] \,\mathrm{Re}\,[1 - \mathcal{S}_{01}]
\end{aligned}$$

Combining the UV terms with the $q\bar{q}$ contribution to σ_L^γ

In dim. reg., the UV subtraction terms can be written as

$$
\begin{aligned}
&\sigma_L^\gamma\big|_{UV,|(a)|^2} + \sigma_L^\gamma\big|_{UV,|(b)|^2} \\
&= 4N_c\,\alpha_{em} \sum_f e_f^2 \int \frac{\mathrm{d}^{D-2}\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^{D-2}\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+\, \delta(k_0^+ + k_1^+ - q^+) \\
&\times \frac{4Q^2}{(q^+)^5}\,(k_0^+ k_1^+)^2 \left[\frac{\overline{Q}^2}{(2\pi)^2\mu^2 x_{01}^2}\right]^{\frac{D}{2}-2} \left[\mathrm{K}_{\frac{D}{2}-2}\Big(|\mathbf{x}_{01}|\,\overline{Q}\Big)\right]^2 \\
&\times \left(\frac{\alpha_s\,C_F}{\pi}\right) \left[\widetilde{\mathcal{V}}^L_{UV,|(a)|^2} + \widetilde{\mathcal{V}}^L_{UV,|(b)|^2}\right] \mathrm{Re}\left[1 - \mathcal{S}_{01}\right]
\end{aligned}
$$

With:

$$
\widetilde{\mathcal{V}}^L_{UV,|(a)|^2} = \Gamma\left(\frac{D}{2}-2\right) \left(\pi\mu^2\mathbf{x}_{01}^2\right)^{2-\frac{D}{2}} \left[\log\left(\frac{k_{\min}^+}{k_0^+}\right) + \frac{3}{4} - \frac{(D-4)}{8}\right]
$$

Combining the UV terms with the $q\bar{q}$ contribution to σ_L^γ

In dim. reg., the UV subtraction terms can be written as

$$\sigma_L^\gamma\big|_{UV,|(a)|^2} + \sigma_L^\gamma\big|_{UV,|(b)|^2}$$
$$= 4N_c\,\alpha_{em} \sum_f e_f^2 \int \frac{\mathrm{d}^{D-2}\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^{D-2}\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+\,\delta(k_0^+ + k_1^+ - q^+)$$
$$\times \frac{4Q^2}{(q^+)^5}\,(k_0^+ k_1^+)^2 \left[\frac{\overline{Q}^2}{(2\pi)^2\mu^2 x_{01}^2}\right]^{\frac{D}{2}-2} \left[\mathrm{K}_{\frac{D}{2}-2}\Big(|\mathbf{x}_{01}|\,\overline{Q}\Big)\right]^2$$
$$\times \left(\frac{\alpha_s\,C_F}{\pi}\right) \left[\widetilde{\mathcal{V}}^L_{UV,|(a)|^2} + \widetilde{\mathcal{V}}^L_{UV,|(b)|^2}\right]\ \mathrm{Re}\left[1 - \mathcal{S}_{01}\right]$$

With:

$$\widetilde{\mathcal{V}}^L_{UV,|(b)|^2} = \Gamma\left(\frac{D}{2}-2\right)\,\left(\pi\mu^2\mathbf{x}_{01}^2\right)^{2-\frac{D}{2}}\ \left[\log\left(\frac{k_{\min}^+}{k_1^+}\right) + \frac{3}{4} - \frac{(D-4)}{8}\right]$$

Combining the UV terms with the $q\bar{q}$ contribution to σ_L^γ

Expanding around $D = 4$:

$$\widetilde{\mathcal{V}}^L_{UV,|(a)|^2} + \widetilde{\mathcal{V}}^L_{UV,|(b)|^2} = -2\left[\frac{1}{\left(2-\frac{D}{2}\right)} - \Psi(1) + \log\left(\pi\, \mathbf{x}_{01}{}^2\, \mu^2\right)\right] \times \left[\log\left(\frac{k^+_{\text{min}}}{\sqrt{k^+_0 k^+_1}}\right) + \frac{3}{4}\right] - \frac{1}{2} + O(D-4)$$

Combining the UV terms with the $q\bar{q}$ contribution to σ_L^γ

Expanding around $D = 4$:

$$\widetilde{\mathcal{V}}^L_{UV,|(a)|^2} + \widetilde{\mathcal{V}}^L_{UV,|(b)|^2} = -2\left[\frac{1}{(2-\frac{D}{2})} - \Psi(1) + \log\left(\pi\,\mathbf{x}_{01}{}^2\,\mu^2\right)\right] \times \left[\log\left(\frac{k^+_{\min}}{\sqrt{k_0^+ k_1^+}}\right) + \frac{3}{4}\right] - \frac{1}{2} + O(D-4)$$

But in the $q\bar{q}$ contribution to σ_L^γ:

$$\widetilde{\mathcal{V}}^L = 2\left[\frac{1}{(2-\frac{D}{2})} - \Psi(1) + \log\left(\pi\,\mathbf{x}_{01}{}^2\,\mu^2\right)\right]\left[\log\left(\frac{k^+_{\min}}{\sqrt{k_0^+ k_1^+}}\right) + \frac{3}{4}\right] + \frac{1}{2}\left[\log\left(\frac{k_0^+}{k_1^+}\right)\right]^2 - \frac{\pi^2}{6} + \frac{5}{2} + \frac{1}{2} + O(D-4)$$

$\Rightarrow$ Cancelation of:

- the UV divergence
- the $k^+_{\min}$ dependence
- the $\pm 1/2$ rational term : strong hint of UV regularization scheme independence

Combining the UV terms with the $q\bar{q}$ contribution to σ_L^γ

Final result for the dipole-like terms:

$$\sigma_L^\gamma\big|_{q\bar{q}} + \sigma_L^\gamma\big|_{UV,|(a)|^2} + \sigma_L^\gamma\big|_{UV,|(b)|^2}$$
$$= 4N_c\,\alpha_{em} \sum_f e_f^2 \int \frac{\mathrm{d}^2\mathbf{x}_0}{2\pi} \int \frac{\mathrm{d}^2\mathbf{x}_1}{2\pi} \int_0^{+\infty} dk_0^+ \int_0^{+\infty} dk_1^+\, \delta(k_0^+ + k_1^+ - q^+)\, \frac{4Q^2}{(q^+)^5}$$
$$\times (k_0^+ k_1^+)^2 \left[\mathrm{K}_0\left(|\mathbf{x}_{01}|\,\overline{Q}\right)\right]^2 \left[1 + \left(\frac{\alpha_s\, C_F}{\pi}\right)\, \widetilde{\mathcal{V}}^L_{\mathrm{reg.}}\right]\ \mathrm{Re}\left[1 - \mathcal{S}_{01}\right]$$

With:

$$\begin{aligned}
\widetilde{\mathcal{V}}^L_{\mathrm{reg.}} &\equiv \widetilde{\mathcal{V}}^L + \widetilde{\mathcal{V}}^L_{UV,|(a)|^2} + \widetilde{\mathcal{V}}^L_{UV,|(b)|^2} \\
&= \frac{1}{2}\left[\log\left(\frac{k_0^+}{k_1^+}\right)\right]^2 - \frac{\pi^2}{6} + \frac{5}{2}
\end{aligned}$$

Final step: BK/JIMWLK resummation

1. Assign $k^+_{\min}$ to the scale set by the target: $k^+_{\min} = \dfrac{Q_0^2}{2x_0\, P^-} = \dfrac{x_{Bj}\, Q_0^2}{x_0\, Q^2}\, q^+$
2. Choose a factorization scale $k_f^+ \lesssim k_0^+, k_1^+$, corresponding to a range for the high-energy evolution $Y_f^+ \equiv \log\left(\dfrac{k_f^+}{k^+_{\min}}\right) = \log\left(\dfrac{x_0\, Q^2\, k_f^+}{x_{Bj}\, Q_0^2\, q^+}\right)$
3. In the LO term in the observable, make the replacement
$$\langle \mathcal{S}_{01}\rangle_0 \quad = \quad \langle \mathcal{S}_{01}\rangle_{Y_f^+} - \int_0^{Y_f^+} dY^+ \left(\partial_{Y^+}\langle \mathcal{S}_{01}\rangle_{Y^+}\right)$$
with both terms calculated with the same evolution equation
4. Combine the second term with the NLO correction to cancel its $k^+_{\min}$ dependence and the associated large logs.

$\Rightarrow$ Works straightforwardly in the case of

- the naive LL BK equation
- the kinematically improved BK equation as implemented in G.B., PRD89 (2014)

  Should also work with the other implementation (Iancu *et al.*, PLB744 (2015)), but might require a bit more work.

Conclusion

1. Direct calculation of $\gamma_{T,L} \to q\bar{q}$ LFWFs at one-gluon-loop order, both in momentum and in mixed space
2. Full NLO corrections to F_L and F_T from the combination of the $q\bar{q}$ and $q\bar{q}g$ contributions, with improved method to cancel UV divergences

Phenomenology outlook : All ingredients soon available for fits to HERA data at NLO+LL accuracy, and hopefully NLO+NLL accuracy, in the dipole factorization, including gluon saturation.

Theory outlook :

- Application of the NLO $\gamma_{T,L} \to q\bar{q}(g)$ LFWFs to calculate other DIS observables at NLO?
- Extension to the case of massive quarks?
- Comparison to other calculations of photon impact factor at NLO ?
  Bartels et al.(2001-2004); Balitsky, Chirilli (2011-2013)