Large-$n_f$ Contributions to the Four-Loop Splitting Functions in QCD


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INTRODUCTION

Deep Inelastic Scattering: a lepton scatters from a proton

Boson: $\gamma, H, Z^0$ (Neutral Current) or $W^\pm$ (Charged Current)

Cross-section: $\sigma \sim \sum_a F_a(x, Q^2) = \sum_a \left[ C_{a,q} \otimes f_q + C_{a,g} \otimes f_g \right]$

$F_a$ – “Structure Function”
$C_{a,j}$ – “Coefficient Function”
$\otimes$ – “Mellin Convolution”
$f_j$ – “Parton Distribution Function”
Inclusive DIS

To compute $C_{a,q}, C_{a,g}$, we use the optical theorem.

Compute forward scattering amplitudes:

$\left| \begin{array}{c} \gamma \\ \end{array} \right| 2 \sim \text{Im} \left| \begin{array}{c} \gamma \\ \end{array} \right|

Use Dim. Reg. ($D = 4 - 2\varepsilon$). Divergences appear as poles in $\varepsilon$.

Renormalization of $a_s$ removes UV poles. “Collinear” poles remain, 

$$\tilde{C}_{a,j} = \tilde{C}_{a,j} (x, a_s, Q^2/\mu^2, \varepsilon).$$
We need to deal with these collinear poles: renormalize the PDF.

\[ F_a = \tilde{C}_{a,j} \otimes \tilde{f}_j = C_{a,j} \otimes Z_{ji} (x, a_s, \mu^2_r/\mu^2, \varepsilon) \otimes \tilde{f}_i = C_{a,j} \otimes f_j. \]

\( C_{a,j} \) is finite. \( Z_{ji} \) contains only poles in \( \varepsilon \).

Factorization at scale \( \mu^2_f \), implies \( f_j \) has scale dependence:

\[ \frac{d}{d \ln \mu^2_f} f_j = \frac{d}{d \ln \mu^2_f} Z_{ji} \otimes \tilde{f}_i = \frac{d}{d \ln \mu^2_f} Z_{jk} \otimes Z^{-1}_{ki} \otimes f_i. \]

- this is the DGLAP evolution equation
- \( P_{ji} \) are the Splitting Functions
**Splitting Functions**

Know $Z_{ji}$ from calculation of $\tilde{C}_{a,j}$, so we can extract $P_{ji}$.

PDFs are universal to all hadron interactions; Splitting Functions are also.

DGLAP evolution: system of $2n_f+1$ coupled equations.

By defining the distributions

$$q_s = \sum_{i=1}^{n_f} (f_i + \bar{f}_i), \quad q_{ns,ij}^\pm = (f_i \pm \bar{f}_i) - (f_j \pm \bar{f}_j), \quad q_v = \sum_{i=1}^{n_f} (f_i - \bar{f}_i),$$

we have the evolution equations, (setting $\mu_f^2 = Q^2$):

$$\frac{d}{d \ln Q^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix},$$

$$\frac{d}{d \ln Q^2} q_{ns,ij}^\pm = P_{ns} q_{ns,ij}^\pm,$$

$$\frac{d}{d \ln Q^2} q_v = P_v q_v.$$
**IN MELLIN SPACE...**

Take the Mellin transform,

\[ F_a(N, Q^2) = \int_0^1 dx \ x^{N-1} \hat{F}_a(x, Q^2). \]

Now all convolutions (\( \otimes \)) are simple products.

We compute **Mellin moments** of \( \tilde{C}_{a,j} \), \( N = 2, 4, 6, \ldots \), **not** an analytic expression for arbitrary \( N \) (which gives \( x \)-space expression via IMT).

- Mellin moments of Splitting Functions \( P_{ij} \).

**Q:** Given some fixed number of Mellin moments of \( P_{ij} \), can we derive an analytic expression for general \( N \)?

- **this is the goal of this project.**
SOFTWARE

**qgraf**: generate diagrams (1.2 million!) [Nogueira ‘93]

**TFORM**: physics, project Mellin moments. [Kuipers, Ueda, Vermaseren, Vollinga ‘13]

Produces 2-point tensor integrals, which must be reduced to masters.

To 3 loops, we can use **MINCER**. [Larin, Tkachov, Vermaseren ‘91]

![Diagram](attachment:image.png)

At 4 loops, **FORCER**. State of the art. [Ruijl, Ueda, Vermaseren]
WHAT DO $P_{ij}$ “LOOK LIKE”?
To $a^3_s$, written in terms of harmonic sums,

$$S_m(N) = \sum_{i=1}^{N} \frac{1}{i^m}, \quad S_{-m}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^m},$$

$$S_{[-]m_1,m_2,...,m_l}(N) = \sum_{i=1}^{N} \frac{[(-1)^i]}{i^m} S_{m_2,...,m_l}(i),$$

and denominators, $D^p_i = \left( \frac{1}{N+i} \right)^p$.

Define

- harmonic weight: $\sum_{i=1}^{l} |m_i|$,  
- overall weight: harmonic weight + $p$.

$$P_{ij} = \sum_{n=0}^{\infty} a_s^{n+1} P_{ij}^{(n)}.$$

To $a^3_s$, $P_{ij}^{(n)}$ written as terms of overall weight up to $(2n + 1)$. 
2-LOOP EXAMPLE

\[ P_{gq}^{(1)} \bigg|_{\text{Canf}} = - \left[ 8(2D_2 - 2D_1 + D_0)S_{-2} + 8(2D_2 - 2D_1 + D_0)S_{1,1} \\
+ 16(D_2^2 - D_1^2)S_1 + 8(4D_2^3 + 2D_1^3 + D_0^3) \right]_{\text{OW3}} \\
- \left[ \frac{4}{3} (44D_2^2 + 12D_1^2 + 3D_0^2) \right]_{\text{OW2}} \\
+ \left[ \frac{4}{9} (20D_{-1} - 146D_2 + 153D_1 - 18D_0) \right]_{\text{OW1}} \]

- At overall weight \( i \), up to factor \((1/3)^{(3-i)}\), coefficients are integers.

Possible basis:

\{S_{-2}, S_{1,1}, S_2\} \cdot \{D_0, D_1, D_2\}

\{S_1\} \cdot \{D_0^{1,2}, D_1^{1,2}, D_2^{1,2}\}

\{1\} \cdot \{D_0^{1,2,3}, D_1^{1,2,3}, D_2^{1,2,3}, D_{-1}\}

Assuming \((1/3)^{(3-i)}\), need to determine 25 integer coefficients.
2-LOOP EXAMPLE

Compute Mellin moments:

\[
P_{qg}^{(1)}\bigg|_{C_{Anf}} (2) = \frac{35}{3^3}
\]
\[
P_{qg}^{(1)}\bigg|_{C_{Anf}} (4) = -\frac{16387}{2^3 \cdot 3^2 \cdot 5^3}
\]
\[
P_{qg}^{(1)}\bigg|_{C_{Anf}} (6) = -\frac{867311}{2^3 \cdot 3^3 \cdot 5 \cdot 7^3}
\]
\[
P_{qg}^{(1)}\bigg|_{C_{Anf}} (8) = -\frac{100911011}{2^6 \cdot 3^6 \cdot 5^3 \cdot 7}
\]

\[\vdots\]

With moments \( N = 2, 4, \ldots, 50 \) we can solve for 25 basis coefficients.

Can we do better?

- Use that the coefficients are **integer**.
- It is a **system of Diophantine equations**.
LATTICE BASIS REDUCTION

Lenstra-Lenstra-Lovász Lattice Basis Reduction: [Lenstra, Lenstra, Lovász ‘82]

- find a short lattice basis in polynomial time
- can be used to find integer solutions to equations

$axb$:

- part of calc [www.numbertheory.org]
- LLL-based solver for systems of Diophantine equations

See also, Mathematica, Maple, fpLLL, ... , many more.

To solve:

$$
\begin{pmatrix}
  b_1(2), \ldots, b_{25}(2) \\
  \vdots \\
  b_1(m), \ldots, b_{25}(m)
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  \vdots \\
  c_{25}
\end{pmatrix}
=
\begin{pmatrix}
  P_{qg}^{(1)} | C_{Anf}(2) \\
  \vdots \\
  P_{qg}^{(1)} | C_{Anf}(m)
\end{pmatrix}
$$

$b_i(N), c_i$: basis elements, coefficients. $c_i \in \mathbb{Z}$. 
2-LOOP EXAMPLE: RECONSTRUCTION

Determines $P_{qg}^{(1)}|_{c_{Anf}}$ (25 integer coefficients) with just 9 Mellin moments.

- solution, $(c_1, \ldots, c_{25}) =$

$(2, 6, 72, 8, 88, 584, 4, 24, -612, -80, 0, 0, 4, 0, -4, 0, 2, 4, -4, 2, 4, -4, 0, 0, 0)$

What if the basis were incorrect? For e.g., leave out $D_{-1}$:

- solve with $N = 2, \ldots, 18,$

  $( -43, 423, 123, 1492, -102, 1332, 4, 24, -612, -15, 437, 102, -2399, 80, 1700, -146, 180, -26, -1065, 670, 579, -919, 490, 605)$

- solve with $N = 2, \ldots, 20,$

  $( -178, 4391, -25712, 412, -10348, -6476, 4, 24, -612, -572, 25401, -2178, -5642, -3526, -20152, -3302, -3161, 6474, -4011, 5092, 3775, -3283, -4617, 11029)$

Claim: these solutions are “obviously bad”. 
FOUR-LOOP SPLITTING FUNCTIONS

Large-$n_f$ contributions:
- subset of diagrams, much easier for FORCER to compute
- smaller reconstruction bases (terms of lower overall weight)

Singlet Splitting Functions, colour factors at $n_f^3$,

\[
\begin{align*}
P^{(3)}_{qq}\{C_Fn_f^3\} & \quad P^{(3)}_{qg}\{C_An_f^3, C_Fn_f^3\} \\
P^{(3)}_{gq}\{C_Fn_f^3\} & \quad P^{(3)}_{gg}\{C_An_f^3, C_Fn_f^3\}
\end{align*}
\]

Guess bases using lower order information. Number of coefficients:

\[
\begin{align*}
P^{(3)}_{qq}\{69\} & \quad P^{(3)}_{qg}\{125, 101\} \\
P^{(3)}_{gq}\{38\} & \quad P^{(3)}_{gg}\{34, 54\}
\end{align*}
\]

Moments used for reconstruction, (check), $N = 2, \ldots$

\[
\begin{align*}
P^{(3)}_{qq}\{30(44)\} & \quad P^{(3)}_{qg}\{\times(\times), 40(54)\} \\
P^{(3)}_{gq}\{18(28)\} & \quad P^{(3)}_{gg}\{20(28), 26(32)\}
\end{align*}
\]
Hardest Singlet Case

\[ P_{qg}^{(3)} |_{c_{An_f}}^3 : \text{Basis with 125 unknown integer coefficients.} \]

\( N = 2, \ldots, 46 \) insufficient to determine a good solution.

Moment calculations become very computationally demanding. Hardest diagram computed at \( N = 46, \)

- \( \sim 2 \) weeks wall-time
- \( \sim 13 \text{TB} \) peak disk usage by TFORM

→ no more moments!

We need to somehow make the basis smaller. Use additional constraints:

- large-\( x \) limit: no irrational constants other than \( \zeta_i \)
- \( \#S_{1,2} = -\#S_{2,1} \)

117 unknowns. Solution with \( N = 2, \ldots, 44, N = 46 \) checks.
**Non-singlet Splitting Functions**

$n_f^3$ terms of $P_{ns}^{(3)},\pm$ are already known. [Gracey '94]

We determine the $n_f^2$ terms of $P_{ns}^{(3)},+\ (\text{even } N)$ and $P_{ns}^{(3)},-\ (\text{odd } N)$.

Colour factors to determine at $n_f^2$:

- $C_F n_f^2$
- $C_A C_F n_f^2$ – diagrams are *very hard* to compute!

Method: *decompose in two ways,*

\[
P_{ns}^{(3)},\pm \{ n_f^2 \{ C_F^2, C_A C_F \} \} = n_f^2 \left( 2C_F^2 A + C_F (C_A - 2C_F)B^\pm \right) = n_f^2 \left( 2C_F^2 (A - B^\pm) + C_F C_A B^\pm \right).
\]

$A$ should be common to both $P_{ns}^\pm$; use both odd and even $N$. Large $n_c$.

Compute (easier) $C_F^2 n_f^2$ diagrams to higher $N$ to determine $(A - B^\pm)$.

From these, determine $B^+$ and $B^-$ and hence $P_{ns}^{(3)},+$ and $P_{ns}^{(3)},-$. √
VERIFICATION

Check against existing results:

- Linear combinations of $n_f^3$ terms of $P^{(3)}_{qq}$, $P^{(3)}_{gq}$, and $P^{(3)}_{gq}$, $P^{(3)}_{gg}$
  
  [Gracey ‘96,’98]

- Large-$N$ prediction of $P^{(3)}_{qq}$, $P^{(3)}_{gg}$
  
  [Dokshitzer, Marchesini, Salam ‘06]

- Small-$x$ Double Log Resummations
  
  [Davies, Kom, Vogt]

- Large-$x$ Double Log Resummations
  
  [Soar, Moch, Vermaseren, Vogt ‘10]

- Cusp Anomalous Dimension at $a_s^4$: Given by $A$ in large-$N$ limit
  
  [Henn, Smirnov, Smirnov, Steinhauser ‘16] [Grozin ‘16]

Everything is in agreement.
OUTLOOK

Using **FORCER**, we determine moments of **4-loop Splitting Functions**.

We have used these moments to derive **analytic all-N expressions** for

- $n_f^3$ terms of $P_{qq}^{(3)}, P_{qg}^{(3)}, P_{gq}^{(3)}, P_{gg}^{(3)}$
- $n_f^2$ terms of $P_{ns}^{(3)}, \pm$ and $P_{V}^{(3)}$.

Using the OPE, [Moch, RUVV, to appear]

- $n_f^1$ and $n_f^0$ terms of $A$.
  \Rightarrow large-$n_c$ $P_{ns}^{(3)}, \pm$ complete.

To come:

- Numerical approx. to (rest of) $P_{ns}^{(3)}, \pm$, using 8 moments.
- Suitable for $N^3\text{LO}$ analysis, at least at $x \gtrsim 10^{-2}$.
BACKUP: NON-SINGLET SPLITTING FUNCTIONS

To determine the basis coefficients,

A:
  ▶ basis with 54 unknown coefficients
  ▶ reconstruct with $N = 2, 3, \ldots, 17$. $N = 18, 19, \ldots, 22$ check.

$(A - B^+)$ and $(A - B^-)$ are harder:
  ▶ bases with 139 unknown coefficients
  ▶ additional constraints reduce to 115, like $P_{qg}^{(3)}$ approach
  ▶ reconstruct $(A - B^+)$ with $N = 2, \ldots, 40$, $N = 42$ checks
  ▶ reconstruct $(A - B^-)$ with $N = 3, \ldots, 37$, $N = 39$ checks.
BACKUP: SIMPLE LLL EXAMPLE

Suppose \( r = 1.61803 \) is a (rounded) solution to a quadratic equation with integer coefficients.

Form the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 10000r^2 \\
0 & 1 & 0 & 10000r \\
0 & 0 & 1 & 10000
\end{pmatrix}.
\]

A new basis consists of vectors of the form \((a, b, c, 10000(ar^2 + br + c))\).

Apply \text{LatticeReduce}[] (Mathematica):
\[
\begin{pmatrix}
-1 & 1 & 1 & 0 \\
-7 & 41 & -48 & 120 \\
-11 & 66 & -78 & -100
\end{pmatrix}.
\]

\[-x^2 + x + 1 = 0 \implies x = 1.61803 \text{ (6 s.f.)} \checkmark\]
\[-7x^2 + 41x - 48 = 0 \implies x = 1.61732 \text{ (6 s.f.)}\]
\[-11x^2 + 66x - 78 = 0 \implies x = 1.61830 \text{ (6 s.f.)}\.]