Multidimensional phase space methods for mass measurements and decay topology determination

Matthew D. Klimek
University of Texas at Austin

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with:
C. Kilic, P. Agrawal, B. Altunkaynak, J.-H. Yu

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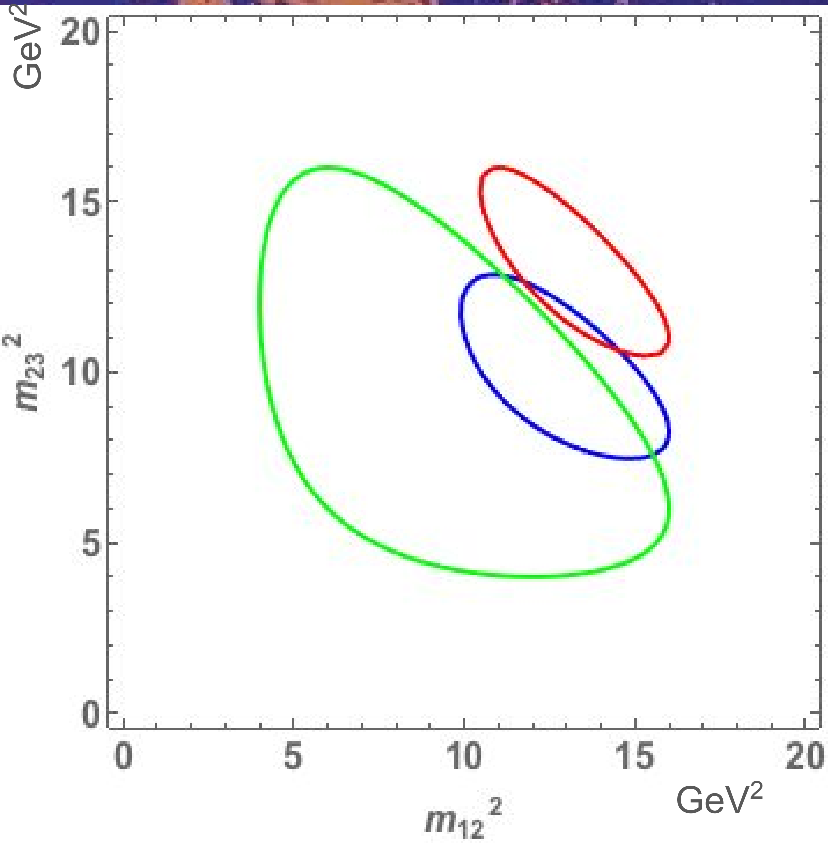
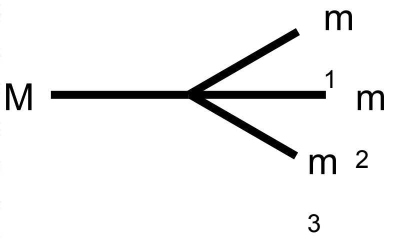
Motivation

- We will demonstrate a technique that allows for the determination of masses and decay topologies in cascade decays by utilizing correlations in the full differential phase space distribution with small samples.
  - No (conclusive) BSM discoveries at LHC yet
  - Discoveries are likely to come with limited numbers of events
  - Essential to be able to make maximal use of small samples
  - This technique will outperform traditional edge/endpoints mass determination

Introduction: the Dalitz plot (3-body decay)

- **Mass spectrum is encoded by the shape of the boundary**
- Matrix element affects distribution of events, but *not* the location of the boundary





$(M^2, m_1^2, m_2^2, m_3^2)$ =

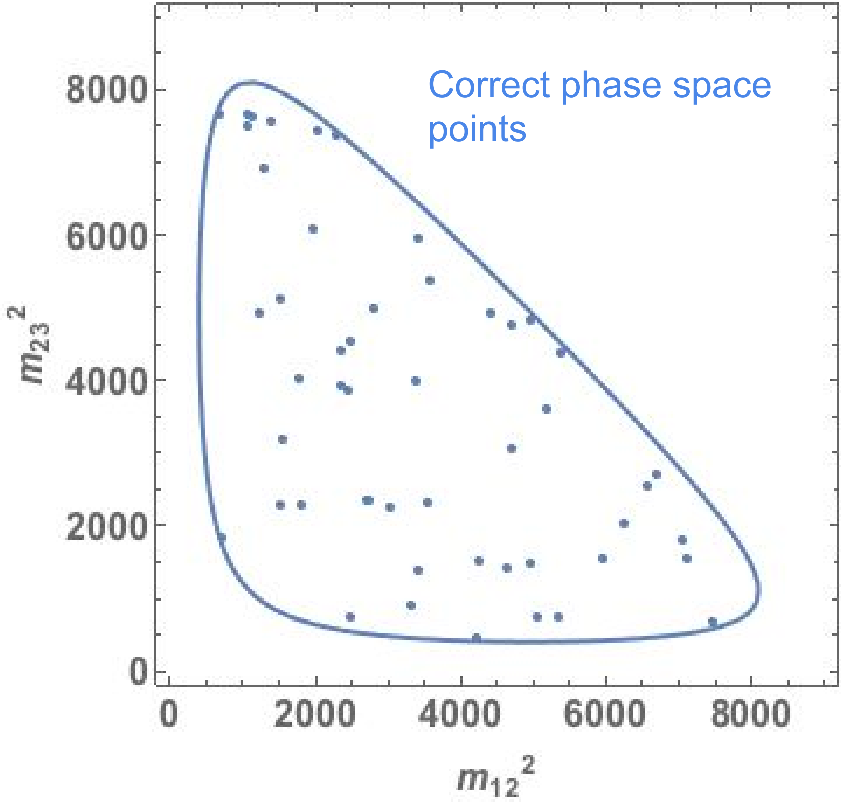
(25, 5, 1, 1) GeV²

(25, 3, 2, 1) GeV²

(25, 1, 1, 1) GeV²

The full phase space advantage

What if you don't know the mass of one of the particles?



The full phase space advantage

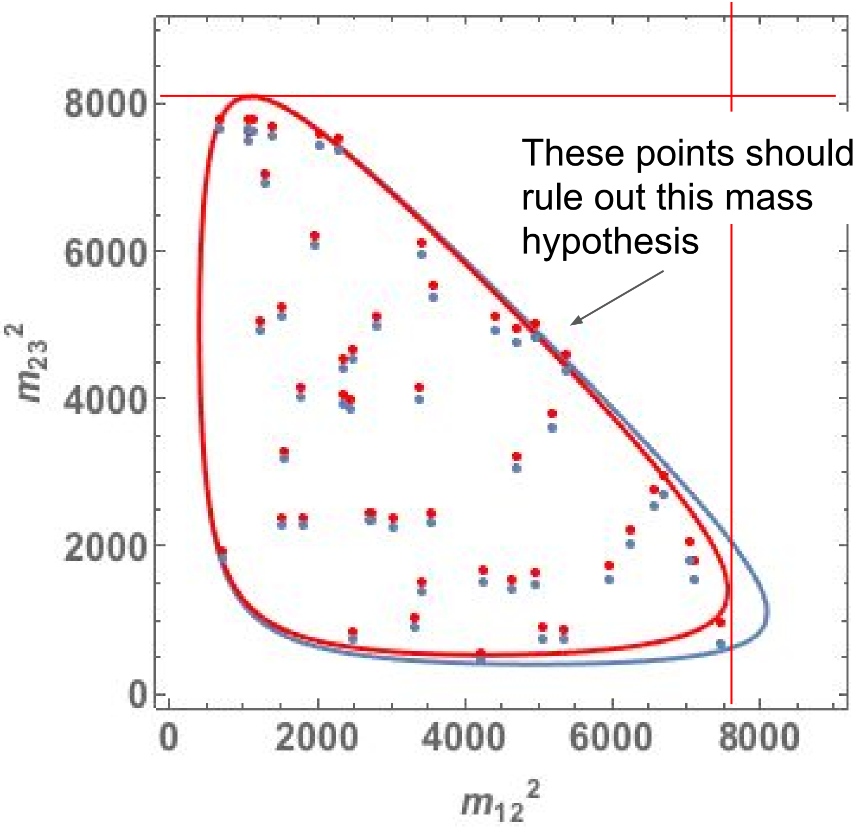
What if you don't know the mass of one of the particles?

Invariant masses and boundary calculated with 30% error in mass of particle 3.

The data then seem to show events outside the boundary (unphysical).

This mass hypothesis should be ruled out!

But you would never notice this if you only looked at endpoints (i.e. projections of phase space).



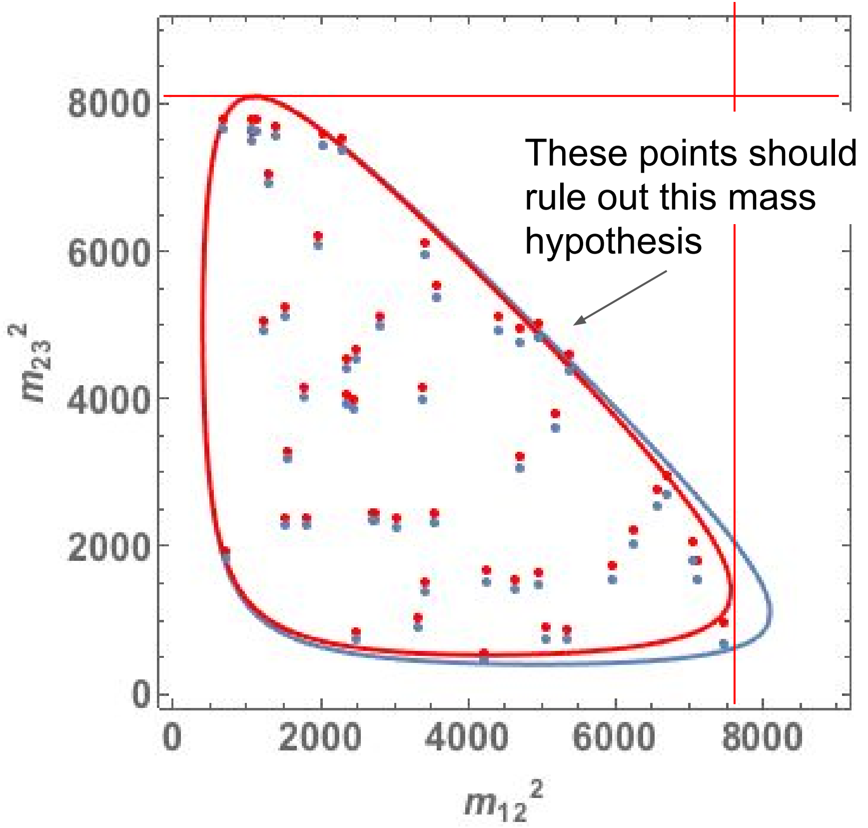
The full phase space advantage

This was a trivial example, for illustration only.

But, at higher multiplicity, there are non-trivial cases with invisible particles where the phase space point can be computed *given a mass hypothesis.*

<u>Moral</u>: If you are measuring unknown masses using a set of edges/endpoints, you are missing important correlations.

A better thing to do is to fit a boundary to the data in the full dimensionality of phase space.

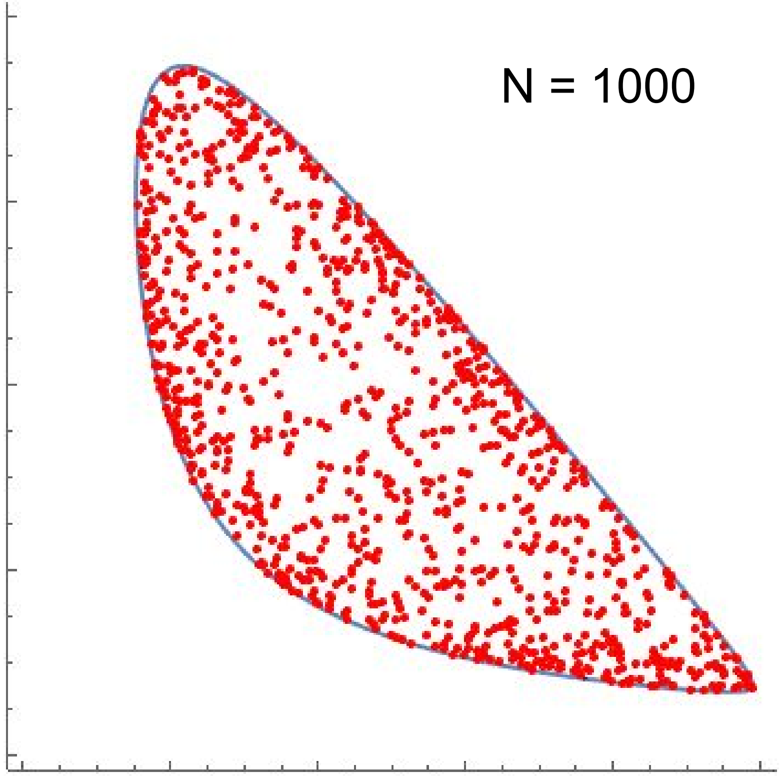


4-body boundary enhancement

- Surprise: For 4-body decays, *the phase space weight is greater near the boundary*! *Yang & Byers 1964*

- The *most useful* events are the *most plentiful.*

- Fitting a boundary to the data, even with a limited number of events, should reconstruct the masses efficiently.

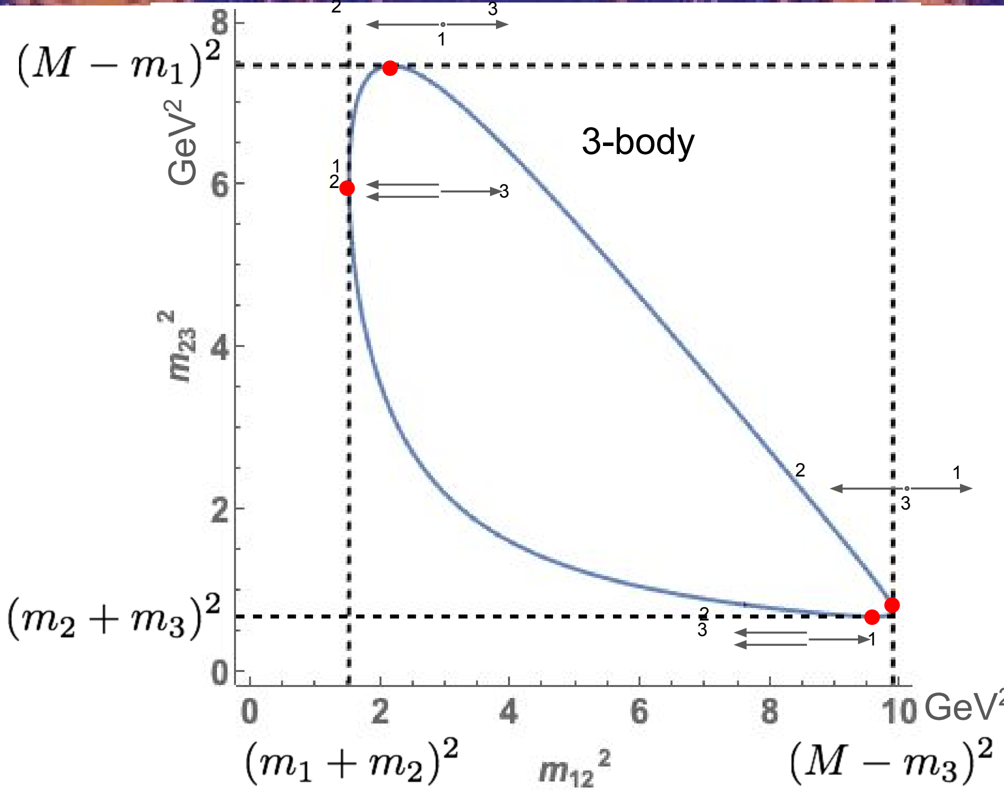


schematic only

4-body boundary enhancement

Note:

- Points on the boundary correspond to momentum configurations which span a *lower dimensional* subspace. (Collinear in the 3-body case.)



Invariant formalism for $1 \to 4$ decay

In order to express the enhancement near the boundary, we need a sort of "radial" coordinate that measures our distance from the boundary.

Recall from Dalitz example: The boundary represents configurations that span a lower dimensional space (a plane for 4-body decays).

Let V be the 4×4 matrix formed by the 4 outgoing 4-momenta $V = |p_1, p_2, p_3, p_4|$

Let $\Delta_4 = (\det V)^2 \geq 0$. $\Delta_4 = 0$ if the vectors in V are not linearly independent.

This happens on the boundary, so this serves as a good radial coordinate.

$\Delta_4 = 0$ on the boundary, $\Delta_4 > 0$ inside

Why is the boundary enhanced?

Of course, any statement about density is a *coordinate-dependent* statement.

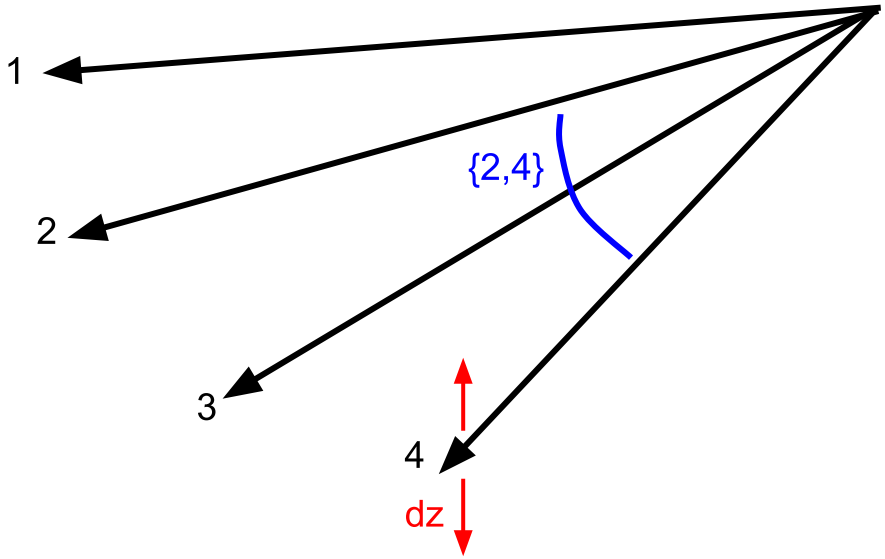
This feature is present in a convenient, Lorentz invariant coordinate basis: $\{i,j\} \equiv p_i \cdot p_j$ (or equivalently, m_{ij}^2).

- But why?

Why is the boundary enhanced?

The dot product variables *{i,j}* are functions of the angles between particles.

These angles are *stationary* with respect to changes in the z-component of one vector when they are all coplanar, i.e. *when det V = 0.*



The coordinate change from dp_n to d{i,j} becomes singular.

Thus, a factor *1/(det V)* must appear in the phase space density in terms of *{i,j}*, which is equivalent to a factor $\Delta_4^{-1/2}$.

$\Delta_4 = (\det V)^2$

Why is the boundary enhanced?

The final result is:

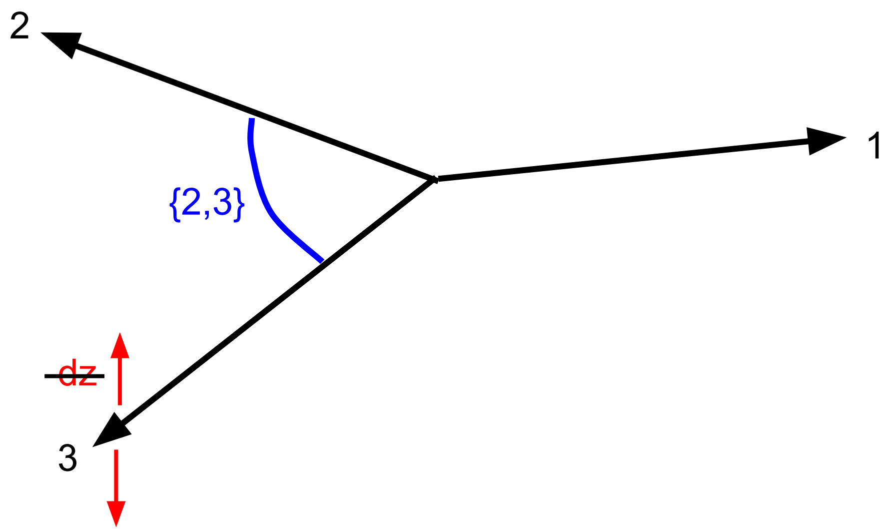
$$dPS_4 = \frac{16\pi^2}{M^2}\Delta_4^{-1/2}\prod_{i<j} d\{i,j\}\delta(\sum\{i,j\} - (M^2 - \sum m_i^2))$$

Yang & Byers 1964

Why doesn't this happen for *n* = *3*?

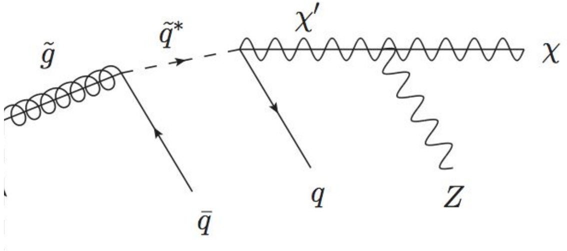
In the parent particle rest frame, all 3-body decays are planar.

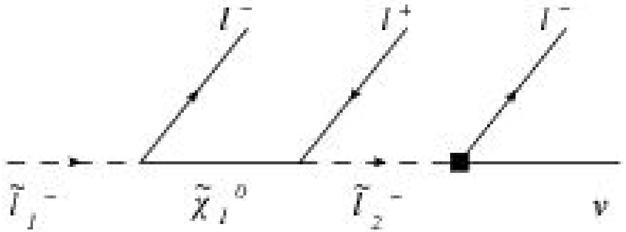
There is no "z-direction" degree of freedom.

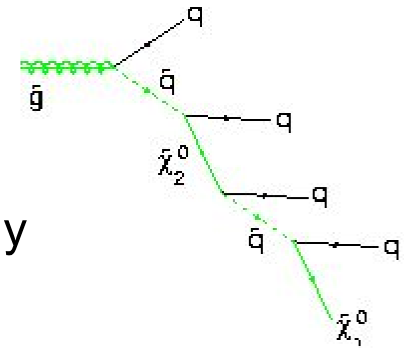


Higher multiplicity decays

4 body







5 body

- One invisible particle
- Depicted as chain of 2-body decays, but intermediate particles could be off-shell giving a 3-body vertex

4-body likelihood functions

$$\mathcal{L}(\tilde{m}) \sim \prod_{\text{events}} \Delta_4^{-1/2}(\tilde{m})\,\Theta(\Delta_4(\tilde{m}))$$

Recall: the goal is to find the mass hypothesis encoding the boundary that just barely contains all the data.

- This likelihood function will be computed for each possible mass spectrum hypothesis, calculating the quantities as if that were the true mass spectrum.
- If any event is outside the boundary specified by a spectrum hypothesis, the likelihood is zero and that spectrum is ruled out (ideal case).
- Otherwise, the hypothesis will be rewarded if its boundary is near many events.

4-body likelihood functions

$$\mathcal{L}(\tilde{m}) \sim \prod_{\text{events}} \Delta_4^{-1/2}(\tilde{m})\,\Theta(\Delta_4(\tilde{m}))$$

- ❑ Event outside → excluded
- ❑ Otherwise, favored if events near boundary
- ❑ The spectrum with the highest likelihood is the winner!

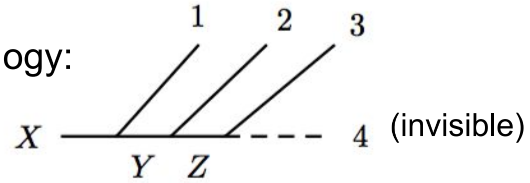
$$Q = \left(\sum_{i=\text{endpts.}} \left(\frac{\mathcal{O}_{i,predicted} - \mathcal{O}_{i,measured}}{\mathcal{O}_{i,measured}} \right)^2 \right)$$

For comparison, a likelihood function based on 1-d distributions, aka endpoints.

- Endpoints are computed for each mass spectrum hypothesis.
- Spectrum excluded if any event is beyond any endpoint predicted by the hypothesis.
- Otherwise, spectrum favored if observed endpoint is near predicted endpoint.

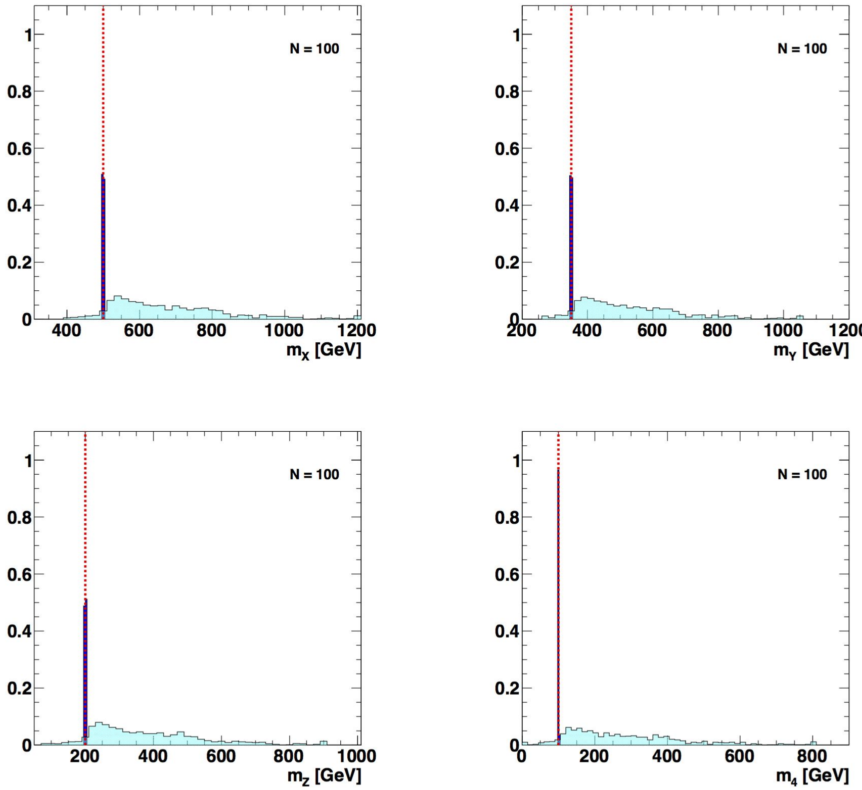
4-body results

Topology:



Histogram of winning mass spectra for many samples of Monte Carlo data, each with 100 events.

True value indicated by red.
Phase space boundary winners in blue.
Endpoint winners in cyan.

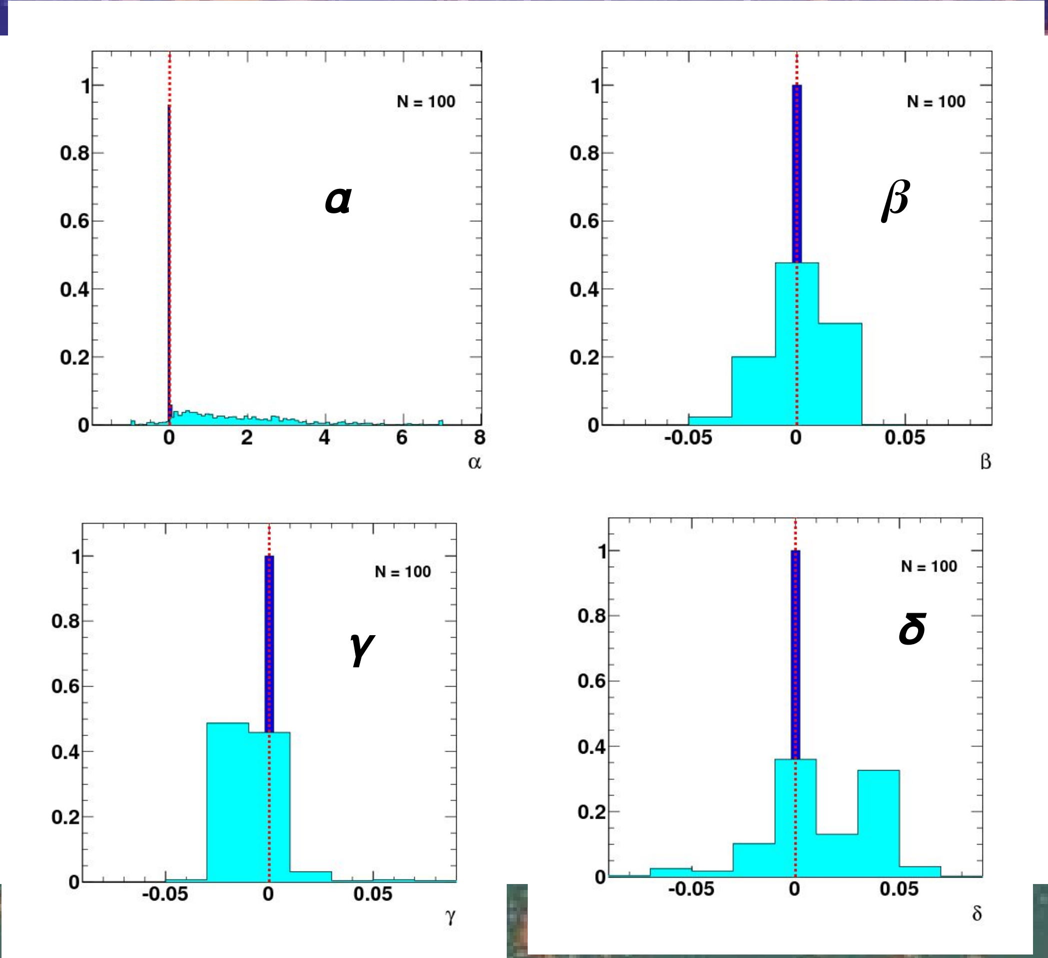


4-body results, overall scale basis

The boundary method gets both the scale and the differences!

Instead of plotting each mass separately, we rotate to a basis where:

- ***α*** parameterizes the mass scale (i.e., a uniform shift of the masses), and
- ***β***, ***γ***, ***δ*** parameterize orthogonal linear combinations of the masses that contain the differences.

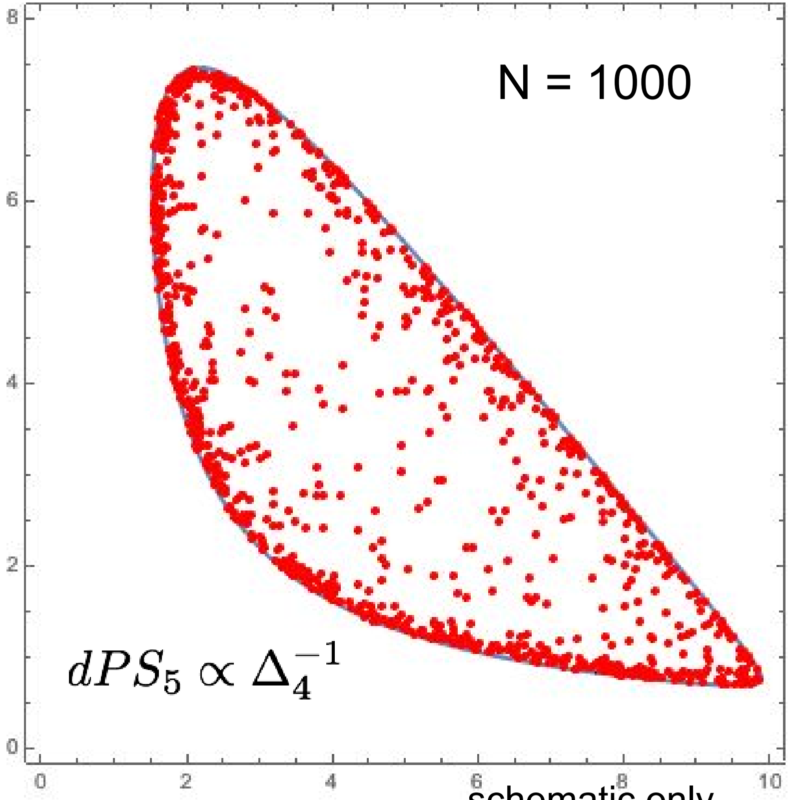


What happens for *n* > 4 ?

Punch line: *The boundary enhancement is even stronger than the 4-body case!*

$$dPS_5 \propto \Delta_4^{-1}$$

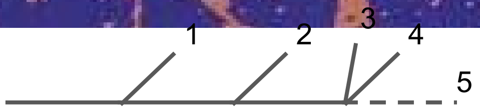
Intuition: when all 5 particles span a plane, the coordinate change is even more singular.



schematic only

5-body results

Topology:



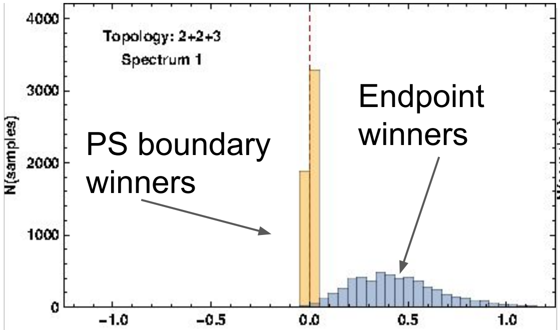
Likelihood-based analysis:

- Disallow hypotheses with events outside boundary
- Favor hypotheses with events near boundary

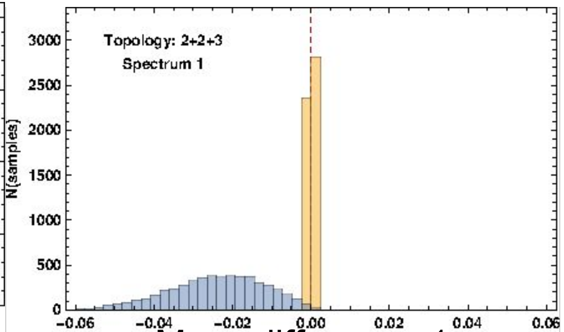
Histogram of winning mass spectra for many samples of Monte Carlo data of 100 events.

True value indicated by red.
Phase space boundary winners in blue.
Endpoint winners in cyan.

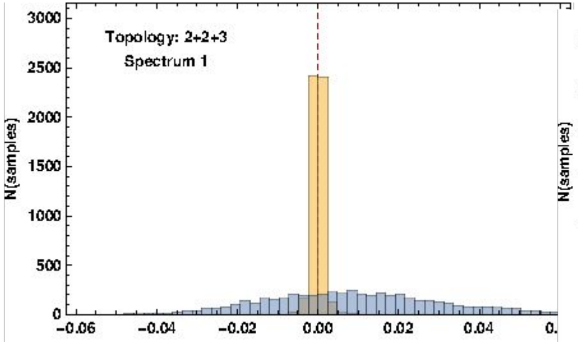
The boundary method rapidly converges on the correct masses, even for this complicated and high-multiplicity final state.



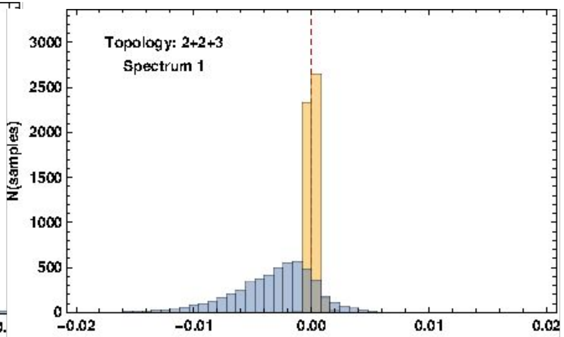
Overall mass scale



Mass difference 1



Mass difference 2

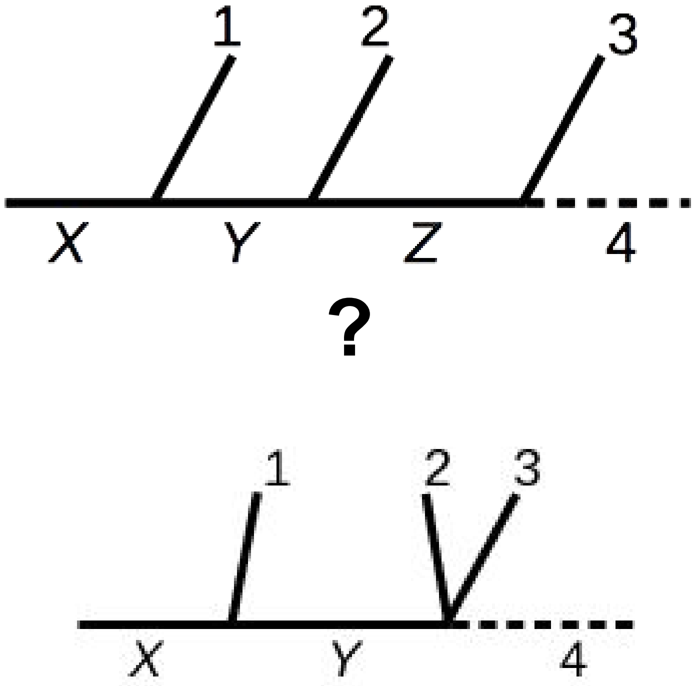


Mass difference 3

Topology determination

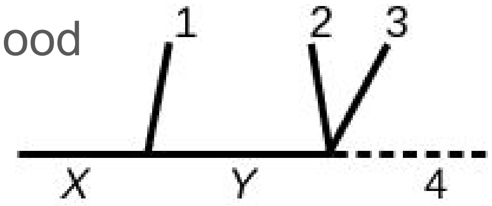
The decay topology of a new particle will not be known a priori.

The likelihood functions used for the mass determination can also be applied to determine the topology.



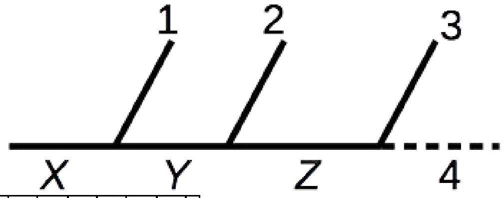
Topology determination

Suppose we try to use the likelihood function appropriate for:

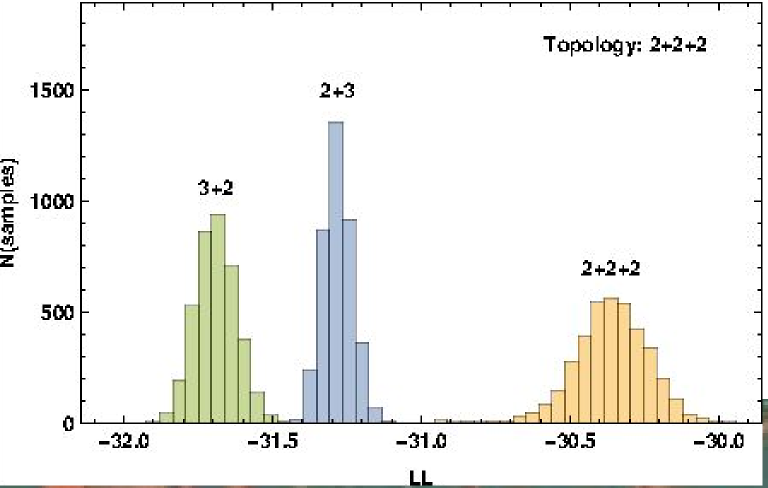


"2+3"

But the true topology is:



"2+2+2"



Histogram of log-likelihood for winning mass spectra under different topology assumptions.

The likelihood function for the correct topology is greatest.

Caveats

- We have ignored all backgrounds
- We have assumed single production of the heavy parent particle and assumed all final state particles are distinguishable, i.e. ignored combinatoric uncertainties
- We have ignored detector effects
- Likelihood function uses the distribution events in the *interior*, requiring knowledge of the matrix element

What we really want is a way to pick "shell-like" features out of a set of points in a space of arbitrary dimension, without reference to the detailed distribution of those points in the vicinity.

Conclusions

- **A likelihood technique that considers the full dimensionality of phase space can then rapidly reconstruct the involved masses with few events**

  - The mass content of particle decays is encoded in the shape of the phase space boundary
  - For $n > 3$ final states, events preferentially cluster on the boundary
  - This effect becomes *stronger* as the multiplicity increases
  - This powerful technique easily outperforms traditional techniques which are based on 1-d projections of phase space
  - Voronoi tesselation or other techniques under investigation may provide a realistic means of accessing this information