Large-n_f Contributions to the Four-Loop Splitting Functions in QCD

Nucl. Phys. B915 (2017) 335-362, arXiv:1610.07477

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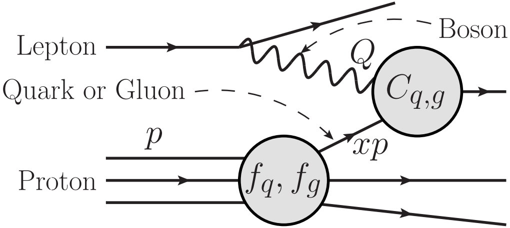
25th International Workshop on Deep Inelastic Scattering and Related Topics
6th April

INTRODUCTION

Deep Inelastic Scattering: a lepton scatters from a proton



Boson: γ, H, Z^0 (Neutral Current) or $W^{\pm}$ (Charged Current)

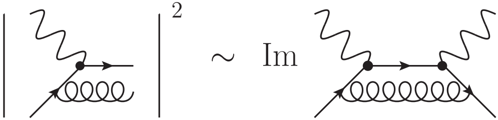
Cross-section: $\sigma \sim \sum_a F_a(x, Q^2) = \sum_a \left[C_{a,q} \otimes f_q + C_{a,g} \otimes f_g\right]$

F_a – *"Structure Function"*
$C_{a,j}$ – *"Coefficient Function"*
$\otimes$ – *"Mellin Convolution"*
f_j – *"Parton Distribution Function"*

INCLUSIVE DIS

To compute $C_{a,q}$, $C_{a,g}$, we use the **optical theorem**.

Compute **forward scattering amplitudes**:



Use Dim. Reg. ($D = 4 - 2\varepsilon$). Divergences appear as poles in ε.

Renormalization of a_s removes UV poles. "Collinear" poles remain,

$$\tilde{C}_{a,j} = \tilde{C}_{a,j}\left(x, a_s, {}^{Q^2}\!/_{\mu_r^2}, \varepsilon\right).$$

Collinear Factorization

We need to deal with these collinear poles: renormalize the PDF.

$$F_a = \tilde{C}_{a,j} \otimes \tilde{f}_j = C_{a,j} \otimes Z_{ji}\left(x, a_\mathrm{s}, \mu_\mathrm{r}^2/\mu_\mathrm{f}^2, \varepsilon\right) \otimes \tilde{f}_i = C_{a,j} \otimes f_j.$$

$C_{a,j}$ is finite. Z_{ji} contains only poles in ε.

Factorization at scale μ_f^2, implies f_j has scale dependence:

$$\frac{d}{d\ln\mu_\mathrm{f}^2} f_j = \frac{d}{d\ln\mu_\mathrm{f}^2} Z_{ji} \otimes \tilde{f}_i = \underbrace{\frac{d}{d\ln\mu_\mathrm{f}^2} Z_{jk} \otimes Z_{ki}^{-1}}_{P_{ji}} \otimes f_i.$$

- this is the DGLAP evolution equation
- P_{ji} are the **Splitting Functions**

Splitting Functions

Know Z_{ji} from calculation of $\tilde{C}_{a,j}$, so we can extract P_{ji}.

PDFs are universal to all hadron interactions; Splitting Functions are also.

DGLAP evolution: system of $2n_f$+1 coupled equations.

By defining the distributions

$$q_s = \sum_{i=1}^{n_f}(f_i + \bar{f}_i), \qquad q^{\pm}_{ns,ij} = (f_i \pm \bar{f}_i) - (f_j \pm \bar{f}_j), \qquad q_V = \sum_{i=1}^{n_f}(f_i - \bar{f}_i),$$

we have the evolution equations, (setting $\mu_{\mathrm{f}}^2 = Q^2$):

$$\frac{d}{d\ln Q^2}\begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix},$$

$$\frac{d}{d\ln Q^2} q^{\pm}_{ns,ij} = P^{\pm}_{ns} q^{\pm}_{ns,ij},$$

$$\frac{d}{d\ln Q^2} q_V = P_V q_V.$$

IN MELLIN SPACE...

Take the **Mellin transform**,

$$F_a(N, Q^2) = \int_0^1 \mathrm{d}x \; x^{N-1} \hat{F}_a(x, Q^2).$$

Now all convolutions ($\otimes$) are simple products.

We compute **Mellin moments** of $\tilde{C}_{a,j}$, $N = 2, 4, 6, \ldots$, **not** an analytic expression for arbitrary N (which gives x-space expression via IMT).

- Mellin moments of Splitting Functions P_{ij}.

Q: Given some fixed number of Mellin moments of P_{ij}, can we derive an analytic expression for general N?

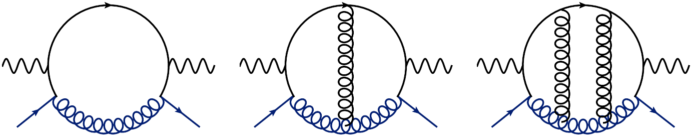
- **this is the goal of this project.**

SOFTWARE

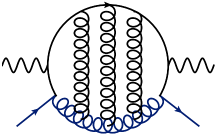
qgraf: generate diagrams (1.2 million!) [Nogueira '93]

TFORM: physics, project Mellin moments. [Kuipers,Ueda,Vermaseren,Vollinga '13]
Produces 2-point tensor integrals, which must be reduced to masters.

To 3 loops, we can use **MINCER**. [Larin,Tkachov,Vermaseren '91]



At 4 loops, **FORCER**. **State of the art**. [Ruijl,Ueda,Vermaseren]



WHAT DO P_{ij} "LOOK LIKE"?

To a_s^3, written in terms of **harmonic sums**,

$$S_m(N) = \sum_{i=1}^{N} \frac{1}{i^m}, \quad S_{-m}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^m},$$

$$S_{[-]m_1,m_2,\ldots,m_l}(N) = \sum_{i=1}^{N} \frac{[(-1)^i]}{i^m} S_{m_2,\ldots,m_l}(i),$$

and **denominators**, $D_i^p = \left(\frac{1}{N+i}\right)^p$.

Define

- **harmonic weight**: $\sum_{i=1}^{l} |m_i|$,
- **overall weight**: harmonic weight + p.

$$P_{ij} = \sum_{n=0}^{\infty} a_s^{n+1} P_{ij}^{(n)}.$$

To a_s^3, $P_{ij}^{(n)}$ written as terms of overall weight up to $(2n+1)$.

2-LOOP EXAMPLE

$$
\begin{aligned}
P_{qg}^{(1)}\Big|_{C_A n_f} = &- \Big[8(2D_2 - 2D_1 + D_0)S_{-2} + 8(2D_2 - 2D_1 + D_0)S_{1,1} \\
&\quad + 16(D_2^2 - D_1^2)S_1 + 8(4D_2^3 + 2D_1^3 + D_0^3)\Big]_{OW3} \\
&- \Big[\frac{4}{3}(44D_2^2 + 12D_1^2 + 3D_0^2)\Big]_{OW2} \\
&+ \Big[\frac{4}{9}(20D_{-1} - 146D_2 + 153D_1 - 18D_0)\Big]_{OW1}
\end{aligned}
$$

- At overall weight i, up to factor $(1/3)^{(3-i)}$, coefficients are **integers**.

Possible basis:

$$
\begin{aligned}
\{S_{-2}, S_{1,1}, S_2\} &\cdot \{D_0, D_1, D_2\} \\
\{S_1\} &\cdot \{D_0^{1,2}, D_1^{1,2}, D_2^{1,2}\} \\
\{1\} &\cdot \{D_0^{1,2,3}, D_1^{1,2,3}, D_2^{1,2,3}, D_{-1}\}
\end{aligned}
$$

Assuming $(1/3)^{(3-i)}$, need to determine **25 integer coefficients**.

2-LOOP EXAMPLE

Compute Mellin moments:

$$
\begin{aligned}
P_{qg}^{(1)}\Big|_{C_A n_f}(2) &= \quad 35/(3^3) \\
P_{qg}^{(1)}\Big|_{C_A n_f}(4) &= -\,16387/(2^3\,3^2\,5^3) \\
P_{qg}^{(1)}\Big|_{C_A n_f}(6) &= -\,867311/(2^3\,3^3\,5^1\,7^3) \\
P_{qg}^{(1)}\Big|_{C_A n_f}(8) &= -\,100911011/(2^6\,3^6\,5^3\,7^1) \\
&\vdots
\end{aligned}
$$

With moments $N = 2, 4, \ldots, 50$ we can solve for 25 basis coefficients.

Can we do better?

- Use that the coefficients are **integer**.
- It is a **system of Diophantine equations**.

LATTICE BASIS REDUCTION

Lenstra-Lenstra-Lovász Lattice Basis Reduction: [Lenstra,Lenstra,Lovász '82]

- find a short lattice basis in polynomial time
- can be used to find integer solutions to equations

`axb`:

- part of `calc` [`www.numbertheory.org`]
- LLL-based solver for **systems of Diophantine equations**

See also, `Mathematica`, `Maple`, `fpLLL`, ... , many more.

To solve:

$$\begin{pmatrix} b_1(2), \ \ldots\ , \ b_{25}(2) \\ \vdots \\ b_1(m), \ \ldots\ , \ b_{25}(m) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{25} \end{pmatrix} = \begin{pmatrix} P^{(1)}_{qg}\big|_{C_A n_f}(2) \\ \vdots \\ P^{(1)}_{qg}\big|_{C_A n_f}(m) \end{pmatrix}$$

$b_i(N), c_i$: basis elements, coefficients. $c_i \in \mathbb{Z}$.

2-Loop Example: Reconstruction

Determines $P_{qg}^{(1)}\big|_{C_A n_f}$ (**25** integer coefficients) with **just 9** Mellin moments.

- solution, $(c_1, \ldots, c_{25}) =$

$$(\underbrace{2, 6, 72, 8, 88, 584, 4, 24, -612, -80}_{SW0}, \underbrace{0, 0, 4, 0, -4, 0}_{SW1}, \underbrace{2, 4, -4, 2, 4, -4, 0, 0, 0}_{SW2})$$

What if the basis were incorrect? For e.g., leave out D_{-1}:

- solve with $N = 2, \ldots, 18$,

$$(-43, 423, 123, 1492, -102, 1332, 4, 24, -612, -15, 437, 102, -2399, 80, 1700, -146, 180, -26, -1065, 670, 579, -919, 490, 605)$$

- solve with $N = 2, \ldots, 20$,

$$(-178, 4391, -25712, 412, -10348, -6476, 4, 24, -612, -572, 25401, -2178, -5642, -3526, -20152, -3302, -3161, 6474, -4011, 5092, 3775, -3283, -4617, 11029)$$

Claim: these solutions are **"obviously bad"**.

FOUR-LOOP SPLITTING FUNCTIONS

Large-n_f contributions:

- subset of diagrams, much easier for **FORCER** to compute
- smaller reconstruction bases (terms of lower overall weight)

Singlet Splitting Functions, colour factors at n_f^3,

$$P_{qq}^{(3)}\{C_F n_f^3\} \quad P_{qg}^{(3)}\{C_A n_f^3, C_F n_f^3\}$$
$$P_{gq}^{(3)}\{C_F n_f^3\} \quad P_{gg}^{(3)}\{C_A n_f^3, C_F n_f^3\}$$

Guess bases using lower order information. Number of coefficients:

$$P_{qq}^{(3)}\{69\} \quad P_{qg}^{(3)}\{125, 101\}$$
$$P_{gq}^{(3)}\{38\} \quad P_{gg}^{(3)}\{34, 54\}$$

Moments used for reconstruction, (check), $N = 2, \ldots$

$$P_{qq}^{(3)}\{30(44)\} \quad P_{qg}^{(3)}\{\times(\times), 40(54)\}$$
$$P_{gq}^{(3)}\{18(28)\} \quad P_{gg}^{(3)}\{20(28), 26(32)\}$$

HARDEST SINGLET CASE

$P_{qg}^{(3)}\big|_{C_A n_f^3}$: Basis with **125** unknown integer coefficients.

$N = 2, \ldots, 46$ insufficient to determine a good solution.

Moment calculations become very computationally demanding.
Hardest diagram computed at $N = 46$,

- $\sim$ **2** weeks wall-time [16 cores, 192GB RAM]
- $\sim$ **13TB** peak disk usage by **TFORM**

$\rightarrow$ **no more moments!**

We need to somehow make the basis smaller.
Use additional constraints:

- large-x limit: no irrational constants other than ζ_i -1 coeff.
- $\#S_{1,2} = -\#S_{2,1}$ -7 coeff.

117 unknowns. Solution with $N = 2, \ldots, 44$, $N = 46$ checks.

Non-singlet Splitting Functions

n_f^3 terms of $P_{ns}^{(3),\pm}$ are already known. [Gracey '94]

We determine the n_f^2 terms of $P_{ns}^{(3),+}$ (even N) and $P_{ns}^{(3),-}$ (odd N).
Colour factors to determine at n_f^2:

- $C_F^2 n_f^2$
- $C_A C_F n_f^2$ – diagrams are **very hard** to compute!

Method: **decompose in two ways**,

$$P_{ns}^{(3),\pm}\{n_f^2\{C_F^2, C_A C_F\}\} = n_f^2\left(2C_F^2 A + C_F(C_A - 2C_F)B^\pm\right)$$
$$= n_f^2\left(2C_F^2(A - B^\pm) + C_F C_A B^\pm\right).$$

A should be **common to both $P_{ns}^\pm$**; use **both odd and even N**. Large n_c.

Compute (easier) $C_F^2 n_f^2$ diagrams to higher N to determine $(A - B^\pm)$.

From these, determine B^+ and B^- and hence $P_{ns}^{(3),+}$ and $P_{ns}^{(3),-}$. ✓

VERIFICATION

Check against existing results:

- Linear combinations of n_f^3 terms of $P_{qq}^{(3)}$, $P_{gq}^{(3)}$, and $P_{gq}^{(3)}$, $P_{gg}^{(3)}$ [Gracey '96,'98]
- Large-N prediction of $P_{qq}^{(3)}$, $P_{gg}^{(3)}$ [Dokshitzer,Marchesini,Salam '06]
- Small-x Double Log Resummations [Davies,Kom,Vogt]
- Large-x Double Log Resummations [Soar,Moch,Vermaseren,Vogt '10]
- Cusp Anomalous Dimension at a_s^4: Given by A in large-N limit [Henn,Smirnov,Smirnov,Steinhauser '16] [Grozin '16]

Everything is in agreement.

OUTLOOK

Using **FORCER**, we determine moments of **4-loop Splitting Functions**.

We have used these moments to derive **analytic all-N expressions** for

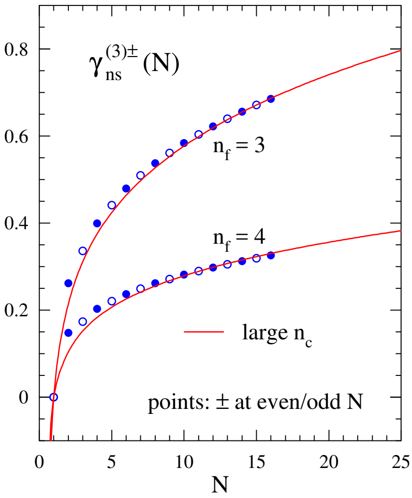
- n_f^3 terms of $P_{qq}^{(3)}, P_{qg}^{(3)}, P_{gq}^{(3)}, P_{gg}^{(3)}$
- n_f^2 terms of $P_{ns}^{(3),\pm}$ and $P_{V}^{(3)}$.

Using the OPE, [Moch, RUVV, to appear]

- n_f^1 and n_f^0 terms of A.
  $\Rightarrow$ large-n_c $P_{ns}^{(3),\pm}$ complete.

To come:

- Numerical approx. to (rest of) $P_{ns}^{(3),\pm}$, using 8 moments.
- Suitable for N^{3}LO analysis, at least at $x \gtrsim 10^{-2}$.



Backup: Non-singlet Splitting Functions

To determine the basis coefficients,

A:

- basis with **54** unknown coefficients
- reconstruct with $N = 2, 3, \dots, 17$. $N = 18, 19, \dots, 22$ check.

$(A - B^+)$ and $(A - B^-)$ are harder:

- bases with **139** unknown coefficients
- **additional constraints** reduce to **115**, like $P_{qg}^{(3)}$ approach
- reconstruct $(A - B^+)$ with $N = 2, \dots, 40$, $N = 42$ checks
- reconstruct $(A - B^-)$ with $N = 3, \dots, 37$, $N = 39$ checks.

Backup: Simple LLL Example

Suppose $r = 1.61803$ is a (rounded) solution to a quadratic equation with integer coefficients.

Form the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 10000r^2 \\ 0 & 1 & 0 & 10000r \\ 0 & 0 & 1 & 10000 \end{pmatrix}.$$

A new basis consists of vectors of the form $(a, b, c, 10000(ar^2 + br + c))$.

Apply `LatticeReduce[]` (`Mathematica`):

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ -7 & 41 & -48 & 120 \\ -11 & 66 & -78 & -100 \end{pmatrix}.$$

$$-x^2 + x + 1 = 0 \implies x = 1.61803 \text{ (6 s.f.)} \checkmark$$

$$-7x^2 + 41x - 48 = 0 \implies x = 1.61732 \text{ (6 s.f.)}$$

$$-11x^2 + 66x - 78 = 0 \implies x = 1.61830 \text{ (6 s.f.)}.$$